Bull. Korean Math. Soc. **45** (2008), No. 2, pp. 277–283

THE VALUES OF AN EULER SUM AT THE NEGATIVE INTEGERS AND A RELATION TO A CERTAIN CONVOLUTION OF BERNOULLI NUMBERS

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Abstract. The paper deals with the values at the negative integers of a certain Dirichlet series related to the Riemann zeta function and with the expression of these values in terms of Bernoulli numbers.

1. Introduction

We consider the function

$$h(s)=\sum_{n=1}^{\infty}\frac{H_n}{n^s}, \tag{1}$$

where

$$H_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}, \tag{2}$$

are the Harmonic numbers and Re $s>1$. The function $h(s)$ was studied by many authors, starting with Euler, who evaluated this series in a closed form when $s=k$ is a positive integer. An elementary derivation of Euler's formula can be found, for instance, in [3]. For general s this function was investigated by Apostol-Vu [1] and Matsuoka [5], who provided an analytic extension to all complex numbers and discussed its values and poles at the negative integers.

In this note we shall find a relation between the values $h(1-n)$ and the numbers A_n, $n=1,2,\ldots$, defined as the convolution

$$A_n=\sum_{k+j=n}\frac{B(k)}{k!\,k}\frac{B(j)}{j!},\ k=1,2,\ldots;\ j=0,1,\ldots, \tag{3}$$

where $B(n)=B_n$ are the Bernoulli numbers for $n\neq 1$ and $B(1)=-B_1=\frac{1}{2}$. Thus

$$\frac{z\,e^z}{e^z-1}=\frac{-z}{e^{-z}-1}=\sum_{n=0}^{\infty}\frac{B(n)}{n!}z^n. \tag{4}$$

Received May 12, 2007.

2000 *Mathematics Subject Classification*. Primary 11M41, Secondary 11B68.

Key words and phrases. Dirichlet series, Euler sum, Bernoulli number, Hankel contour integration.

The notation $B(n)$ is used here in order to avoid the negative sign in B_1 and thus make possible the representation (4). We have also the expansion

$$\log\left(\frac{e^z-1}{z}\right)=\sum_{n=1}^{\infty}\frac{B(n)}{n!\,n}\,z^n\,. \tag{5}$$

The product of the functions in (4) and (5) is the generating function of the numbers A_n,

$$\frac{z\,e^z}{e^z-1}\log\left(\frac{e^z-1}{z}\right)=\sum_{n=1}^{\infty}A_n\,z^n\,. \tag{6}$$

The relation between A_n and the values of $h(s)$ is based on the evaluation of the following integral

$$F(s)=\frac{\Gamma(1-s)}{2\pi i}\int_L\frac{z^{s-1}\,e^z}{e^z-1}\log\left(\frac{e^z-1}{z}\right)\,dz\,, \tag{7}$$

where L is the Hankel contour consisting of three parts: $L=L_-\cup L_+\cup L_\varepsilon$, with $L_-$the "lower side" (i.e., $\arg(z)=-\pi$) of the ray $(-\infty,-\varepsilon),\varepsilon>0$, traced left to right, and L_+ the "upper side" ($\arg(z)=\pi$) of this ray traced right to left. Finally, $L_\varepsilon=\{z=\varepsilon\,e^{\theta i}:-\pi\le\theta\le\pi\}$ is a small circle traced counter-clockwise and connecting the two sides of the ray. This contour was used, for instance, in [2].

We note that convolutions like (3) appear in the Matiyasevich version of Miki's identity - see [6]; see also Yu. Matiyasevich, Identities with Bernoulli numbers, http://logic.pdmi.ras.ru/~yumat/Journal/Bernoulli/bernulli.htm

General reference for the Bernoulli numbers, the Riemann-zeta function, and the Gamma and digamma functions is [7].

2. Main results

The main results of this article are given in the following theorem and the three corollaries.

Theorem 1. *For* Re $s>1$,

$$F(s)=h(s)-\zeta(s+1)+\psi(s)\zeta(s)+\zeta'(s), \tag{8}$$

where $\zeta(s)$ is the Riemann-zeta function and $\psi(s)=\Gamma'(s)/\Gamma(s)$ is the digamma function.

As $F(s)/\Gamma(1-s)$ is an entire function (from (7)), this provides an extension of the right hand side in (8) to all complex s.

The proof of the theorem is given in Section 3.

It is easy to see that when s is a negative integer or zero, the integration in (7) can be reduced to L_ε only, as the integrals on L_+ and L_- cancel each other. This way for the coefficients A_n of the Taylor series (6) we have

$$(n-1)!\,A_n=F(1-n) \tag{9}$$

for $n = 1, 2, \ldots$. We shall evaluate the right hand side of (8) when $s = 1 - n$ by considering the three cases: $n > 1$ odd, $n = 1$, and n even. The results are organized in three corollaries. Before listing these corollaries, we recall two properties of the Riemann zeta-function. For $m = 1, 2, \ldots$, $\zeta(-2m) = 0$ and $\zeta(1-2m) = -\frac{B_{2m}}{2m}$.

We first consider the case when n is odd.

Corollary 1. *Let $n = 2m + 1$, $m > 0$. Then*

$$(10)\qquad (2m)!\, A_{2m+1} = h(-2m) - \zeta(1-2m) = \frac{1}{2}(1 + \frac{1}{2m})\, B_{2m}\,.$$

Proof. From (9) we have $(2m)!\, A_{2m+1} = F(-2m)$. In order to evaluate $F(-2m)$ we use the well-known property of the digamma function

$$(11)\qquad \psi(s) = \psi(1-s) - \pi\cot\pi s$$

to write

$$(12)\qquad \psi(s)\zeta(s) = \psi(1-s)\zeta(s) - \zeta(s)\pi\cot\pi s.$$

Now, for $s = -2m$ we have $\psi(1 + 2m)\zeta(-2m) = 0$ and

$$(13)\qquad \zeta(s)\pi\cot\pi s\,|_{s=-2m} = \zeta'(-2m).$$

This follows from the Taylor expansion around $s = -2m$,

$$(14)\qquad \zeta(s)\pi\cot\pi s = \zeta'(-2m) + \frac{1}{2}\zeta''(-2m)(s+2m) + O((s+2m)^2).$$

Thus from (8) we find

$$(15)\qquad F(-2m) = h(-2m) - \zeta(1-2m).$$

The values $h(-2m)$ were computed by Matsuoka [5] as

$$(16)\qquad h(-2m) = -\frac{B_{2m}}{4m} + \frac{B_{2m}}{2}.$$

(Note that Matsuoka worked with the function $f(s) = h(s) - \zeta(s+1)$). Therefore, equation (10) follows from (15) and (16).

$h(-2m)$ was also evaluated in [1], but incompletely (missing the second term on the right hand side in (16)). □

Now let us consider the case $s = 0$ in (8), that is, $n = 1$ in (9).

Corollary 2. *In a neighborhood of zero,*

$$(17)\qquad h(s) = \frac{1}{2s} + \frac{1}{2}(1+\gamma) + O(s),$$

where $\gamma = -\psi(1)$ is the Euler constant.

Proof. As found in [1] and [5], the function $h(s)$ has a simple pole at $s = 0$ with residue $\frac{1}{2}$. In order to establish (17) we need to evaluate $h(s) - \frac{1}{2s}$ at zero. The functions $\zeta(s+1)$ and $\psi(s)\zeta(s)$ have residues 1 and $\frac{1}{2}$ respectively, at zero, and so the function

$$\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s} \tag{18}$$

does not have a pole at $s = 0$. Moreover, one easily finds that around $s = 0$

$$\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s} = \frac{\gamma}{2} + \zeta'(0) + O(s). \tag{19}$$

Next we rewrite (8) in the form

$$h(s) - \frac{1}{2s} = F(s) + (\zeta(s+1) - \psi(s)\zeta(s) - \frac{1}{2s}) - \zeta'(s) \tag{20}$$

and also compute the coefficient $A_1 = \frac{1}{2} = F(0)$ from (3). From (19) and (20) we find

$$\left(h(s) - \frac{1}{2s}\right)|_{s=0} = \frac{1}{2}(1+\gamma), \tag{21}$$

which proves (17). □

Finally, we compute $F(1-n)$ for $n = 2m$.

Corollary 3. *For $m = 2, 3, \ldots$, in a neighborhood of $s = 1 - 2m$ the function $h(s)$ is represented as*

$$h(s) = \frac{\zeta(1-2m)}{s+2m-1} + (2m-1)!\, A_{2m} - \psi(2m)\zeta(1-2m) + O(s+2m-1), \tag{22}$$

and in a neighborhood of $s = -1$,

$$h(s) = \frac{-1}{12(s+1)} - \frac{1}{8} + \frac{\gamma}{12} + O(s+1). \tag{23}$$

Proof. Apostol–Vu [1] and Matsuoka [5] showed that the function $h(s)$ has simple poles at the negative odd integers $s = 1 - 2m$ with residues $\zeta(1 - 2m)$. The same is true for the function $\zeta(s)\pi \cot \pi s$, as follows from the Taylor expansion at $s = 1 - 2m$,

$$\zeta(s)\pi \cot \pi s = \zeta(1-2m)\frac{1}{s+2m-1} + \zeta'(1-2m) + O(s+2m-1). \tag{24}$$

Using (12) in (8), we obtain the representation

$$h(s) = \zeta(s)\pi \cot \pi s + F(s) + \zeta(s+1) - \psi(1-s)\zeta(s) - \zeta'(s), \tag{25}$$

and substituting (24) in this, we get

$$\begin{aligned} h(s) - \frac{\zeta(1-2m)}{s+2m-1} &= F(s) + \zeta(s+1) - \psi(1-s)\zeta(s) \\ &\quad -\zeta'(s) + \zeta'(1-2m) + O(s+2m-1)\,. \end{aligned} \tag{26}$$

Now, evaluating both sides of (26) at $s = 1 - 2m$,

$$\left[h(s) - \frac{\zeta(1-2m)}{s+2m-1} \right] \Bigg|_{s=1-2m}$$
$$= F(1-2m) + \zeta(2-2m) - \psi(2m)\zeta(1-2m) + O(s+2m-1) \tag{27}$$

and as $F(1-2m) = (2m-1)!\, A_{2m}$ and $\zeta(2-2m) = 0$, we obtain (22).

When $m = 1$, we have $\zeta(2-2m) = \zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $\psi(2) = 1 - \gamma$, and by direct computation from (3), $A_2 = \frac{7}{24}$. Thus (23) follows from (27). □

3. Proof of Theorem 1

Here we evaluate the integral in (7)

$$I(s) = \frac{1}{2\pi i} \int_L \frac{z^{s-1}\, e^z}{e^z - 1} \log\left(\frac{e^z - 1}{z}\right) dz, \tag{28}$$

where the contour L is as described in Section 1. We choose $\operatorname{Re} s > 1$ and set $\varepsilon \to 0$. The integral over L_ε becomes zero, as the function

$$\frac{z\, e^z}{e^z - 1} \log\left(\frac{e^z - 1}{z}\right) \tag{29}$$

is holomorphic in a neighborhood of zero. Noticing that $z = x\, e^{-\pi i}$ on L_- and $z = x\, e^{\pi i}$ on L_+, we find that

$$\begin{aligned} -I(s) &= \frac{e^{-\pi i s}}{2\pi i} \int_\infty^0 \frac{x^{s-1}\, e^{-x}}{1 - e^{-x}} \log\left(\frac{1 - e^{-x}}{x}\right) dx \\ &\quad + \frac{e^{\pi i s}}{2\pi i} \int_0^\infty \frac{x^{s-1}\, e^{-x}}{1 - e^{-x}} \log\left(\frac{1 - e^{-x}}{x}\right) dx \\ &= \frac{\sin \pi s}{\pi} \int_0^\infty \frac{x^{s-1}\, e^{-x}}{1 - e^{-x}} \log\left(\frac{1 - e^{-x}}{x}\right) dx. \end{aligned} \tag{30}$$

Next,

$$\int_0^\infty \frac{x^{s-1}\, e^{-x}}{1 - e^{-x}} \log\left(\frac{1 - e^{-x}}{x}\right) dx$$
$$= \int_0^\infty \frac{x^{s-1}\, e^{-x}}{1 - e^{-x}} \log(1 - e^{-x})\, dx - \int_0^\infty \frac{x^{s-1}}{e^x - 1} \log x\, dx. \tag{31}$$

We shall evaluate the two integrals on the right hand side in (31) one by one. First we use the expansion

$$\frac{\log(1 - e^{-x})}{1 - e^{-x}} = -\sum_{n=1}^\infty H_n\, e^{-nx} \tag{32}$$

(see [4], (7.57), p. 352). Multiplying this by $x^{s-1}e^{-x}$ and integrating from zero to infinity, we find that

$$\begin{aligned}\int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}\log(1-e^{-x})\,dx &= -\sum_{n=1}^\infty H_n\int_0^\infty x^{s-1}e^{-(n+1)x}\,dx\\ &= -\Gamma(s)\sum_{n=1}^\infty \frac{H_n}{(n+1)^s}\\ &= -\Gamma(s)(h(s)-\zeta(s+1)).\end{aligned} \tag{33}$$

Next, differentiating for s the representation

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}dx, \tag{34}$$

we obtain

$$\int_0^\infty \frac{x^{s-1}}{e^x-1}\log x\,dx = \Gamma'(s)\zeta(s)+\Gamma(s)\zeta'(s) = \Gamma(s)(\psi(s)\zeta(s)+\zeta'(s)). \tag{35}$$

From (31), (33) and (35)

$$\int_0^\infty \frac{x^{s-1}e^{-x}}{1-e^{-x}}\log\left(\frac{1-e^{-x}}{x}\right)dx = -\Gamma(s)(h(s)-\zeta(s+1)+\psi(s)\zeta(s)+\zeta'(s)), \tag{36}$$

and therefore,

$$I(s) = \frac{1}{\pi}\Gamma(s)\sin(\pi s)(h(s)-\zeta(s+1)+\psi(s)\zeta(s)+\zeta'(s)). \tag{37}$$

Finally, (8) follows from here in view of the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}. \tag{38}$$

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