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ON A CONJECTURE OF S. ELLIS CONCERNING THE NON-EXISTENCE OF JACK FIELDS

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ABSTRACT. We prove that a full jack field does not exist on the sum of a trivial bundle and the canonical bundle on $G_{2,4}$, the Grassmanian of 2-planes in 4-space.

1. Introduction

Fix a positive integer n. An *ordered orthogonal k-jack* in n-space is a k-tuple $\langle L_1, \ldots, L_k \rangle$ of one dimensional subspaces of $\mathbb{R}^n$ which are pairwise orthogonal to each other. As we will deal only with orthogonal jacks in this paper, we will drop the adjective "orthogonal" from now on. An *unordered k-jack* is a set $\{L_1, \ldots, L_k\}$ of k one dimensional subspaces which are pairwise orthogonal to each other.

The set $J_{(k),n}$ of ordered k-jacks is a closed subset of the k-fold product of the $(n-1)$-dimensional real projective space. The set $J_{\{k\},n}$ of unordered k-jacks is the quotient of $J_{(k),n}$ by the evident action of Σ_k, the symmetric group on k letters. We give it the quotient topology. Note that the orthogonal group O_n acts on both spaces.

Unless otherwise specified, we will assume that all spaces are paracompact and have the homotopy type of CW-complexes.

Given an n-dimensional vector bundle with fiberwise inner product over a space B, we can take the space of ordered or unordered jacks over each fiber to get the associated jack bundles. If the vector bundle is associated with a principal O_n-bundle ξ, then these can be idenified with $\xi[J_{(k),n}] = E \times_{O_n} J_{(k),n} \to B$ and $\xi[J_{\{k\},n}] = E \times_{O_n} J_{\{k\},n} \to B$ respectively, where E is the total space of ξ. An ordered k-jack field of ξ is a cross-section of $\xi[J_{(k),n}]$ and an unordered k-jack field is a cross-section of $\xi[J_{\{k\},n}]$.

Ellis [1] considered the question of continuity of maps that would form factor analysis as defined in statistics and showed that it involves the question of finding continuous n-jack fields in n-vector bundles. Then he considered this question for ζ_n, the Whitney sum of the $(n-2)$ dimensional trivial bundle and

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the canonical 2-bundle on $G_{2,4}$, the Grassmanian of 2-planes in $\mathbb{R}^4$ and proved the following result.

Theorem 1.1. *For any $n \geq 2$, ζ_n does not have a global ordered n-jack field. Also, if $n = 2$ or $n = 3$, then ζ_n does not have a global unordered n-jack field.*

He conjectured that the second result above is true for any n. In this note, we will show that this is true, that is we prove

Theorem 1.2. *Suppose that $n \geq 3$. Then ζ_n does not have a global unordered n-jack field.*

2. Proof

Note that the actions of O_n on $J_{(k),n}$ and $J_{\{k\},n}$ are transitive. Let D_k be the subgroup of diagonal matrices in O_k, and let B_k be the subgroup of O_k generated by D_k and the permutation matrices (so that $B_k \cong \Sigma_k D_k$). Then the isotropy subgroup of the ordered jack of first k coordinate axes in $\mathbb{R}^n$ is $D_k \times O_{n-k}$ and the isotropy subgroup of the corresponding unordered jack is $B_k \times O_{n-k}$. This gives us the next lemma.

Lemma 2.1. *A an O_n-space, $J_{(k),n}$ is homeomorphic to $O_n/(D_k \times O_{n-k})$, and $J_{\{k\},n}$ is homeomorphic to $O_n/(B_k \times O_{n-k})$.*

Lemma 2.2. *Let ξ be a principal O_n bundle over a space B, and let $f_\xi : B \to BO_n$ be its classifying map. Then $\xi[J_{\{k\},n}]$ has a cross-section if and only if f_ξ factors through $B(B_k \times O_{n-k})$.*

Proof. This follows from Theorems 6.2.3 and 6.5.1 in [2]. □

Let i be the inclusion of O_2 into O_n.

Lemma 2.3. *If $n \geq 3$, the composition*

$$G_{2,4} \to G_{2,\infty} = BO_2 \xrightarrow{Bi} BO_n$$

induces a non-trivial homomorphism on π_2's.

Proof. We have the following diagram where the rows are fibration sequences:

$$\begin{array}{ccccc}
O_2 & \longrightarrow & G_{2,4} & \longrightarrow & G_{2,4} \\
\| & & \downarrow & & \downarrow \\
O_2 & \longrightarrow & EO_2 & \longrightarrow & BO_2 \\
{\scriptstyle i}\downarrow & & {\scriptstyle Ei}\downarrow & & \downarrow{\scriptstyle Bi} \\
O_n & \longrightarrow & EO_n & \longrightarrow & BO_n
\end{array}$$

Consider the following portion of the homotopy long exact sequences of the above fibrations, and the induced homomorphisms:

$$\begin{array}{ccccccc}
\pi_2 G_{2,4} & \longrightarrow & \pi_2 G_{2,4} & \longrightarrow & \pi_1 O_2 & \longrightarrow & \pi_1 G_{2,4} = 0 \\
\downarrow & & \downarrow & & \| & & \downarrow \\
\pi_2 EO_2 = 0 & \longrightarrow & \pi_2 BO_2 & \xrightarrow{\cong} & \pi_1 O_2 = \mathbb{Z} & \longrightarrow & \pi_1 EO_2 = 0 \\
\downarrow & & {\scriptstyle Bi_*}\downarrow & & {\scriptstyle i_*}\downarrow & & \downarrow \\
\pi_2 EO_n = 0 & \longrightarrow & \pi_2 BO_n & \xrightarrow{\cong} & \pi_1 O_n = \mathbb{Z}/(2) & \longrightarrow & \pi_1 EO_n = 0
\end{array}$$

[See, for example, [2, pp. 91–93] for the identification of the groups made above.]

The fact that i_* is an epimorphism and the exactness of the rows of the diagram above imply that the composition $\pi_2 G_{2,4} \xrightarrow{\pi}_2 BO_2 \xrightarrow{Bi_*} \pi_2 BO_n$ is an epimorphism. □

Proof of Theorem 1.2. Suppose that ζ_n has a global n-jack field. Then its classifying map f must factor via BB_n. The latter is a $K(\pi, 1)$, and so has a trivial π_2. So homorphism on π_2 induced by f is trivial, contradicting the previous lemma. □

Remark 2.4. From a topological point of view, existence of k-jack fields, for $k \leq n-2$, on n-dimensional vector bundles is a more interesting question. This will be studied elsewhere.

References

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