Bull. Korean Math. Soc. **45** (2008), No. 2, pp. 221–230

A CHARACTERIZATION OF THE VANISHING OF THE SECOND PLURIGENUS FOR NORMAL SURFACE SINGULARITIES

Koukichi Wada

Abstract. In the study of normal (complex analytic) surface singularities, it is interesting to investigate the invariants. The purpose of this paper is to give a characterization of the vanishing of δ_2. In [11], we gave characterizations of minimally elliptic singularities and rational triple points in terms of the second plurigenera δ_2 and γ_2. In this paper, we also give a characterization of rational triple points in terms of a certain computation sequence. To prove our main theorems, we give two formulae for δ_2 and γ_2 of rational surface singularities.

1. Introduction

Let (X,x) be a normal (complex analytic) surface singularity, $\pi : (\widetilde{X}, A) \to (X,x)$ the minimal good resolution of the singularity (X,x), i.e., the smallest resolution for which A consists of non-singular curves intersecting transversally, with no three through one point. We always assume that X is Stein and sufficiently small. The geometric genus of (X,x) is defined by $p_g(X,x) = \dim_{\mathbb{C}} H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. This number has been studied from many viewpoints (e.g., [1], [3], [4], and so on). In particular, some numerical characterizations of the vanishing of p_g were given. In Section 2, we recall those results. The following two kinds of typical plurigenera are defined by Knöller [2] and Watanabe [13] respectively as follows:

$$\gamma_m(X,x) = \dim_{\mathbb{C}} \Gamma(\widetilde{X} - A, \mathcal{O}_{\widetilde{X}}(mK))/\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK)),$$

$$\delta_m(X,x) = \dim_{\mathbb{C}} \Gamma(\widetilde{X} - A, \mathcal{O}_{\widetilde{X}}(mK))/\Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(mK + (m-1)A)),$$

where $m \in \mathbb{N}$ and K denotes the canonical divisor on $\widetilde{X}$. Note that $p_g = \gamma_1 = \delta_1 \le \delta_2 \le \gamma_2$. Knöller [2] proved that (X,x) is a rational double point if and only if $\gamma_m(X,x) = 0$ for all $m \in \mathbb{N}$. Watanabe [13] proved that (X,x) is a quotient singularity if and only if $\delta_m(X,x) = 0$ for all $m \in \mathbb{N}$. Okuma [8] obtained a formula for δ_m of normal surface singularities. He [7] also gave a

Received July 7, 2005.

2000 *Mathematics Subject Classification.* Primary 14J17; Secondary 32S12.

Key words and phrases. second plurigenus, rational surface singularity, rational triple points.

formula for δ_2 of Gorenstein surface singularities and proved relations among the invariants δ_2, p_g, μ, τ (Milnor number and Tjurina number of (X,x)) and the modality. We [11] gave characterizations of minimally elliptic singularities and rational triple points in terms of the second plurigenera δ_2 and γ_2. In particular, we have that (X,x) is a rational double point or a rational triple point if and only if $\gamma_2(X,x)=0$. Concerning δ_2, we do not have the fact. In general, it is difficult to determine all the minimal (or minimal good) resolution graphs of normal surface singularities with $\delta_2=0$ (cf. [10]). Our purpose is to give a topological criterion (on the minimal resolution) for the vanishing of the second plurigenus δ_2.

Let (X,x) be a normal surface singularity and $\pi:(\widetilde{X},A)\to(X,x)$ the minimal resolution and $A=\bigcup_{i=1}^n A_i$ the decomposition of the exceptional set A into irreducible components. We define the unique smallest positive cycle "W" satisfying the property $(-K-W)\cdot A_i\geq 0$ for all A_i. The positive cycle W can be computed inductively as in [3]. We prove the following main theorems.

Theorem 1.1. *In the situation above, $\delta_2(X,x)=0$ if and only if (X,x) is rational and for any (or some) computation sequence $\{W_i\}_{i=0}^l$ from A to W, we have $(K+W_{j-1})\cdot A_{i_j}=1$ for $j>0$, where $W_0=A$.*

The following theorem is a characterization of rational triple points in terms of the computation sequence.

Theorem 1.2. *In the situation above, (X,x) is a rational triple point if and only if (X,x) is rational and there exists a computation sequence $\{W_i\}_{i=0}^l$ from A_{i_1} to W with the property $(K+W_{j-1})\cdot A_{i_j}=1$ for $j>0$, where $W_0=0$.*

2. Preliminaries

Let $\pi:(\widetilde{X},A)\to(X,x)$ be a resolution of a normal surface singularity and $A=\bigcup_{i=1}^n A_i$ the decomposition of the exceptional set A into irreducible components. A cycle D is an integral combination of the A_i, i.e., $D=\sum_{i=1}^n d_iA_i$ with $d_i\in\mathbb{Z}$, where $\mathbb{Z}$ denotes the rational integer. There is a natural partial ordering between cycles defined by comparing the coefficients. A cycle D is said to be effective if $d_i\geq 0$ for all i. In particular, a cycle D is said to be positive if D is effective and $D\neq 0$. For any two positive cycles V and W, there exists an exact sequence

$$0\to\mathcal{O}_W\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_{\widetilde{X}}(-V)\to\mathcal{O}_{V+W}\to\mathcal{O}_V\to 0.$$

Notation 2.1. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{\widetilde{X}}$-modules, D a divisor on $\widetilde{X}$ and E an effective cycle on $\widetilde{X}$. We use the following notation: $\mathcal{F}(D)=\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_{\widetilde{X}}(D)$, $\mathcal{F}_E=\mathcal{F}\otimes_{\mathcal{O}_{\widetilde{X}}}\mathcal{O}_E$, $H^i(\mathcal{F})=H^i(\widetilde{X},\mathcal{F})$, $H^i{}_A(\mathcal{F})=H^i{}_A(\widetilde{X},\mathcal{F})$, $h^i(\mathcal{F})=\dim_{\mathbb{C}}H^i(\mathcal{F})$ and $h^i{}_A(\mathcal{F})=\dim_{\mathbb{C}}H^i{}_A(\mathcal{F})$. If $E=0$, then we put $\mathcal{F}_E=0$.

We denote by K the canonical divisor on $\widetilde{X}$. The Riemann-Roch theorem implies for any positive cycle V and any invertible sheaf $\mathcal{F}$ on $\widetilde{X}$, that $\chi(\mathcal{O}_V)=$

$h^0(\mathcal{O}_V) - h^1(\mathcal{O}_V) = -V \cdot (V + K)/2$ and $\chi(\mathcal{O}_V \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{F}) = h^0(\mathcal{O}_V \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{F}) - h^1(\mathcal{O}_V \otimes_{\mathcal{O}_{\tilde{X}}} \mathcal{F}) = \mathcal{F} \cdot V + \chi(\mathcal{O}_V)$. There exists the unique smallest positive cycle Z satisfying $Z \cdot A_i \leq 0$ for all i ([1], p.131). The cycle Z is called a *fundamental cycle*. The fundamental cycle can be computed inductively as follows [3]: put $Z_0 = A_{i_0}$. Given Z_k there are two possibilities:

(1) if there is an $A_{i_{k+1}}$ such that $Z_k \cdot A_{i_{k+1}} > 0$, then put $Z_{k+1} = Z_k + A_{i_{k+1}}$,

(2) otherwise we are finished and the fundamental cycle $Z = Z_k$.

For a computation sequence $\{Z_j\}_{j=0}^{l+1}$ for Z, by the Riemann-Roch theorem, we have

$$p(Z) = \sum_{j=0}^{l+1} p(A_{i_j}) + \sum_{j=1}^{l+1} (Z_{j-1} \cdot A_{i_j} - 1),$$

where we put $p(V)=1 - \chi(\mathcal{O}_V)$ for any positive cycle V and $\sum_{j=1}^{0}\{\cdots\} = 0$. Artin [1] proved that (X, x) is rational (i.e., $p_g = 0$) if and only if $p(Z) = 0$ and that if (X, x) is rational, then we have the multiplicity $\operatorname{mult}(X, x) = -Z^2$ and the embedding dimension $\operatorname{emb\,dim}(X, x) = -Z^2 + 1$. Laufer [3] proved the following topological criterion.

Theorem 2.1 ([3], Theorem 4.2). *The singularity (X, x) is rational if and only if each component A_i of A is a non-singular rational curve and for any (or some) computation sequence $\{Z_j\}_{j=0}^{l+1}$ for Z, we have $Z_{j-1} \cdot A_{i_j} = 1$ for $j > 0$.*

Hence rational surface singularities are characterized by their weighted dual graphs. We use the following theorem.

Theorem 2.2 ([7], Corollary 2.5). *Let (X, x) be a Gorenstein or a Du Bois singularity and $\pi : (\tilde{X}, A) \to (X, x)$ the minimal good resolution. Then we have $\delta_2(X, x) = h^1(\mathcal{O}_{\tilde{X}}(-K - A))$.*

Note that rational surface singularities are Du Bois.

3. A characterization of the vanishing of the second plurigenus

We follow the notation of the preceding section. The purpose of this section is to give a formula for δ_2 of rational surface singularities and a topological criterion for the vanishing of δ_2. For a rational surface singularity, we have relations among δ_2, $\operatorname{mult}(X, x)$ and $\operatorname{emb\,dim}(X, x)$ as follows:

Proposition 3.1. *Let $\pi : (\tilde{X}, A) \to (X, x)$ be the minimal resolution. Then we have $\delta_2(X, x) \geq \operatorname{mult}(X, x) - K \cdot A - 2 = \operatorname{emb\,dim}(X, x) - K \cdot A - 3$.*

Proof. Since (X, x) is rational, we can construct a computation sequence $Z_0, \ldots, Z_k = A, \ldots, Z_{l+1} = Z$, where $Z_0 = A_{i_0}, Z_1 = Z_0 + A_{i_1}, \ldots, Z_k = Z_{k-1} + A_{i_k}, \ldots, Z_{l+1} = Z_l + A_{i_{l+1}}$ with the property $Z_{j-1} \cdot A_{i_j} = 1$ by Theorem 2.1. If

$Z = A$, then it is obvious. Hence we can assume that $Z \neq A$. For each $j > k$, we have the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-K - Z_j) \to \mathcal{O}_{\widetilde{X}}(-K - Z_{j-1}) \to \mathcal{O}_{A_{i_j}}(-K - Z_{j-1}) \to 0.$$

By the Riemann-Roch theorem, we have inductively

$$h^1(\mathcal{O}_{\widetilde{X}}(-K - Z_{j-1})) = h^1(\mathcal{O}_{\widetilde{X}}(-K - Z_j)) + K \cdot A_{i_j}.$$

Since $\operatorname{mult}(X, x) - 2 = \operatorname{emb}\dim(X, x) - 3 = K \cdot Z$, we have

$$\delta_2(X, x) \geq \operatorname{mult}(X, x) - K \cdot A - 2 = \operatorname{emb}\dim(X, x) - K \cdot A - 3$$

by Theorem 2.2. □

Corollary 3.2. *Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution. If $\delta_2(X, x) = 0$, then we have* $\operatorname{mult}(X, x) = K \cdot A + 2$ *and* $\operatorname{emb}\dim(X, x) = K \cdot A + 3$.

Proof. Since $p_g \leq \delta_2$, (X, x) is rational. The assertion follows from Proposition 3.1. □

Let (X, x) be a normal surface singularity and $\pi : (\widetilde{X}, A) \to (X, x)$ the minimal resolution. Since K is π-nef, by the well-known vanishing theorem, we have the exact sequence

$$0 \to H^0(\mathcal{O}_{\widetilde{X}}(2K)) \to H^0(\widetilde{X} - A, \mathcal{O}_{\widetilde{X}}(2K)) \to H^1{}_A(\mathcal{O}_{\widetilde{X}}(2K)) \to 0,$$

i.e., $\gamma_2(X, x) = h^1(\mathcal{O}_{\widetilde{X}}(-K))$ by the Serre duality.

Proposition 3.3 (cf. [11], Lemma 4.3). *In the situation above, $\gamma_2(X, x) = 0$ if and only if (X, x) is a rational double point or a rational triple point.*

Proof. Assume that $\gamma_2(X, x) = 0$. Consider the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-K - Z) \to \mathcal{O}_{\widetilde{X}}(-K) \to \mathcal{O}_Z(-K) \to 0.$$

Since $\gamma_2(X, x) \geq h^1(\mathcal{O}_Z(-K)) \geq K \cdot Z - \chi(\mathcal{O}_Z)$ and $p_g \leq \gamma_2$, (X, x) is a rational double point or a rational triple point. If (X, x) is a rational double point, then $\gamma_2(X, x) = 0$. Hence it is enough to prove that $\gamma_2(X, x) = 0$ if (X, x) is a rational triple point. But we may prove the following if we use the argument above.

(i) $H^0(\mathcal{O}_Z(-K)) = 0$,

(ii) $H^1(\mathcal{O}_{\widetilde{X}}(-K - Z)) = 0$.

(i) Since (X, x) is a rational triple point, there exists a non-singular rational curve A_{i_0} such that $K \cdot A_{i_0} = 1$. We can construct a computation sequence $Z_0 = A_{i_0}, \ldots, Z_k = Z_{k-1} + A_{i_k}, \ldots, Z_{l+1} = Z_l + A_{i_{l+1}} = Z$ with the property $Z_{j-1} \cdot A_{i_j} = 1$ by Theorem 2.1. Consider the exact sequences for $k \geq 1$,

$$0 \to \mathcal{O}_{A_{i_k}}(-K - Z_{k-1}) \to \mathcal{O}_{Z_k}(-K) \to \mathcal{O}_{Z_{k-1}}(-K) \to 0.$$

Since $(-K - Z_{k-1}) \cdot A_{i_k} \leq -1$ and $-K \cdot A_{i_0} = -1$, by induction, we have $H^0(\mathcal{O}_Z(-K)) = 0$.

(ii) By the theorem on formal functions, we have

$$H^1(\mathcal{O}_{\tilde{X}}(-K-Z)) = \varprojlim_{D>0} H^1(\mathcal{O}_D(-K-Z)).$$

Hence we may prove that $H^1(\mathcal{O}_D(-K-Z)) = 0$ for all positive cycles D. In the situation above, consider the infinite sequence: $Z_0 = A_{i_0}, \ldots, Z_k, \ldots, Z = Z_{l+1}, Z+Z_0, \ldots, Z+Z_k, \ldots, 2Z, \ldots, tZ, tZ+Z_0, \ldots, tZ+Z_k, \ldots, (t+1)Z, \ldots$. Consider the exact sequences for $k \geq 1$

$$0 \to \mathcal{O}_{A_{i_k}}(-K-Z-tZ-Z_{k-1}) \to \mathcal{O}_{tZ+Z_k}(-K-Z)$$
$$\to \mathcal{O}_{tZ+Z_{k-1}}(-K-Z) \to 0.$$

Similarly, by induction, we have $h^1(\mathcal{O}_{tZ+Z_k}(-K-Z)) = 0$ for all t and k. This fact implies $H^1(\mathcal{O}_D(-K-Z)) = 0$ for all D. Hence we obtain the assertion. □

In general, it is difficult to determine all the minimal (or minimal good) resolution graphs of normal surface singularities with $\delta_2 = 0$ (cf. [10]). Next, we prove our main theorems. First, we have the following.

Definition-Lemma 3.4. *Let (X,x) be a normal surface singularity and $\pi : (\tilde{X}, A) \to (X,x)$ the minimal resolution. Then there exists the unique smallest positive cycle W satisfying the property*

$$(*) \qquad (-K-W)\cdot A_i \geq 0 \quad \text{for all } i.$$

Proof. If (X,x) is a rational double point, then we can set $W = Z$. Assume that (X,x) is not a rational double point. Since the intersection matrix $(A_i \cdot A_j)_{1\leq i,j\leq n}$ is negative definite, there exists a unique $\mathbb{Q}$-cycle $Z_K = \sum_{i=1}^n d_iA_i \in \sum_i \mathbb{Q}A_i$ such that $-K\cdot A_j = (\sum_{i=1}^n d_iA_i)\cdot A_j$ for all j. Let d be the absolute value of the determinant $\det(A_i\cdot A_j)_{1\leq i,j\leq n}$. Since π is minimal, we have that $(-K-dZ_K)\cdot A_i \geq 0$ for all i and dZ_K is a positive cycle satisfying Supp $(dZ_K) = A$. Let $W_j = \sum_{i=1}^n d_{ji}A_i$ $(j = 1,2)$ be positive cycles satisfying the property $(*)$. Note that Supp $(W_j) = A$. Let $d_k{}' = \min\{d_{1k}, d_{2k}\}$. Let $W' = \sum_{k=1}^n d_k{}'A_k$. Assume that $d_i{}' = d_{1i}$. Then we have

$$\begin{aligned}(-K-W')\cdot A_i &= -K\cdot A_i - (d_i{}'A_i{}^2 + \sum_{k\neq i} d_k{}'A_k\cdot A_i)\\ &\geq -K\cdot A_i - (d_{1i}A_i{}^2 + \sum_{k\neq i} d_{1k}A_k\cdot A_i)\\ &= (-K-W_1)\cdot A_i.\end{aligned}$$

The positive cycle W' satisfies the property $(*)$. □

The argument of the minimal property above has already been seen in Artin [1]. In general, we have $A \leq Z \leq W$. The positive cycle "W" can be computed inductively as in [3]: put $W_0 = A$ (resp. $W_0 = 0$, $W_1 = A_{i_1}$). Given W_k (resp. W_k $(k \geq 1)$) there are two possibilities:

(1) if there is an $A_{i_{k+1}}$ such that $(-K - W_k) \cdot A_{i_{k+1}} < 0$, then put $W_{k+1} = W_k + A_{i_{k+1}}$,

(2) otherwise, we are finished and $W = W_k$.

We can always construct a "computation sequence from A to W" (resp. from A_{i_1} to W).

Lemma 3.5. *In the situation above, $W = A$ if and only if A satisfies one of the following cases.*

(1) A is a non-singular elliptic curve.

(2) A is a rational curve with a node singularity.

(3) A is a rational curve with a cusp singularity.

(4) A is a cycle of non-singular rational curves.

(5) A is a chain of non-singular rational curves of arbitrary length.

(6) A is the sum of two non-singular rational curves which have first order tangential contact at one point.

(7) A is the sum of three non-singular rational curves all meeting transversely at the same point.

Proof. Let $t_i = (A - A_i) \cdot A_i$. By the property $(*)$, we have $2 - t_i \geq 2p(A_i) \geq 0$ for all i. If $t_i = 0$, then $A = A_i$ and A satisfies (1), (2), (3) or (5). Assume that $1 \leq t_i \leq 2$ for all i. Then A_i is a non-singular rational curve. If there exists an A_{i_0} with $t_{i_0} = 1$, then A satisfies (5). If $t_i = 2$ for all i, then A satisfies (4), (6) or (7). The converse is now obvious. □

If A satisfies (1), (2), (3), (4), (6) or (7) (resp. (5)), then (X, x) is a minimally elliptic (resp. cyclic quotient) singularity and $\delta_2(X, x) = 1$ (resp. 0) (cf. [4], [7]).

Proposition 3.6. *Let (X, x) be a rational surface singularity and $\{W_i\}_{i=0}^{l}$ a computation sequence from A to W as follows : $W_0 = A, W_1 = W_0 + A_{i_1}, \ldots, W_l = W_{l-1} + A_{i_l} = W$ with the property $(-K - W_{j-1}) \cdot A_{i_j} < 0$ for $j > 0$. Then $\delta_2(X, x) = \sum_{j=1}^{l} \Big((K + W_{j-1}) \cdot A_{i_j} - 1\Big) = (W - A) \cdot (K + A) - \chi(\mathcal{O}_{W-A})$.*

Proof. If $W = A$, then it is obvious by Lemma 3.5. Assume that $W > A$. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}(-K - W) \to \mathcal{O}_{\tilde{X}}(-K - A) \to \mathcal{O}_{W-A}(-K - A) \to 0.$$

By ([5], (12.1)) and Theorem 2.2, we have

$$\delta_2(X, x) = h^1(\mathcal{O}_{W-A}(-K - A)).$$

Next, we set $V_0 = W_0 - A, V_1 = W_1 - A, \ldots$ and $V_l = W_l - A$. For each $j > 0$, we have the exact sequence

$$0 \to \mathcal{O}_{A_{i_j}}(-K - W_{j-1}) \to \mathcal{O}_{V_j}(-K - A) \to \mathcal{O}_{V_{j-1}}(-K - A) \to 0.$$

By the Riemann-Roch theorem, we have inductively

(a) $h^1(\mathcal{O}_{V_j}(-K - A)) = h^1(\mathcal{O}_{V_{j-1}}(-K - A)) + \Big((K + W_{j-1}) \cdot A_{i_j} - 1\Big)$.

(b) $H^0(\mathcal{O}_{W-A}(-K-A)) = 0$.

The first equality follows from (a). Using (b) and the Riemann-Roch theorem again, we obtain that

$$\delta_2(X,x) = -\chi(\mathcal{O}_{W-A}(-K-A)) = (W-A)\cdot(K+A) - \chi(\mathcal{O}_{W-A}).$$

□

Theorem 3.7. *Let (X,x) be a normal surface singularity. Then $\delta_2(X,x) = 0$ if and only if (X,x) is rational and for any (or some) computation sequence $\{W_i\}_{i=0}^{l}$ from A to W, we have $(K+W_{j-1})\cdot A_{i_j} = 1$ for $j > 0$.*

Proof. Assume that $\delta_2(X,x) = 0$. Since $p_g \leq \delta_2$, (X,x) is rational. If (X,x) is a cyclic quotient singularity, then it is obvious by Lemma 3.5. If (X,x) is not a cyclic quotient singularity, then we can construct a computation sequence $\{W_j\}_{j=0}^{l(l\geq 1)}$ from A to W by Lemma 3.5. By Proposition 3.6, we have $(K+W_{j-1})\cdot A_{i_j} = 1$ for $j > 0$. Similarly, we have the converse. □

Since rational surface singularities are characterized by their weighted dual graphs, we obtain a topological criterion on the minimal resolution for the vanishing of δ_2. We can consider that Theorem 3.7 is a δ_2-version of Laufer's result. Next, we show a similar formula for γ_2 of rational surface singularities. If (X,x) is a rational surface singularity which is not a rational double point, then there exists a non-singular rational curve A_{i_1} with $-K\cdot A_{i_1} < 0$. Hence we can construct a computation sequence $\{W_i\}_{i=0}^{l}$ from A_{i_1} to W as follows: $W_0 = 0, W_1 = W_0 + A_{i_1}, \ldots, W_l = W_{l-1} + A_{i_l} = W$ with the property $(-K - W_{j-1})\cdot A_{i_j} < 0$ for $j > 0$. Then we obtain the following.

Corollary 3.8. *In the situation above, we have the following:*

(i) $\gamma_2(X,x) = \sum_{j=1}^{l}\Big((K+W_{j-1})\cdot A_{i_j} - 1\Big) = K\cdot W - \chi(\mathcal{O}_W)$,

(ii) $\delta_2(X,x) = \gamma_2(X,x) - K\cdot A + 1$ (*cf.* [11], *Corollary* 3.7).

Proof. (i) As in the proof of Proposition 3.6, we obtain the following two formulae:

(c) $\gamma_2(X,x) = \sum_{j=1}^{l}\Big((K+W_{j-1})\cdot A_{i_j} - 1\Big)$.

(d) $\gamma_2(X,x) = h^1(\mathcal{O}_W(-K)) = -\chi(\mathcal{O}_W(-K)) = K\cdot W - \chi(\mathcal{O}_W)$.

(ii) Since $\chi(\mathcal{O}_W) = \chi(\mathcal{O}_{W-A}) + \chi(\mathcal{O}_A) - (W-A)\cdot A$, by the formula (d) and Proposition 3.6, we have

$$\delta_2(X,x) - \gamma_2(X,x) = -K\cdot A + \chi(\mathcal{O}_A).$$

Since (X,x) is rational, we have $\chi(\mathcal{O}_A) = 1$. Hence we obtain the assertion. □

Let (X,x) be a rational surface singularity and $\pi : (\widetilde{X}, A) \to (X,x)$ the minimal resolution. Let A' be a connected cycle such that $0 < A' \leq A$. If we contract a connected cycle $A' \subset \widetilde{X}$, then we obtain a unique singular point; this will be denoted by $(\widetilde{X}/A', p)$.

Corollary 3.9 (cf. [13], Theorem 2.8). *In the situation above, we have* $\gamma_2(X,x) \geq \gamma_2(\widetilde{X}/A',p)$.

Proof. We may assume that $0 < A' = \sum_{i=1}^{k} A_i < A = \sum_{i=1}^{n(>k)} A_i$ and the singularities (X,x) and $(\widetilde{X}/A',p)$ are not rational double points. Since $(\widetilde{X}/A',p)$ is not a rational double point, there exists a non-singular rational curve $A_{i_1} \leq A'$ with $-K \cdot A_{i_1} < 0$. Let $\{W_0 = 0, W_1 = A_{i_1}, \ldots, W_s = W(A')\}$ be a computation sequence from A_{i_1} to $W(A')$ with respect to the singularity $(\widetilde{X}/A',p)$. Continuing this process, we can construct a computation sequence $\{W_0 = 0, W_1 = A_{i_1}, \ldots, W_s, \ldots, W_l = W\}$ from A_{i_1} to W. By (i) of Corollary 3.8, we obtain the assertion. □

Corollary 3.10. *In the situation above, if* $A - A'$ *consists of* (-2)*-curves, then we have* $\delta_2(X,x) \geq \delta_2(\widetilde{X}/A',p)$.

Proof. The assertion follows from (ii) of Corollary 3.8 and Corollary 3.9. □

Rational triple points are classified into 9 classes according to the dual graphs in [1]. In [9], Tjurina showed that any given rational triple point can be realized as a singularity on an explicitly described surface in complex four dimensional space $\mathbb{C}^4$. In [11], we gave characterizations of rational triple points in terms of the second plurigenera δ_2 and γ_2. In this paper, we have the following.

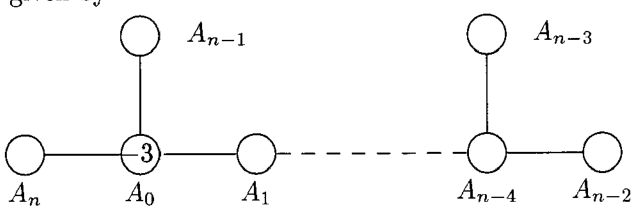
Theorem 3.11. *Let* (X,x) *be a normal surface singularity and* $\pi : (\widetilde{X},A) \to (X,x)$ *the minimal resolution. Then* (X,x) *is a rational triple point if and only if* (X,x) *is rational and there exists a computation sequence* $\{W_i\}_{i=0}^{l}$ *from* A_{i_1} *to* W *with the property* $(K + W_{j-1}) \cdot A_{i_j} = 1$ *for* $j > 0$.

Proof. If (X,x) is a rational triple point, by Proposition 3.3, then $\gamma_2(X,x) = 0$ and there exists a A_{i_1} with $-K \cdot A_{i_1} = -1$. Hence we can construct any computation sequence $\{W_i\}_{i=0}^{l}$ from A_{i_1} to W. By (i) of Corollary 3.8, we have $(K + W_{j-1}) \cdot A_{i_j} = 1$ for $j > 0$. Similarly, we have the converse. □

4. Application of the formulae for the second plurigenera of rational surface singularities

By Proposition 3.6 and Corollary 3.8, we can compute the second plurigenera of rational surface singularities. Finally, we give the following example.

Example 4.1. (1) Let (X,x) be a normal surface singularity whose weighted dual graph is given by



$(n \geq 4)$

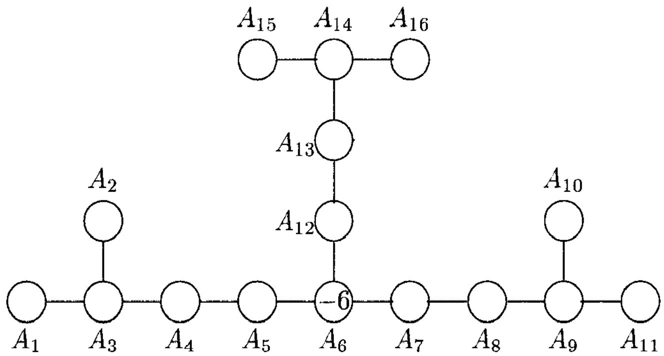
where A_0 is a (-3)-curve. Then (X, x) is rational (see Section 2). We can construct a computation sequence from A to W as follows:

(i) if $n = 4$, then $W_0 = A$, $W_1 = W_0 + A_0 = W$,

(ii) if $n > 4$, then $W_0 = A$, $W_1 = W_0 + A_0, \ldots, W_{n-3} = W_{n-4} + A_{n-4} = W$.

By Proposition 3.6 and (ii) of Corollary 3.8, we have that $\delta_2(X, x) = \gamma_2(X, x) = 1$.

(2) Let (X, x) be a normal surface singularity whose weighted dual graph is given by



where A_6 is a (-6)-curve. Then (X, x) is rational. We can construct a computation sequence $\{W_i\}_{i=0}^{10}$ from A to W as follows: $W_0 = A$, $W_1 = W_0 + A_3$, $W_2 = W_1 + A_4$, $W_3 = W_2 + A_5$, $W_4 = W_3 + A_9$, $W_5 = W_4 + A_8$, $W_6 = W_5 + A_7$, $W_7 = W_6 + A_{14}$, $W_8 = W_7 + A_{13}$, $W_9 = W_8 + A_{12}$, $W_{10} = W_9 + A_6 = W$. By Proposition 3.6 and (ii) of Corollary 3.8, we have that $\delta_2(X, x) = 3$ and $\gamma_2(X, x) = 6$.

Acknowledgements. The author would like to thank the referee for pointing out errors in the earlier version of this paper. I also thank the Managing Editor Professor Gyo Taek Jin.

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THE SECOND HIGH SCHOOL
TOKYO UNIVERSITY OF AGRICULTURE
TAKASAKI-SHI GUNMA, 370-0864, JAPAN
E-mail address: kw_nk2_1223@yahoo.co.jp or wadak1219@softbank.ne.jp