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MENDELSOHN TRIPLE SYSTEMS EXCLUDING CONTIGUOUS UNITS WITH $\lambda = 1$

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Abstract. We obtain a necessary and sufficient condition for the existence of Mendelsohn triple systems excluding contiguous units with $\lambda = 1$. Also, we obtain the spectrum for cyclic such systems.

1. Introduction

If $X = \{x_0, x_1, \ldots, x_{v-1}\}$ is a cyclically ordered set of points, then two points x_i and x_{i+1} are said to be *contiguous points* for all i such that $0 \le i \le v-2$, as are x_{v-1} and x_0. Otherwise, they are *non-contiguous*. A *triple sampling plan excluding contiguous units* $TSEC(v, \lambda)$ *of order* v and *index* λ is a pair $(X, \mathfrak{B})$ where X is a cyclically ordered v-set of points (units) and $\mathfrak{B}$ is a collection of 3-subsets of X, called *triples*, such that any two contiguous points of X do not appear in any triple while any two non-contiguous distinct points appear in exactly λ triples of $\mathfrak{B}$. When any two distinct points appears in precisely λ triples of $\mathfrak{B}$, it is a *triple system* $TS(v, \lambda)$. There exists a $TS(v, \lambda)$ if and only if $\lambda \equiv 0 \pmod{\gcd(v-2, 6)}$ and $v \ne 2$ [8], and a $TSEC(v, \lambda)$ exists if and only if $v \in \{0, 3\}$ or $v \ge 9$ and $\lambda(v-3) \equiv 0 \pmod 6$ [6].

Hung and Mendelsohn [10] considered triple systems in which the triples are *ordered*. A *transitive triple* $[x, y, z]$ is taken to contain the ordered pairs $(x, y), (x, z)$ and (y, z). A *directed triple system* $DTS(v, \lambda)$ is a pair $(X, \mathfrak{B})$ where X is a v-set of points and $\mathfrak{B}$ is a collection of transitive triples of X such that every ordered pair (x, y) of distinct points of X appears in precisely λ transitive triples in $\mathfrak{B}$. Necessary and sufficient conditions for the existence of directed triple systems have been established by Hung and Mendelsohn for $\lambda = 1$ [10], and by Seberry and Skillicorn for all λ [12]; they found that the conditions for the existence of a $DTS(v, \lambda)$ are the *same* as those for the existence of a $TS(v, 2\lambda)$. Mendelsohn [11] also considered triple systems in which the triples are *ordered* slightly different from transitive triples. A *cyclic triple*

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(x, y, z) is a 3-set of the ordered pairs $(x, y), (y, z)$ and (z, x). A *Mendelsohn triple system* $MTS(v, \lambda)$ is a pair $(X, \mathfrak{B})$ where X is a v-set of points and $\mathfrak{B}$ is a collection of cyclic triples of X such that every ordered pair (x, y) of distinct points of X appears in precisely λ cyclic triples in $\mathfrak{B}$. The existence of a $MTS(v, \lambda)$ was settled by Mendelsohn for $\lambda = 1$ [11], and by Bennett for all λ [2]. There exists a $DTS(6, 1)$ while there does not exists a $MTS(6, 1)$.

Analogously, if X is a cyclically ordered v-set of points, then we may define a *directed* [*Mendelsohn*] sampling plan excluding contiguous units $DTSEC(v, \lambda)$ [$MTSEC(v, \lambda)$] based on X as a collection $\mathfrak{B}$ of transitive [cyclic] triples of X such that any ordered pair of two contiguous points of X does not appear in any transitive [cyclic, respectively] triple in $\mathfrak{B}$ while each ordered pair of two non-contiguous distinct points appears in exactly λ members of $\mathfrak{B}$.

Naturally, if we treat the transitive [cyclic] triples of a $DTS(v,\lambda)$ [$MTS(v,\lambda)$] as unordered 3-subsets, we obtain a $TS(v, 2\lambda)$ which is called the *underlying triple system* of the $DTS(v, \lambda)$ [$MTS(v, \lambda)$, respectively]. The underlying triple system of a $DTS(v, \lambda)$ [$MTS(v, \lambda)$] is termed *directable* [*cyclable*, respectively].

Theorem 1.1 ([4, 5, 9]). *Every $TS(v, 2\lambda)$ is directed.*

From Theorem 1.1, every $TSEC(v, 2\lambda)$ is obviously directed, and hence the existence of a $DTSEC(v, \lambda)$ is equivalent to the existence of a $TSEC(v, 2\lambda)$. From [6], we have the following theorem.

Theorem 1.2. *There exists a $DTSEC(v, \lambda)$ if and only if $v \in \{0, 3\}$ or $v \geq 9$ and $\lambda v \equiv 0 \pmod 3$.*

There exists a $TS(6, 2)$, but not a $MTS(6, 1)$. Thus not every $TS(v, 2\lambda)$ is cyclical and hence every $TSEC(v, k, 2\lambda)$ is not cyclical. Therefore, the existence of a $MTSEC(v, \lambda)$ is in doubt. In this paper, we obtain a necessary and sufficient condition for the existence of a $MTSEC(v, \lambda)$ with $\lambda = 1$. We simply denote $MTSEC(v)$ for $MTSEC(v, 1)$. Since the existence of a $MTSEC(v, \lambda)$ implies the existence of a $TSEC(v, 2\lambda)$, we have the following necessary condition.

Lemma 1.3. *If there exists a $MTSEC(v, \lambda)$, then $v \in \{0, 3\}$ or $v \geq 9$ and $\lambda v \equiv 0 \pmod 3$. When $\lambda = 1$, it is $v \equiv 0 \pmod 3$ and $v \neq 6$.*

In the next section, we will show that there exists a $MTSEC(v)$ for all $v \equiv 0 \pmod 3$ and $v \neq 6$ (we will omit the trivial case $v \in \{0, 3\}$.)

2. The existence of $MTSEC(v)$s

We will first construct a $MTSEC(v)$ for $v = 12, 18, 24, 30, 36, 42, 48, 60$ and 78, and then establish the existence of a $MTSEC(v)$ by employing a recursive method for all $v \equiv 0 \pmod 3$ and $v \notin \{0, 3, 6\}$. An *automorphism* of a $MTSEC(v)$, $(X, \mathfrak{B})$, is a permutation α on X such that each cyclic triple of $\mathfrak{B}$ maps onto a cyclic triple. A $MTSEC(v)$ is said to be *bicyclic* if it admits an automorphism consisting of exactly two cycles of length $\frac{v}{2}$. Such an automorphism is said to be *bicyclic*. If $(X, \mathfrak{B})$ is a bicyclic $MTSEC(v)$ with a bicyclic automorphism α, then the group $\langle\alpha\rangle$ generated by α acts on $\mathfrak{B}$. Thus $\mathfrak{B}$ is partitioned into mutually disjoint *orbits*. A collection of cyclic triples, called *base blocks*, which are taken each of the orbits exactly once represents the bicyclic $MTSEC(v)$ together with the bicyclic automorphism. In order to avoid the presentation of many large systems, whenever possible, we present a collection of base blocks for a bicyclic $MTSEC(v)$. It is convenient to name the points as ordered pairs from

$$\left\{0, 1, \ldots, \frac{v-2}{2}\right\} \times \{1, 2\},$$

taking $(i, 1)$ and $(j, 2)$ to be contiguous when $i = j$, or $i \equiv j + 1 \pmod{\frac{v}{2}}$. We write shortly x_i for the ordered pair (x, i).

We will construct that our bicyclic $MTSEC(v)$ is based on $\left\{0, 1, \ldots, \frac{v-2}{2}\right\} \times \{1, 2\}$ and the corresponding bicyclic automorphism is

$$\left(0_1, 1_1, \ldots, \left(\frac{v}{2} - 1\right)_1\right)\left(0_2, 1_2, \ldots, \left(\frac{v}{2} - 1\right)_2\right).$$

Lemma 2.1. *There exists a bicyclic* $MTSEC(v)$ *for* $v =$ 12, 18, 24, 30, 36, 42, 48, 60 *and* 78.

Proof. A collection of base blocks for a $MTSEC(v)$ is
when $v = 12$:

$(1_1, 0_1, 4_2)$, $(0_2, 1_2, 5_1)$, $(0_1, 4_1, 1_2)$, $(0_2, 2_1, 4_2)$, $(0_1, 2_1, 5_1)$,
$(0_2, 5_2, 2_2)$,

when $v = 18$:

$(0_1, 1_1, 4_1)$, $(0_1, 4_1, 1_1)$, $(0_2, 1_2, 3_2)$, $(0_1, 2_1, 7_2)$, $(7_2, 2_1, 0_1)$,
$(4_2, 3_2, 0_1)$, $(3_2, 1_2, 0_1)$, $(0_1, 1_2, 4_2)$, $(2_2, 6_2, 0_1)$, $(6_2, 2_2, 0_1)$,

when $v = 24$:

$(0_1, 1_1, 10_1)$, $(0_1, 3_1, 8_1)$, $(0_1, 6_1, 2_1)$, $(0_2, 1_2, 5_2)$, $(1_1, 0_1, 9_2)$,
$(5_1, 0_1, 10_2)$, $(8_2, 7_2, 0_1)$, $(7_2, 3_2, 0_1)$, $(3_2, 1_2, 0_1)$, $(1_2, 6_2, 0_1)$,
$(6_2, 9_2, 0_1)$, $(2_2, 4_2, 0_1)$, $(4_2, 10_2, 0_1)$, $(5_2, 2_2, 0_1)$,

when $v = 30$:

$(0_1, 1_1, 4_1)$, $(0_1, 4_1, 1_1)$, $(0_1, 2_1, 8_1)$, $(0_1, 8_1, 2_1)$, $(1_2, 0_2, 4_2)$,
$(0_1, 5_1, 10_2)$, $(5_1, 0_1, 10_2)$, $(2_2, 3_2, 0_1)$, $(3_2, 9_2, 0_1)$, $(9_2, 11_2, 0_1)$,
$(11_2, 6_2, 0_1)$, $(6_2, 13_2, 0_1)$, $(13_2, 7_2, 0_1)$, $(7_2, 12_2, 0_1)$, $(12_2, 8_2, 0_1)$,
$(8_2, 1_2, 0_1)$, $(1_2, 4_2, 0_1)$, $(4_2, 2_2, 0_1)$,

when $v = 36$:

$(0_1,1_1,15_1)$, $(0_1,2_1,12_1)$, $(0_1,9_1,3_1)$, $(0_1,7_1,5_1)$, $(0_1,8_1,1_1)$,
$(0_1,4_1,16_2)$, $(0_1,5_1,7_2)$, $(0_2,1_2,9_2)$, $(6_2,5_2,0_1)$, $(5_2,11_2,0_1)$,
$(11_2,4_2,0_1)$, $(4_2,9_2,0_1)$, $(9_2,1_2,0_1)$, $(1_2,8_2,0_1)$, $(8_2,2_2,0_1)$,
$(7_2,3_2,0_1)$, $(3_2,6_2,0_1)$, $(16_2,13_2,0_1)$, $(13_2,15_2,0_1)$, $(15_2,10_2,0_1)$,
$(10_2,14_2,0_1)$, $(14_2,12_2,0_1)$,

when $v = 42$:

$(0_1,1_1,5_1)$, $(0_1,5_1,1_1)$, $(0_1,2_1,10_1)$, $(0_1,10_1,2_1)$, $(0_1,3_1,9_1)$,
$(0_1,9_1,3_1)$, $(0_1,7_1,14_2)$, $(7_1,0_1,14_2)$, $(1_2,0_2,4_2)$, $(11_2,19_2,0_1)$,
$(19_2,11_2,0_1)$, $(4_2,6_2,0_1)$, $(6_2,4_2,0_1)$, $(3_2,8_2,0_1)$, $(8_2,3_2,0_1)$,
$(2_2,9_2,0_1)$, $(9_2,2_2,0_1)$, $(13_2,16_2,0_1)$, $(16_2,17_2,0_1)$, $(17_2,13_2,0_1)$,
$(1_2,10_2,0_1)$, $(10_2,1_2,0_1)$, $(5_2,15_2,0_1)$, $(15_2,5_2,0_1)$, $(12_2,18_2,0_1)$,
$(18_2,12_2,0_1)$,

when $v = 48$:

$(0_1,1_1,3_1)$, $(0_1,3_1,1_1)$, $(0_1,13_1,4_1)$, $(0_1,12_1,5_1)$, $(0_1,14_1,6_1)$,
$(0_1,4_1,13_1)$, $(0_1,6_1,14_1)$, $(0_1,5_1,12_2)$, $(0_1,7_1,19_2)$, $(0_2,1_2,12_2)$,
$(19_2,22_2,0_1)$, $(22_2,20_2,0_1)$, $(20_2,15_2,0_1)$, $(15_2,17_2,0_1)$, $(17_2,16_2,0_1)$,
$(16_2,21_2,0_1)$, $(21_2,18_2,0_1)$, $(18_2,7_2,0_1)$, $(1_2,11_2,0_1)$, $(11_2,1_2,0_1)$,
$(2_2,10_2,0_1)$, $(10_2,2_2,0_1)$, $(3_2,9_2,0_1)$, $(9_2,3_2,0_1)$, $(4_2,8_2,0_1)$,
$(8_2,4_2,0_1)$, $(5_2,14_2,0_1)$, $(14_2,5_2,0_1)$, $(6_2,13_2,0_1)$, $(13_2,6_2,0_1)$,

when $v = 60$:

$(0_1,1_1,25_1)$, $(0_1,15_1,2_1)$, $(0_1,3_1,22_1)$, $(0_1,4_1,16_1)$, $(0_1,6_1,17_1)$,
$(0_1,27_1,7_1)$, $(0_1,18_1,8_1)$, $(0_1,26_1,5_1)$, $(1_1,0_1,27_2)$, $(0_1,2_1,28_2)$,
$(0_1,7_1,23_2)$, $(9_1,0_1,12_2)$, $(14_1,0_1,22_2)$, $(0_2,15_2,14_2)$, $(0_2,1_2,9_2)$,
$(24_2,16_2,0_1)$, $(23_2,17_2,0_1)$, $(17_2,21_2,0_1)$, $(21_2,27_2,0_1)$, $(28_2,24_2,0_1)$,
$(5_2,10_2,0_1)$, $(10_2,5_2,0_1)$, $(4_2,11_2,0_1)$, $(11_2,4_2,0_1)$, $(3_2,12_2,0_1)$,
$(2_2,13_2,0_1)$, $(13_2,2_2,0_1)$, $(6_2,9_2,0_1)$, $(9_2,6_2,0_1)$, $(8_2,22_2,0_1)$,
$(19_2,7_2,0_1)$, $(7_2,19_2,0_1)$, $(25_2,15_2,0_1)$, $(15_2,25_2,0_1)$, $(14_2,1_2,0_1)$,
$(1_2,14_2,0_1)$, $(20_2,18_2,0_1)$, $(18_2,20_2,0_1)$,

when $v = 78$:

$(0_1, 1_1, 17_1)$,	$(0_1, 17_1, 1_1)$,	$(0_1, 2_1, 20_1)$,	$(0_1, 20_1, 2_1)$,	$(0_1, 3_1, 11_1)$,
$(0_1, 11_1, 3_1)$,	$(0_1, 4_1, 8_1)$,	$(0_1, 8_1, 4_1)$,	$(0_1, 5_1, 12_1)$,	$(0_1, 12_1, 5_1)$,
$(0_1, 6_1, 15_1)$,	$(0_1, 15_1, 6_1)$,	$(0_1, 13_1, 26_2)$,	$(13_1, 0_1, 26_2)$,	$(1_2, 0_2, 4_2)$,
$(25_2, 31_2, 0_1)$,	$(31_2, 25_2, 0_1)$,	$(29_2, 32_2, 0_1)$,	$(32_2, 37_2, 0_1)$,	$(37_2, 33_2, 0_1)$,
$(33_2, 28_2, 0_1)$,	$(28_2, 36_2, 0_1)$,	$(36_2, 34_2, 0_1)$,	$(34_2, 35_2, 0_1)$,	$(35_2, 27_2, 0_1)$,
$(27_2, 29_2, 0_1)$,	$(1_2, 20_2, 0_1)$,	$(20_2, 1_2, 0_1)$,	$(2_2, 19_2, 0_1)$,	$(19_2, 2_2, 0_1)$,
$(3_2, 18_2, 0_1)$,	$(18_2, 3_2, 0_1)$,	$(4_2, 17_2, 0_1)$,	$(17_2, 4_2, 0_1)$,	$(5_2, 16_2, 0_1)$,
$(16_2, 5_2, 0_1)$,	$(6_2, 15_2, 0_1)$,	$(15_2, 6_2, 0_1)$,	$(7_2, 14_2, 0_1)$,	$(14_2, 7_2, 0_1)$,
$(8_2, 24_2, 0_1)$,	$(24_2, 8_2, 0_1)$,	$(9_2, 23_2, 0_1)$,	$(23_2, 9_2, 0_1)$,	$(10_2, 22_2, 0_1)$,
$(22_2, 10_2, 0_1)$,	$(11_2, 21_2, 0_1)$,	$(21_2, 11_2, 0_1)$,	$(12_2, 30_2, 0_1)$,	$(30_2, 12_2, 0_1)$.

$\square$

A *group divisible design* (*GDD*) *of order* v and *index* λ is a triple $(X, \mathcal{G}, \mathfrak{B})$ which satisfies the following properties:

(1) X is a v-set of points,

(2) $\mathcal{G}$ is a partition of X whose members are called *groups*, and

(3) $\mathfrak{B}$ is a collection of subsets of X, called *blocks*, such that any block and any group contain at most one common point, and every pair of points from distinct groups occurs in exactly λ blocks.

The *group-type* (*type*) of the *GDD* is the multiset $\{|G| \ : \ G \in \mathcal{G}\}$. We use the notation for group-type: $g_1^{u_1} g_2^{u_2} \cdots g_s^{u_s}$ indicates that there are u_i groups of size g_i for $1 \leq i \leq s$. The set $K = \{|B| \ : \ B \in \mathfrak{B}\}$ is the set of block sizes of the *GDD*, and the notation *K-GDD* is used to denote a *GDD* whose block sizes lie in the set K. When $K = \{k\}$, we write k-*GDD* for $\{k\}$-*GDD*.

A *Latin square of side* n is a $n \times n$ array based on a set S of n symbols with the property that every row and every column contains every symbol exactly once. Two Latin squares $A = (a_{ij})$ and $B = (b_{ij})$ of the same side n are said to be *orthogonal* if the n^2 ordered pairs (a_{ij}, b_{ij}), the pairs formed superimposing one square on the other, are all different. There exist three mutually orthogonal Latin squares of side n for all $n \neq 2, 6, 10$ [13,14]. The existence of a 4-*GDD* of n^4 and index 1 is equivalent to the existence of three mutually orthogonal Latin squares of side n [1]. Thus there exists a 4-*GDD* of n^4 and index 1 for all n except for $n = 2, 3, 6, 10$.

A 3-*GDD* with index λ, $(X, \mathcal{G}, \mathfrak{B})$, is called a *Mendelsohn group divisible design* (*MGDD*) if each block of $\mathfrak{B}$ is considered as a cyclic triple and every ordered pair of points from distinct groups occurs in exactly λ blocks. It is not hard to construct a *MGDD* of type 2^i, $3 \leq i \leq 4$, and index 1. Namely, taking groups $\{1, 2\}, \{3, 4\}, \{5, 6\}$ and cyclic triples

$$(1,3,5), (1,5,3), (1,4,6), (1,6,4), (2,3,6), (2,6,3), (2,4,5), (2,5,4)$$

we have a $MGDD$ of type 2^3 and index 1, and groups $\{1,2\},\{3,4\},\{5,6\},\{7,8\}$ and cyclic triples

$$(1,3,6),\ (1,4,8),\ (1,5,7),\ (2,3,7),\ (2,4,5),\ (2,6,8),\ (3,5,8),\ (4,6,7),$$
$$(1,6,3),\ (1,8,4),\ (1,7,5),\ (2,7,3),\ (2,5,4),\ (2,8,6),\ (3,8,5),\ (4,7,6)$$

form a $MGDD$ of type 2^4 and index 1. Consequently, there exists a $MGDD$ of type 2^i, $3 \le i \le 4$, and index $\lambda \ge 1$.

A *partial triple system* $PTS(v,\lambda)$ *of order* v and *index* λ is a pair $(V,\mathfrak{B})$ where V is a v-set of points and $\mathfrak{B}$ is a collection of 3-subsets of V, called *triples*, such that every 2-subset of V appears in at most λ triples of $\mathfrak{B}$. The *leave* of a $PTS(v,\lambda)$ is the collection of pairs of points, which appear *fewer* than λ times in the triples of the $PTS(v,\lambda)$ and if a pair $\{x,y\}$ appears in $s(<\lambda)$ triples then it appears $\lambda - s$ times in the leave. Let $V = \{x_1,x_2,\ldots,x_{2m+1}\}$ be a $(2m+1)$-set. If $2m+1 \equiv 1,5 \pmod 6$, then Colbourn and Rosa [7] show that there exists a $PTS(2m+1,1)$ whose leave is the set $\{\{x_1,x_2\},\{x_2,x_3\},\ldots,\{x_{2m-1},x_{2m}\},\{x_{2m},x_1\}\}$. The following lemma is slightly modified the Lemma 3.1 of Colbourn and Ling [6].

Lemma 2.2. (1) *Let* $m \ne 2,3,6,10$ *and* $x = 0$ *or* $5 \le x \le m$. *If there exist both a* $MTSEC(2m,\lambda)$ *and a* $MTSEC(2x,\lambda)$, *then there exists a* $MTSEC(6m+2x,\lambda)$.

(2) *Let* $m \ne 2,3,6,10$, $4 \le x \le m$ *and* $2m+1 \equiv 1,5 \pmod 6$. *If there exists a* $MTSEC(2x+1,\lambda)$, *then there exists a* $MTSEC(6m+2x+1,\lambda)$.

Proof. Since $m \ne 2,3,6,10$, there exists a 4-GDD of type m^4 and index 1. Let us have a 4-GDD of type m^4 and index 1, whose groups are G_1,G_2,G_3 and G_4, and blocks $\mathfrak{B}$. Partition G_4 into two disjoint subsets A and B so that $|A| = x$, and $|A| > 0$ whenever $x > 0$. For Case (1), a $MTSEC(6m+2x,\lambda)$ to be constructed has points

$$(G_1 \cup G_2 \cup G_3 \cup A) \times \{1,2\},$$

and in Case (2), a $MTSEC(6m+2x+1,\lambda)$ to be constructed has the same points together with an additional point ∞.

Choose one block $D \in \mathfrak{B}$, so that when A is nonempty, $D \cap A \ne \emptyset$ (when $A = \emptyset$, one has $x = 0$; in this case choose any block to serve as D).

For each block $\{u,v,w,z\} \in \mathfrak{B}$ other than D with $z \in G_4$, set

$$\sigma = 0 \text{ if } z \in B; \text{ and } \sigma = 2 \text{ if } z \in A.$$

Then form a $MGDD$ of type $2^3\sigma^1$ and index λ with groups

$$\{(u,1),(u,2)\},\qquad \{(v,1),(v,2)\},$$
$$\{(w,1),(w,2)\},\qquad \{(z,i)|i=1,\sigma\}(=\emptyset \text{ if } \sigma = 0).$$

Next, we handle the block $D = \{a_{11},a_{12},a_{13},a_{14}\}$ with $a_{1i} \in G_i$, $i = 1,2,3,4$. For $i = 1,2,3$, we assume that

$$G_i \times \{1,2\} = \{(a_{1i},1),(a_{1i},2),(a_{2i},1),(a_{2i},2),\ldots,(a_{mi},1),(a_{mi},2)\}$$

is cyclically ordered, and if $a_{14} \in A$, then

$$A \times \{1,2\} = \{(a_{14},1),(a_{14},2),(a_{24},1),(a_{24},2),\ldots,(a_{x4},1),(a_{x4},2)\ \}$$

is assumed to be cyclically ordered for Case (1). In Case (2),

$$A \times \{1,2\} \cup \{\infty\} = \{(a_{14},1),(a_{14},2),(a_{24},1),(a_{24},2),\ldots,(a_{x4},1),(a_{x4},2),\infty\}$$

is assumed to be cyclically ordered. If $a_{14} \in B$, form a $MGDD$ of type 2^3 and index λ with groups

$$\{(a_{m1},2),(a_{12},1)\},\quad \{(a_{m2},2),(a_{13},1)\},\quad \{(a_{m3},2),(a_{11},1)\}.$$

If $a_{14} \notin B$, then, by assumption, $a_{14} \in A$. In Case (1), place the blocks of a $MGDD$ of type 2^4 and index λ with groups

$$\{(a_{m1},2),(a_{12},1)\},\{(a_{m2},2),(a_{13},1)\},\{(a_{m3},2),(a_{14},1)\},\{(a_{x4},2),(a_{11},1)\};$$

in Case (2), with groups

$$\{(a_{m1},2),(a_{12},1)\},\{(a_{m2},2),(a_{13},1)\},\{(a_{m3},2),(a_{14},1)\},\{\infty,(a_{11},1)\}.$$

We now treat the cases separately for the rest, observing Case (1) first. For $i = 1,2,3$, form a $MTSEC(2m,\lambda)$ on the points $G_i \times \{1,2\}$ with the given cyclically ordering.

Next, if $x > 0$, form a $MTSEC(2x,\lambda)$ on the points $A \times \{1,2\}$ with the given cyclically ordering. The resulting cyclic triples form a required $MTSEC(6m+2x,\lambda)$.

Let us turn to Case (2). For $i = 1,2,3$, form a $PTS(2m+1,1)$ on the points $(G_i \times \{1,2\}) \cup \{\infty\}$ whose leave is the cycle of $2m$ points of $G_i \times \{1,2\}$, that is,

$$\{\{(a_{1i},1),(a_{1i},2)\},\{(a_{1i},2),(a_{2i},1)\},\{(a_{2i},1),(a_{2i},2)\},\ldots,$$
$$\{(a_{mi},1),(a_{mi},2)\},\{(a_{mi},2),(a_{1i},1)\}\},$$

then form λ times cyclic triples (a,b,c) and (a,c,b) each for each block $\{a,b,c\}$ of the $PTS(2m+1,1)$.

Next, form a $MTSEC(2x+1,\lambda)$ on the points $A \times \{0,1\} \cup \{\infty\}$ with the given cyclically ordering. The resulting cyclic triples form a required $MTSEC(6m+2x+1,\lambda)$. □

If we replace each triple $\{x,y,z\}$ of a $TSEC(v,1)$ by two cyclic triples (x,y,z) and (x,z,y), the resulting cyclic triples form a $MTSEC(v)$. Since there exists a $TSEC(v,1)$ for $v \equiv 3 \pmod 6$ [6], so does a $MTSEC(v)$ for such all v. Thus we have the following lemma.

Lemma 2.3. *If* $v \equiv 3 \pmod 6$, *then there exists a* $MTSEC(v)$.

Lemma 2.4. *If* $v \equiv 0 \pmod 6$ *and* $v \neq 6$, *then there exists a* $MTSCE(v)$.

Proof. Let $v \equiv 0 \pmod 6$ and $v \neq 6$. By Lemma 2.1, there exists a $MTSEC(v)$ for $v = 12,18,24,30,36,42,48,60,78$. Now, Lemma 2.2 is applied to be existing of a $MTSEC(v)$ for $v = 54,66,72,84$. Write $v = 6m + 2x$ where $m \equiv 0$

(mod 3), $m \geq 12$ and $x \in \{0, 12, 6\}$. Then, by Lemma 2.2, there exists a $MTSEC(v)$. □

Lemmas 1.3, 2.3 and 2.4 together yield the following theorem.

Theorem 2.5. *There exists a* $MTSCE(v)$ *if and only if* $v \equiv 0 \pmod 3$ *and* $v \neq 6$.

3. Concluding remarks

A $TSEC(v, \lambda)$ is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length v. Wei [14] shows that there exists a cyclic $TSEC(v, 1)$ if and only if $v \equiv 3 \pmod 6$, and a cyclic $TSEC(v, 2)$ exists if and only if $v \equiv 0, 3, 9 \pmod{12}$. Colbourn [3] shows that every cyclic $DTS(v, 2\lambda)$ is directable, so there exists a cyclic $TSEC(v, 1)$ if and only if $v \equiv 0, 3, 9 \pmod{12}$.

Let (a, b, c) be a base block of a cyclic $MTSEC(v, 1)$ based on the cyclically ordered set $\{0, 1, \ldots, v-1\}$. Define the *difference triple* $[x, y, z]$ corresponding to (a, b, c) so that

$$x \equiv b - a \pmod v, \quad y \equiv c - b \pmod v, \quad z \equiv a - c \pmod v.$$

Then we see that the existence of a cyclic $MTSEC(v, 1)$ is equivalent to partition of the set $\{2, 3, \ldots, v-2\}$ into disjoint difference triples $[x, y, z]$ such that $x + y + z \equiv \pmod v$. Thus, if there exists a cyclic $MTSEC(v, 1)$, then

$$\sum_{i=2}^{v-2} i = \frac{v(v-3)}{2} \equiv 0 \pmod v, \text{ equivalently, } v - 3 \equiv 0 \pmod 2.$$

Thus v cannot be even. Therefore, if there exists a cyclic $MTSEC(v)$, then $v \equiv 3 \pmod 6$ since v must be odd and $v \equiv 0 \pmod 3$ and $v \neq 6$. Since there exists a cyclic $TSEC(v, 1)$ for $v \equiv 3 \pmod 6$ [15], we have the following theorem.

Theorem 3.1. *There exists a cyclic* $MTSCE(v, 1)$ *if and only if* $v \equiv 3 \pmod 6$.

By Lemma 1.3, a necessary condition for the existence of a $MTSEC(v, \lambda)$ is

$$\lambda \equiv 1, 2 \pmod 3 \quad \text{and} \quad v \equiv 0 \pmod 3,\ v \neq 6, \text{ or}$$
$$\lambda \equiv 0 \pmod 3 \quad \text{and} \quad v \in \{0, 3\} \text{ or } v \geq 9.$$

Since the union of a $MTSEC(v, \lambda_1)$ and a $MTSEC(v, \lambda_2)$ is a $MTSEC(v, \lambda_1 + \lambda_2)$, it suffices to establish the existence of a $MTSEC(v, \lambda)$ for the minimum value of λ, namely for

$$\lambda = 1 \quad \text{and} \quad v \equiv 0 \pmod 3,\ v \neq 6, \text{ or}$$
$$\lambda \equiv 3 \quad \text{and} \quad v \equiv 1, 2 \pmod 3,\ v \geq 9.$$

To complete the existence of a $MTSEC(v, \lambda)$ for all λ, we need the existence of a $MTSEC(v, 3)$ for $v \equiv 1, 2 \pmod 3$ with $10 \leq v \leq 50$ which is unsettled and then Lemma 2.2 is applied.

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