East Asian Math. J. **24** (2008), No. 4, pp. 415–419

ON GENERIC SUBMANIFOLDS WITH SASAKIAN STRUCTURE OF $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$

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Abstract. Let M be a generic submanifold of $S^n \times S^n$. If M admits an Sasakian structure, then M is a Brieskorn manifold.

1. Diferential geometry of $S^n \times S^n$

In 1973, K. Yano [1] proved that the (f, g, u, v, λ)-structure induced on $S^n \times S^n$. In this paper, we consider the global form of generic submanifolds with sasakian structure of $S^n \times S^n$. Consider an $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ in E^{2n+2} covered by a system of coordinate neighborhoods $\{U \times V; x^h\}$, where here and in the sequel the indices $h, i, j, k, l, m, n, \ldots$ run over the range $\{1, 2, 3, \ldots, 2n\}$ and denote by ∇_i the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$ formed with g_{ji}.

Then we have, so called an (f, g, u, v, λ)-structure,

$$\begin{cases} f_j^i f_i^h = -\delta_j^h + u_j u^h + v_j v^h, \\ u_i f_j^i = \lambda v_j, \ f_i^h u^i = -\lambda v^h, \ v_i f_j^i = -\lambda u_j, \ f_i^h v^i = \lambda u^h, \\ u_i u^i = v_i v^i = 1 - \lambda^2, \ u_i v^i = 0, \\ f_j^m f_i^l g_{ml} = g_{ji} - u_j u_i - v_j v_i. \end{cases} \tag{1.1}$$

where $f_{ji} = f_j^l g_{li}$ is skew-symmetric in j and i.

Denoting by h_{ji} and k_{ji} the component with respect to the unit normals, then we have, $h_{ji} = g_{ji}$,

$$\begin{cases} \nabla_j f_i^h = -g_{ji} u^h + \delta_j^h u_i - k_{ji} v^h + k_j^h v_i, \\ \nabla_j u_i = f_{ji} - \lambda k_{ji}, \\ \nabla_j v_i = -k_{jl} f_i^l + \lambda g_{ji}, \\ \nabla_j \lambda = -2 v_j. \end{cases} \tag{1.2}$$

Y. H. Shin and T. H. Kang [2] researched the condition that a real hypersurface of $S^n \times S^n$ becomes a Brieskorn manifold.

Received July 2, 2008; Accepted September 5, 2008.
2000 *Mathematics Subject Classification.* 51M99,53C25.
Key words and phrases. generic submanifold, sasakian structure.

We introduce the following Theorem A for later use.

Theorem A ([2]). *Let M be a hypersurface of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ $(n > 1)$ with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure, and let M admits an almost contact metric structure (f_b^a, g_{cb}, p^a), p^a being a killing vector. Then M as a submanifold of codimension 3 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$, that is, a Brieskorn manifold B^{2n-1}.*

2. Generic submanifolds of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ admitting an almost contact metric structure

Let M be an m-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{\widetilde{U}; \eta^a\}$ and isometrically immersed in $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ by the immersion

$$\iota : M \to S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}}),$$

where, here and in the sequel, indices a, b, c, d and e run the range $\{1, 2, \ldots, n\}$. We identify $\iota(M)$ with M itself and represent the immersion locally by

$$X^h = X^h(\eta^a).$$

If we put $B_c^h = \partial_c X^h (\partial_c = \partial/\partial\eta^c)$, then B_c^h are m linearly independent vectors of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ tangent to M which span the tangent space at every point of M.

Denoting by g_{cb} the Riemannian metric tensor of M, we have $g_{cb} = g_{ji} B_c^j B_b^i$ since the immersion is isometric.

We denote by C_x^h $2n - m$ mutually orthogonal unit normals of M. (In the sequel, the indicies x, y, z and u run over the range $\{m+1, \cdots, 2n\}$.)

$$g_{ji} B_b^j C_x^i = 0$$

and the metric tensor g^* induced on the normal bundle of M from the metric tensor g_{ji} of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ has components g_{xy} given by

$$g_{xy} = g_{ji} C_x^j C_y^j = \delta_{xy},$$

δ_{xy} being the kronecker delta.

By the denoting ∇_c the operator of covariant differentiation with respect to g_{cb} the equations of Gauss and Weingarten are respectively given by

$$\nabla_c B_b^h = h_{cb}^x C_x^h, \nabla_c C_y^h = -h_{c\,y}^{\,a} B_a^h, \tag{2.1}$$

where h_{ch}^x are components of the second fundamental tensor of M with respect to the normals C_x^h and

$$h_{cy}^a = h_{cb}^x g^{ba} g_{xy},$$

g^{ba} being contravariant components of the metric tensor of M.

Now, we consider the submanifold M of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ which satisfies

$$N_p(M) \perp f(N_p(M))$$

at each point p of M, where $N_p(M)$ denotes the normal space of M at p, f being the structure tensor of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$.

Such a submanifold is called generic (or antiholomorphic) submanifold ([3]).

From now on, we consider generic submanifold immersed in $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$. Then we can put in each coordinate neighborhood.

$$f_j^h B_c^j = f_c^a B_a^h - f_c^x c_x^h, \tag{2.2}$$

$$f_j^h C_x^j = f_x^a B_a^h, \tag{2.3}$$

$$u^h = u^a B_a^h + u^x C_x^h, \tag{2.4}$$

$$v^h = v^a B_a^h + v^x C_x^h, \tag{2.5}$$

where f_c^a is a tensor field of type (1,1) defined on M, f_c^x a local 1-form for each fixed index x,v^a and v^a vector fields, u^x and v^x functions for fixed index x, and

$$f_x^a = f_c^y g^{ac} g_{yx}.$$

Applying f to (2.2)-(2.5) succesively and using (1.1), we find respectibely

$$f_c^a f_a^b = -\delta_c^b + u_c u^b + v_c v^b + f_c^x f_x^b, \tag{2.6}$$

$$f_c^e f_e^x = -(u_c u^x + v_c v^x), \tag{2.7}$$

$$f_x^e f_e^a = u^a u_x + v^a v_x, \tag{2.8}$$

$$f_x^e f_e^y = \delta_x^y - u_x u^y - v_x v^y, \tag{2.9}$$

$$u^e f_e^a = -\lambda v^a - u^x f_x^a, \tag{2.10}$$

$$u^e f_e^x = \lambda v^x, \tag{2.11}$$

$$v^e f_e^a = \lambda u^a - u^x f_x^a, \tag{2.12}$$

$$v^e f_e^x = -\lambda u^x, \tag{2.13}$$

$$u_a u^a + u_x u^x = v_a v^a + v_x v^x = 1 - \lambda^2, \quad u_a v^a + u_x v^x = 0. \tag{2.14}$$

Putting $f_{cb} = f_c^a g_{ab}$, $f_{cx} = f_c^y g_{yx}$ and $f_{xc} = f_x^a g_{ac}$, we can easily find

$$f_{cb} = -f_{bc}, \ f_{cx} = f_{xc} \tag{2.15}$$

By differentiating (2.4) covariantly, we obtain

$$\nabla_c u^a = f_c^a - \lambda k_c^a + h_{cx}^a u^x \tag{2.16}$$

by means of (1.2) and (2.1).

Suppose that the generic submanifold M of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ admits an almost contact metric structure (f_c^a, g_{cb}, p^a). Then, we have

$$\begin{cases} f_b^e f_e^a = -\delta_b^a + p_b p^a, \ f_e^a p^e = 0, \ p_e f_b^e = 0, \\ p_e p^e = 1, \ g_{de} f_c^d f_b^e = g_{cb} - p_c p_b, \end{cases} \tag{2.17}$$

where p_c is a 1-form associated with the vector field p^a given by $p_c = p^a g_{ac}$.

On the other hand, comparing (2.6) with the first equation of (2.17), we find

$$p_b p^a = u_b u^a + u_b v^a + f_b^x f_x^a. \tag{2.18}$$

Transvecting this with p^a, we get

$$p_b = Au_b + Bv_b + C_x f_b^x, \tag{2.19}$$

where we have put $A = p_e u^e, B = p_e v^e$ and $C_x = p_a f_x^a$. Also, transvecting p^b gives

$$1 = A^2 + B^2 + C_x C^x, \tag{2.20}$$

because of (2.17) contraction (2.18) with respect to the indices b and a yields

$$1 = u_e u^e + v_e v^e + f_{bx} f^{bx}$$

or, using (2.9) and (2.14),

$$\lambda^2 + u_x u^x + v_x v^x = \frac{1}{2}(2n - m + 1). \tag{2.21}$$

If we transvect (2.18) with $u^b u_a$ and make use of (2.11) and (2.14), then we find

$$A^2 = (1 - \lambda^2 - u_x u^x)^2 + (u_x v^x)^2 + \lambda^2 (v_x v^x). \tag{2.22}$$

Similarly, transvecting (2.18) with $v^b v_a$ and taking account of (2.13) and (2.14), we get

$$B^2 = (u_x v^x)^2 + (1 - \lambda^2 - v_x v^x)^2 + \lambda^2 (u_x u^x). \tag{2.23}$$

Transvecting (2.18) with $f_y^b f_a^y$ and using (2.9), (2.11) and (2.13), we have

$$\begin{aligned} C_y C^y &= (\lambda^2 - 2)(v_x v^x + u_x u^x) \\ &\quad + 2(u_x u^x)^2 + (u_x u^x)^2 + (v_x v^x)^2 + 2n - m, \end{aligned} \tag{2.24}$$

where C_y denotes $C_y = f_y^a p_a$.

Tranvecting the second equation of (2.17) with f_a^y, we find

$$Au_x + Bv_x = 0$$

with the aid of (2.8), from which,

$$\begin{cases} A(u_x u^x) + B(u_x v^x) = 0, \\ A(u_x v^x) + B(v_x v^x) = 0. \end{cases} \tag{2.25}$$

Substituting (2.22), (2.23) and (2.24) into (2.20) gives

$$\begin{aligned} 1 &= 2 + 2n - m + 2(\lambda^4 - 2\lambda^2) \\ &\quad + 2\{2(u_x v^x)^2 + (u_x u^x)^2 + (v_x v^x)^2\} \\ &\quad + 4(\lambda^2 - 1)(u_x u^x + v_x v^x). \end{aligned} \tag{2.26}$$

Let's set (2.21) by

$$\lambda^2 + u_x u^x + v_x v^x = \frac{1}{2}(2n - m + 1) = 1 + \beta, \tag{2.27}$$

where β is a nonegative constant. Thus (2.26) reduces to

$$\beta(1+\beta) = 2\left[(u_x u^x)(v_y v^y) - (u_x v^x)^2\right].$$

If β is positive, then we have

$$(u_x u^x)(v_y v^y) - (u_x v^x)^2 > 0. \tag{2.28}$$

So, it follows from (2.25) that

$$A = B = 0.$$

Hence we have from (2.22), $u_a = 0$. Therefore, this together with (2.16) gives

$$f_c^a = 0,$$

which contradict to the fact that f_c^a has a maximal rank. And consequently β must be zero. Hence we can see from(2.27) that M is a hypersurface of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$. Thus we have

Theorem 2.1. *Let M be a generic submanifold of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$. If M admits an almost contact metric structure, then M is a hypersurface of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$.*

Finally, let M admits a Sasakian structure, that is, the given structure admits an almost metric structure (f_c^a, g_{cb}, p^a) and

$$\nabla_c f_b^a = -g_{cb} p^a + \delta_c^a p_b. \tag{2.29}$$

From (2.17) and (2.29), we get

$$\nabla_c p_a = f_{ca},$$

which shows that p^a is a killing vector because f_{ca} is skew-symmtric with respect to a and c. Combing Theorem A and Theorem 2.1 with the fact that p^a is a killing vector, we find

Theorem 2.2. *Let M be a generic submanifold of $S^n(\frac{1}{\sqrt{2}}) \times S^n(\frac{1}{\sqrt{2}})$ $(n > 1)$. If M admits a Sasakian structure (f_b^a, g_{cb}, p^a), then M as a submanifold of codimension 3 of a $(2n+2)$-dimensional Euclidean space E^{2n+2} is an intersection of a complex cone with generator C and a $(2n+1)$-dimensional unit sphere $S^{2n+1}(1)$, that is, a Brieskorn manifold B^{2n-1}.*

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