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LIGHTLIKE SUBMANIFOLDS OF INDEFINITE COSYMPLECTIC MANIFOLDS

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Abstract. We study invariant lightlike submanifolds and almost contact CR-lightlike submanifolds of an indefinite cosymplectic manifold.

1. Introduction

In [7], K. L. Duggal and B. Sahin introduced a general notion of contact Cauchy-Riemann (CR) lightlike submanifolds and studied the geometry of leaves of their distributions. This concept is similar to that of lightlike CR-submanifolds of indefinite Kaehler manifolds ([6]). But this new class of submanifolds includes neither invariant submanifolds nor screen real submanifolds. To include these two subcases they introduced a class, i. e., contact screen Cauchy-Riemann (SCR) lightlike submanifolds. This is a contact lightlike version of the CR-submanifolds ([1]) of a Kaehler manifold. On the other hand, the odd dimensional counterparts of indefinite Kaehler manifolds are indefinite cosymplectic manifolds. The lightlike hypersurfaces of indefinite Sasakian (resp. cosymplectic) manifolds are contained in the class of (almost) contact CR-lightlike submanifolds of indefinite Sasakian (resp. cosymplectic) manifolds ([8], [9]). In this context, we study invariant lightlike submanifolds and obtain analogous results to those in contact case. In addition, we define an almost contact CR-lightlike submanifold of indefinite cosymplectic manifolds and study the geometry of leaves of their distributions.

2. Preliminaries

In this section, we recall briefly some results from the general theory of lightlike submanifolds (cf. [6]). An m-dimensional submanifold M immersed in a semi-Riemannian manifold $(\bar{M}, \bar{g})$ of dimension $n+m$ is called a *lightlike submanifold* if it admits a degenerate metric g induced from $\bar{g}$ whose radical

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distribution $Rad(TM) := TM \cap TM^{\perp}$ is of rank $r(1 \leq r \leq m)$, where

$$TM^{\perp} := \cup_{x \in M}\{u \in T_x\bar{M} \ ; \ \bar{g}(u, v) = 0, \forall v \in T_xM\}.$$

Let $S(TM)$ be a *screen distribution* which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, i. e., $TM = Rad(TM) \perp S(TM)$, where the symbol $\perp$ denotes the orthogonal direct sum. We consider a *screen transversal vector bundle* $S(TM^{\perp})$, which is a semi-Riemannian complementary vector bundle of $Rad(TM)$ in $TM^{\perp}$. Let $(M, g, S(TM))$ be a lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. For any local basis $\{E_i\}$ of $Rad(TM)$, there exists a local frame $\{N_i\}$ of sections with values in the orthogonal compelment of $S(TM^{\perp})$ in $[S(TM)]^{\perp}$ such that $\bar{g}(E_i, N_j) = \delta_{ij}$ and $\bar{g}(N_i, N_j) = 0$. It follows that there exists a *lightlike transversal vector bundle* $ltr(TM)$ locally spanned by $\{N_i\}$.

Let $tr(TM)$ (called a *tranaversal vector bundle*) be complementary (but not orthogonal) vector bundle to TM in $T\bar{M} \mid_M$. Then we have decompositions.

$$\begin{aligned} tr(TM) &= ltr(TM) \perp S(TM^{\perp}), \\ T\bar{M} \mid_M &= S(TM) \perp \{Rad(TM) \oplus ltr(TM)\} \perp S(TM^{\perp}) \\ &= TM \oplus tr(TM). \end{aligned}$$

We note that the lightlike second fundamental forms of a lightlike submanifold M do not depend on $S(TM), S(TM^{\perp})$ and $ltr(TM)$.

We say that a submanifold $(M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is

Case 1: r-lightlike if $r < min\{m, n\}$;
Case 2: co-isotropic if $r = n < m$, $S(TM^{\perp}) = \{0\}$;
Case 3: isotropic if $r = m < n$, $S(TM) = \{0\}$;
Case 4: totally lightlike if $r = m = n$, $S(TM) = \{0\}$ and $S(TM^{\perp}) = \{0\}$.

It is clear from the decomposition of $T\bar{M}$ that the Gauss and Weingarten equations are given by

$$(1) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \ \ \forall X, Y \in \Gamma(TM),$$

$$(2) \qquad \bar{\nabla}_X U = -A_U X + \nabla^t_X U, \ \ \forall X \in \Gamma(TM), \ U \in \Gamma(tr(TM)),$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$, respectively. Moreover, we have

$$(3) \qquad \bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \ \ \forall X, Y \in \Gamma(TM),$$

$$(4) \qquad \bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \ \ N \in \Gamma(ltr(TM)),$$

$$(5) \qquad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \ \ W \in \Gamma(S(TM^{\perp})).$$

It is known ([6]) that $h^l = 0$ on $Rad(TM)$. In this point of view, we say that a lightlike submanifold $(M, g, S(TM), S(TM^{\perp}))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *minimal* (cf. [2], [7]) if $h^s = 0$ on $Rad(TM)$ and trace h=0, where trace is written with respect to g restricted to $S(TM)$. This definition is independent of $S(TM)$ and $S(TM^{\perp})$, but it depends on the choice of the transversal bundle $tr(TM)$ ([2]). Denote the projection of TM on $S(TM)$ by $\bar{P}$. Then, by using (1), (3), (4), (5) and the fact that $\bar{\nabla}$ is a metric connection, we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{6}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(N, A_W X). \tag{7}$$

From the decomposition of tangent bundle of lightlike submanifold, we have

$$\nabla_X \bar{P}Y = \nabla^*_X \bar{P}Y + h^*(X, \bar{P}Y), \tag{8}$$

$$\nabla_X E = -A^*_E X + \nabla^{*t}_X E, \tag{9}$$

for $X, Y \in \Gamma(TM)$ and $E \in \Gamma(Rad(TM))$, where

$$\{\nabla^*_X \bar{P}Y, A^*_E X\} \quad \text{and} \quad \{h^*(X, \bar{P}Y), \nabla^{*t}_X E\}$$

belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. It follows that ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. By using the above equations we obtain

$$\bar{g}(h^l(X, \bar{P}Y), E) = g(A^*_E X, \bar{P}Y), \tag{10}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{11}$$

$$\bar{g}(h^l(X,E),E) = 0, \qquad A^*_E E = 0. \tag{12}$$

In general, the induced connection ∇ on M is not metric connection, since

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y), \tag{13}$$

which is easily obtained by using (3) and the fact that $\bar{\nabla}$ is a metric connection. However, it is important to note that ∇^* is a metric connection on $S(TM)$.

3. Invariant Lightlike Submanifolds

An odd dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *almost contact metric manifold* (cf. [3], [5], [7], [8], [9], [11], [12]) if there are a (1,1) tensor field $\bar{\phi}$, a vector field ξ, called *characteristic vector field* and a 1-form η such that for any vector fields X, Y on $\bar{M}$,

$$\bar{g}(\bar{\phi}X, \bar{\phi}Y) = \bar{g}(X,Y) - \epsilon\eta(X)\eta(Y), \tag{14}$$

$$\bar{g}(\xi,\xi) = \epsilon, \qquad \epsilon = 1 \quad or \quad \epsilon = -1, \tag{15}$$

$$\bar{\phi}^2(X) = -X + \eta(X)\xi, \qquad \bar{g}(X,\xi) = \epsilon\eta(X). \tag{16}$$

Then $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an *almost contact metric structure* of $\bar{M}$. It follows that

$$\bar{\phi}(\xi) = 0, \quad \eta \circ \bar{\phi} = 0, \quad \eta(\xi) = \epsilon. \tag{17}$$

Also, the almost contact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is normal if $N_{\bar{\phi}} + d\eta \otimes \xi = 0$, where $N_{\bar{\phi}}$ is the Nijenhuis tensor field, which is defined by $N_{\bar{\phi}}(X,Y) = \bar{\phi}^2[X,Y] + [\bar{\phi}X, \bar{\phi}Y] - \bar{\phi}[\bar{\phi}X, Y] - \bar{\phi}[X, \bar{\phi}Y]$. Define a 2-form Φ on $\bar{M}$ by $\Phi(X,Y) = \bar{g}(X, \bar{\phi}Y)$. A normal almost contact metric structure $(\bar{\phi}, \xi, \eta, \bar{g})$ on

$\bar{M}$ is called a *cosymplectic structure* if $d\Phi = 0$ and $d\eta = 0$. It is characterized by

$$(18) \qquad \bar{\nabla}_X \bar{\phi} = 0, \quad \bar{\nabla}_X \eta = 0$$

for any vector field X on $\bar{M}$, where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$ (cf. [3], [8], [11]). A semi-Riemannian manifold $\bar{M}$ with a cosymplectic structure $(\bar{\phi}, \xi, \eta, \bar{g})$ is called an *indefinite cosymplectic manifold*.

Lemma 3.1. *For an indefinite cosymplectic manifold $\bar{M}$, we have*

$$(19) \qquad \bar{\nabla}_X \xi = 0, \quad \forall X \in \Gamma(T\bar{M}),$$

where ξ is the characteristic vector field.

Proof. Differentiating $\bar{\phi}(\xi) = 0$, we get $\bar{\phi}(\bar{\nabla}_X \xi) = 0$. Transvecting this with $\bar{\phi}$ and using (16), we complete the proof. □

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \xi, \eta, \bar{g})$. For any vector field X tangent to M, we put

$$(20) \qquad \bar{\phi}X = \phi X + FX,$$

where ϕX and FX are the tangential and transversal parts of $\bar{\phi}X$, respectively. Moreover, ϕ is skew symmetric on $S(TM)$.

It is known ([4]) that if M is tangent to the structure vector field ξ, then ξ belongs to $S(TM)$. Using this, we say that M is *invariant* in $\bar{M}$ if M is tangent to the structure vector field ξ and

$$(21) \qquad \bar{\phi}X = \phi X, \quad \text{i. e.,} \quad \bar{\phi}X \in \Gamma(TM), \quad \forall X \in \Gamma(TM).$$

From (1), (17), (18), (19) and (21) we get

$$(22) \qquad \nabla_X \xi = 0, \quad h^l(X, \xi) = 0 \quad \text{and} \quad h^s(X, \xi) = 0,$$

$$(23) \qquad h(X, \phi Y) = \bar{\phi} h(X, Y) = h(\phi X, Y), \quad \forall X, Y \in \Gamma(TM).$$

Remark 3.2. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. If M is invariant, then the distributions $Rad(TM)$ and $S(TM)$ are $\bar{\phi}$-invariant. It follows that distributions $S(TM^{\perp})$, $ltr(TM)$ and $tr(TM)$ are also $\bar{\phi}$-invariant. Therefore the aggregate (ϕ, ξ, η, g) defined on M forms a singular cosymplectic structure in a sense that the metric tensor g of M is degenerate. If $S(TM)$ is integrable, then each leaves of $S(TM)$ admits a cosymplectic structure.

Proposition 3.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an invariant lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g}, \bar{\phi}, \xi, \eta)$. Then $S(TM)$ is integrable if and only if*

$$h^*(X, \phi Y) = h^*(\phi X, Y), \ \forall X, Y \in \Gamma(S(TM)).$$

Proof. For an invariant lightlike submanifold M, we put

$$\phi X = sX + rX,$$

where $sX \in \Gamma(S(TM))$ and $rX \in \Gamma(Rad(TM))$. Differentiating this, we get from (8), (18) and (21)

$$r([X,Y]) = h^*(X,\phi Y) - h^*(\phi Y, X),$$

which completes the proof. □

Theorem 3.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an invariant lightlike submanifold of codimension two of an indefinite cosymplectic manifold $\bar{M}$. Then $Rad(TM)$ defines a totally geodesic foliation on M.*

Proof. Since $rank(Rad(TM)) = 2$, any $X, Y \in \Gamma(Rad(TM))$ can be written as a linear combination of E and $\bar{\phi}E$, respectively, i. e., $X = A_1E + B_1\bar{\phi}E$, $Y = A_2E + B_2\bar{\phi}E$. Thus by direct calculation, using (1) and (3) we obtain

$$\begin{aligned}
g(\nabla_X Y, \bar{P}Z) &= \bar{g}(\bar{\nabla}_X Y - h(X,Y), \bar{P}Z)\\
&= X(\bar{g}(Y,\bar{P}Z)) - \bar{g}(Y, \bar{\nabla}_X \bar{P}Z)\\
&= -\bar{g}(Y, h^l(X,\bar{P}Z))\\
&= -\bar{g}(A_2E + B_2\bar{\phi}E, h^l(A_1E + B_1\bar{\phi}E, \bar{P}Z))\\
&= -A_2A_1\bar{g}(E, h^l(E,\bar{P}Z)) - A_2B_1\bar{g}(E, h^l(\bar{\phi}E,\bar{P}Z))\\
&\quad -B_2A_1\bar{g}(\bar{\phi}E, h^l(E,\bar{P}Z)) - B_2B_1\bar{g}(\bar{\phi}E, h^l(\bar{\phi}E,\bar{P}Z))\\
&= 0,
\end{aligned}$$

where the last equality follows from (12). Hence the screen ditribution part of $\nabla_X Y$ vanishes, which means that $Rad(TM)$ defines a totally geodesic foliation. □

Theorem 3.5. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an invariant lightlike submanifold of codimension two of an indefinite cosymplectic manifold $\bar{M}$. Suppose that (M', g') is a non-degenerate submanifold of $\bar{M}$ such that M' is a leaf of integrable $S(TM)$. Then M is totally geodesic with an induced metric connection if M' is immersed as a submanifold of M.*

Proof. Since $dim(Rad(TM)) = dim(ltr(TM)) = 2$, $h^l(X,Y) = A_1N + B_1\bar{\phi}N$, where A_1 and B_1 are functions on M. Thus $h^l(X,E) = 0$ if and only if $\bar{g}(h^l(X,E),E) = 0$ and $\bar{g}(h^l(X,E),\bar{\phi}E) = 0$ for any $X \in \Gamma(TM)$ and $E \in \Gamma(Rad(TM))$. From (12), we have $\bar{g}(h^l(X,E),E) = 0$. Using (23), we get $\bar{g}(h^l(X,E),\bar{\phi}E) = -\bar{g}(h^l(\bar{\phi}X,E),E) = 0$. Similarly, we also have $h^l(X,\bar{\phi}E) = 0$.

On the other hand, for M', we write

$$\bar{\nabla}_X Y = \nabla'_X Y + h'(X,Y),\ \forall X, Y \in \Gamma(TM'),$$

where ∇' is a metric connection of M' and h' is the second fundamental form of M'. It is known ([6]) that ∇' is a metric connection of M' is equivalent to

$h' = 0$. Note that $g'(X,Y) = g(X,Y)$ for $X,Y \in \Gamma(S(TM))$. Thus we have $0 = h'(X,Y) = h^*(X,Y) + h^l(X,Y)$, $\forall X,Y \in \Gamma(TM')$. Summing up, we have $h^l(X,Y) = 0$ for any $X,Y \in \Gamma(TM)$, which completes the proof. $\square$

Theorem 3.6. *Let $(M, g, S(TM), S(TM^\perp))$ be an invariant lightlike submanifold of an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. Then M is minimal in $\bar{M}$ if and only if $D^l(X,W) = 0$ for $X \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM))$.*

Proof. Since M is invariant, it follows from Remark 3.2 that $S(TM)$ is $\bar{\phi}$-invariant. Thus we can take a local $\bar{\phi}$-basis $\{e_i, \bar{\phi}e_i, \xi\}_{i=1,\cdots,(m-r-1)/2}$ on $S(TM)$, where r denotes the rank of $Rad(TM)$. It is clear from (22) and (23) that

$$\text{trace } h = \sum_i \epsilon_i\{h(e_i,e_i) + h(\bar{\phi}e_i, \bar{\phi}e_i)\} + \epsilon h(\xi,\xi) = 0,$$

where $\epsilon_i = +1$ or -1. From (6), we have

$$\bar{g}(h^s(X,Y), W) = -\bar{g}(Y, D^l(X,W))$$

for any $X,Y \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$. Hence by definition, we complete the proof. $\square$

A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\bar{M},\bar{g})$ is *totally umbilical* (cf. [7]) in $\bar{M}$ if there is a smooth transversal vector field $\mathbf{H} \in \Gamma(tr(TM))$ on M, called the *transversal curvature vector field* of M, such that, for all $X,Y \in \Gamma(TM)$,

$$h(X,Y) = \mathbf{H}g(X,Y). \tag{24}$$

It is easy from (3) and (6) to see that M is totally umbilical if and only if on each coordinate neighbourhood $\mathcal{U}$ there exist smooth vector fields $\mathbf{H}^l \in \Gamma(ltr(TM))$ and $\mathbf{H}^s \in \Gamma(S(TM^\perp))$ such that

$$h^l(X,Y) = \mathbf{H}^l g(X,Y), \quad D^l(X,W) = 0, \tag{25}$$

$$h^s(X,Y) = \mathbf{H}^s g(X,Y), \quad \forall X,Y \in \Gamma(TM), W \in \Gamma(S(TM^\perp)). \tag{26}$$

Theorem 3.7. *Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold, tangent to the structure vector field ξ, of an indefinite cosymplectic manifold $(\bar{M},\bar{g})$. If M is totally umbilical, then M is totally geodesic.*

Proof. Since ξ is tangent to M, we have from Lemma 3.1 and (3)

$$h^s(X,\xi) = 0 = h^l(X,\xi), \ \forall X \in \Gamma(TM). \tag{27}$$

It follows from (25) and (26) that

$$h^l(\xi,\xi) = \mathbf{H}^l g(\xi,\xi) = \epsilon\mathbf{H}^l = 0, \qquad \text{so, } \mathbf{H}^l = 0,$$
$$h^s(\xi,\xi) = \mathbf{H}^s g(\xi,\xi) = \epsilon\mathbf{H}^s = 0, \qquad \text{so, } \mathbf{H}^s = 0,$$

since M is totally umbilical and ξ is a non-null vector field. Thus we get

$$h^l(X,Y) = \mathbf{H}^l g(X,Y) = 0,$$
$$h^s(X,Y) = \mathbf{H}^s g(X,Y) = 0, \ \forall X,Y \in \Gamma(TM),$$

which imply $h^l = 0$ and $h^s = 0$. So M is totally geodesic. □

Any lightlike totally geodesic submanifold is minimal. Hence it follows from Theorem 3.6 that any totally umbilical lightlike submanifold with structure vector field tangent to submanifold is minimal.

4. Almost Contact CR-lightlike Submanifolds

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold, tangent to the structure vector field ξ, immersed in an indefinite cosymplectic manifold $(\bar{M}, \bar{g})$. We say that M is an *almost contact CR-lightlike submanifold* of $\bar{M}$ if the following conditions (i) and (ii) are satisfied.
(i) $Rad(TM)$ is a distribution on M such that

$$Rad(TM) \cap \bar{\phi}(Rad(TM)) = \{0\}.$$

(ii) There exist vector bundles D_0 and D' over M such that

$$S(TM) = \{\bar{\phi}(Rad(TM)) \oplus D'\} \perp D_0 \perp \xi,$$
$$\bar{\phi}(D_0) = D_0, \;\; \bar{\phi}(D') = L_1 \perp ltr(TM),$$

where D_0 is non-degenerate and L_1 is a vector subbundle of $S(TM^{\perp})$.

Thus, we have the following decomposition

$$TM = D \oplus D' \oplus \xi, \;\; D = Rad(TM) \perp \bar{\phi}(Rad(TM)) \perp D_0. \tag{28}$$

An almost contact CR-lightlike submanifold is *proper* if $D_0 \neq \{0\}$ and $L_1 \neq \{0\}$. We note that any almost contact CR-lightlike 3-dimensional submanifold is 1-lightlike.

Remark 4.1. From this definition and the decomposition of $T\bar{M}$ as appeared in section 2, we obtain that a vector subbundle $L_1^{\perp}$ in $S(TM^{\perp})$ is $\bar{\phi}$-invariant and $D' = \bar{\phi}L_1 \perp \bar{\phi}(ltr(TM))$.

Example 4.2. Let (M, g) be a lightlike hypersurface of $\bar{M}$([8]). Then for a nonzero local section $E \in \Gamma(Rad(TM)) = TM^{\perp}$, $\bar{g}(\bar{\phi}E, E) = 0$, which implies that $\bar{\phi}E$ is tangent to M. Hence we get a distribution $\bar{\phi}(Rad(TM))$ of rank one such that $\bar{\phi}(Rad(TM)) \cap Rad(TM) = \{0\}$. Now we choose a screen distribution $S(TM)$ such that it contains both $\bar{\phi}E$ and ξ. For any local section $E \in \Gamma(Rad(TM))$, there exists a unique lightlike local section N in $\Gamma(ltr(TM))$ such that $\bar{g}(N, E) = 1$. On the other hand, we obtain $\bar{g}(\bar{\phi}N, E) = -\bar{g}(N, \bar{\phi}E) = 0$ and $\bar{g}(N, N) = 0$, which mean that $\bar{\phi}N \in \Gamma(S(TM))$. Taking $D' = \bar{\phi}(tr(TM))$, we get a non-degenerate vector subbundle $\bar{\phi}(TM^{\perp}) \oplus \bar{\phi}(tr(TM))$ of $S(TM)$ of rank 2. Then there exists a non-degenerate distribution D_0 on M such that

$$S(TM) = \{\bar{\phi}(Rad(TM)) \oplus \; D'\} \perp D_0 \perp \xi,$$

where D_0 is an invariant distribution with respect to $\bar{\phi}$, i. e., $\bar{\phi}(D_0) = D_0$. Moreover $\bar{\phi}(D') = tr(TM)$. Thus M is an almost contact CR-lightlike hypersurface.

Proposition 4.3. *There exist no isotropic or totally lightlike almost contact CR-lightlike submanifolds on $\bar{M}$.*

Proof. If M is isotropic or totally lightlike, then $S(TM) = \{0\}$. Hence conditions (i) and (ii) of the definition are not satisfied. □

Denote the orthogonal complement subbundle to the vector subbundle L_1 in $S(TM^\perp)$ by $L_1^\perp$. For an almost contact CR-lightlike submanifold M we put

$$\bar{\phi}X = fX + \omega X, \ \ \forall X \in \Gamma(TM), \tag{29}$$

where $fX \in \Gamma(D)$ and $\omega X \in \Gamma(L_1 \perp ltr(TM))$. Similarly, we have

$$\bar{\phi}W = BW + CW, \ \ \forall W \in \Gamma(S(TM^\perp)), \tag{30}$$

where $BW \in \Gamma(\bar{\phi}L_1)$ and $CW \in \Gamma(L_1^\perp)$.

Proposition 4.4. *Let M be an almost contact CR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Then we have the followings :*

(i) $D \oplus \xi$ is integrable if and only if the second fundamental form of M satisfies

$$h(X, \bar{\phi}Y) = h(\bar{\phi}X, Y), \quad \forall X, Y \in \Gamma(D \oplus \xi).$$

(ii) The totally real distribution D' is integrable if and only if the shape operator of M satisfies

$$A_{\bar{\phi}X}Y = A_{\bar{\phi}Y}X, \quad \forall X, Y \in \Gamma(D').$$

Proof. It is clear that $\omega[X, Y] = 0$ for any $X, Y \in \Gamma(D \oplus \xi)$ if and only if $D \oplus \xi$ is integrable. Differentiating (29) along $Y \in \Gamma(D \oplus \xi)$ and taking transversal parts, we get from Remark 4.1 $\omega(\nabla_Y X) = -Ch^s(X, Y) + h(Y, \bar{\phi}X)$, which implies that $\omega[X, Y] = h(Y, \bar{\phi}X) - h(X, \bar{\phi}Y)$ for any $X, Y \in \Gamma(D \oplus \xi)$, where we have used (1), (2), (18) and (30). Thus we prove (i). For the proof of (ii) we also obtain that $f[X, Y] = A_{\bar{\phi}X}Y - A_{\bar{\phi}Y}X$, which implies (ii). □

Proposition 4.5. *Let M be an almost contact CR-lightlike submanifold of an indefinite cosymplectic manifold $\bar{M}$. Then $D \oplus \xi$ is totally geodesic foliation if and only if*

$$h^l(X, \bar{\phi}Y) = 0, \quad \textit{and} \quad h^s(X, Y) \textit{ has no components in } L_1. \tag{31}$$

Proof. Since $D \oplus \xi$ defines a totally geodesic foliation if and only if $g(\nabla_X Y, \bar{\phi}E) = 0 = g(\nabla_X Y, W)$ for $X, Y \in \Gamma(D \oplus \xi)$ and $W \in \Gamma(\bar{\phi}L_1)$, we have from (3) and (18)

$$\begin{aligned} g(\nabla_X Y, \bar{\phi}E) &= -\bar{g}(\bar{\phi}\bar{\nabla}_X Y, E) - \bar{g}(\bar{\nabla}_X \bar{\phi}Y, E) \\ &= -\bar{g}(\nabla_X \bar{\phi}Y + h(X, \bar{\phi}Y), E) \\ &= -\bar{g}(h^l(X, \bar{\phi}Y), E). \end{aligned} \tag{32}$$

In a similar way, we have

$$g(\nabla_X \bar{\phi}Y, W) = -\bar{g}(h^s(X, Y), \bar{\phi}W). \tag{33}$$

Hence the proof follows from (32) and (33). □

A lightlike submanifold M of a semi-Riemannian manifold is an *irrotational submanifold* ([10]) if $\bar{\nabla}_X E \in \Gamma(TM)$ for any $X \in \Gamma(TM)$ and $E \in \Gamma(Rad(TM))$.

From (3) we concluded that M is an irrotational lightlike submanifold if and only if the followings hold :

$$h^s(X, E) = 0, \;\; h^l(X, E) = 0, \; \forall X \in \Gamma(TM).$$

We say that a lightlike submanifold M of an indefinite cosymplectic manifold $\bar{M}$, is a *screen real submanifold* ([7]) if $Rad(TM)$ and $S(TM)$ are invariant and anti-invariant with respect to $\bar{\phi}$, respectively.

Proposition 4.6. *An almost contact CR-lightlike submanifolds are nontrivial.*

Proof. Suppose M is an invariant lightlike submanifold of an indefinite cosymplectic manifold. Then we see from Remark 3.2 that radical distribution is invariant, which is not consistent with condition (i) of the definition. Similarly, the case of the screen real lightlike can be also argued. □

We know from Proposition 4.6 that almost contact CR-lightlike submanifolds exclude the invariant and screen real subcases.

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