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OSCILLATION OF NONLINEAR SECOND ORDER NEUTRAL DELAY DYNAMIC EQUATIONS ON TIME SCALES

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Abstract. In this paper, we establish some oscillation criteria for nonautonomous second order neutral delay dynamic equations

$$(x(t) \pm r(t)x(\tau(t)))^{\triangle\triangle} + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) = 0$$

on a time scale $\mathbb{T}$. Oscillatory behavior of such equations is not studied before. This is a first paper concerning these equations. The results are not only can be applied on neutral differential equations when $\mathbb{T} = \mathbb{R}$, neutral delay difference equations when $\mathbb{T} = \mathbb{N}$ and for neutral delay $q-$difference equations when $\mathbb{T} = q^{\mathbb{N}}$ for $q > 1$, but also improved most previous results. Finally, we give some examples to illustrate our main results. These examples are not discussed before and there is no previous theorems determine the oscillatory behavior of such equations.

1. Introduction

In the recent years, the theory of time scales has received a lot of attention which was introduced by Stefan Hilger in his Ph. D. thesis in 1988 in order to unify continuous and discrete analysis (see [10]). In fact there has been much activities concerning the oscillation and nonoscillation of solutions of dynamic equations on time scales (or measure chains). We refer the reader to recent papers [1-5, 8, 11, 13-15, 17] and the references cited therein. A book on the subject of time scales, by Bohner and Peterson [7] summarizes and organizes much of time scales calculus, see also the book by Bohner and Peterson [6] for advances in dynamic equations on time scales.

In this paper, we are concerned with the oscillation of the second-order nonlinear dynamic equations

$$\text{(A)} \qquad (x(t) + r(t)x(\tau(t)))^{\triangle\triangle} + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) = 0$$

and

$$\text{(B)} \qquad (x(t) - r(t)x(\tau(t)))^{\triangle\triangle} + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) = 0$$

on a time scale $\mathbb{T}$. Since we are interested in asymptotic behavior of solutions, we will suppose that the time scale $\mathbb{T}$ under consideration is not bounded above,

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i.e., it is a time scale interval of the form $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$. Through this paper, we assume that:

(H_1) $r \in C_{rd}(\mathbb{T}, \mathbb{R}^+)$, nonincreasing, h and $g \in C_{rd}(\mathbb{T}, \mathbb{T}), \tau(t) < t, h_1(t) < t$, $h_2(t) < \tau(t)$, $\lim_{t \to \infty} \tau(t) = \infty$, $\lim_{t \to \infty} h_1(t) = \infty$, $\lim_{t \to \infty} h_2(t) = \infty$ and $0 \leq r(t) \leq r < 1$, $C_{rd}(\mathbb{T}, \mathbb{S})$ denotes the set of all functions $f : \mathbb{T} \to \mathbb{S}$ which are *right-dense continuous* on $\mathbb{T}$.

(H_2) $H(t, u, v) \in C(\mathbb{T} \times \mathbb{R}^2, \mathbb{R})$ for each $t \in \mathbb{T}$ which are nondecreasing in u and v and $H(t, u, v) > 0$ for $uv > 0$.

(H_3) $|H(t, u, v)| \geq \alpha(t) |u|^\lambda + \beta(t) |v|^\lambda$ where $\alpha(t), \beta(t) \geq 0$ and $0 \leq \lambda = \frac{p}{q} < 1$ with p, q are odd integers.

By a solution of equation (A & B), we mean a nontrivial real valued function $x(t)$ which has the properties $(x(t) \pm r(t)x(\tau(t)) \in C^2_{rd}[t_x, \infty), t_x > t_0$ and satisfying equation (A & B) for all $t > t_x$. Our attention is restricted to those solutions of equation (A & B) which exist on some half line $[t_x, \infty)$ and satisfy $\sup\{|x(t)| : t > t_1\} > 0$ for any $t_1 > t_x$.

A solution $x(t)$ of (A & B) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is called nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Note that if $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = \rho(t) = t$, $f^\Delta(t) = f'(t)$, and (A), (B) become respectively as second-order neutral delay differential equations

$$[x(t) + r(t)x(\tau(t))]'' + H(t, x(h_1(t)), x'(h_2(t))) = 0 \tag{1.1}$$

and

$$[x(t) - r(t)x(\tau(t))]'' + H(t, x(h_1(t)), x'(h_2(t))) = 0. \tag{1.2}$$

As a special case of equation (1.2), Wong in [16] considered the second order sublinear neutral differential equation

$$[x(t) - rx(t - \tau)]'' + a(t)f(x(t - \sigma)) = 0, \quad t \geq t_0, \tag{1.3}$$

where $0 < r < 1, a(t) \in C\ ([0, \infty), [0, \infty))$, $f \in C^1((-\infty, \infty), (-\infty, \infty))$ satisfying $x\ f(x) > 0$ for $x \neq 0$, $f'(x) \geq 0$ and that

$$0 < \int_0^\epsilon \frac{dx}{f(x)}, \quad \int_{-\epsilon}^0 \frac{dx}{-f(x)} < \infty, \quad \forall \epsilon > 0$$

and

$$f(uv) \geq f(u)f(v), \text{ if } uv \geq 0 \text{ and } |v| \geq M$$

for some large $M > 0$. It was proved that equation (1.3) is oscillatory if and only if

$$\int^\infty a(t)f(t)dt = \infty. \tag{1.4}$$

Recently, X. Lin [13] considered also a sublinear neutral differential equation which is a special case of equation (1.2) in the form

$$[x(t) - r(t)x(t-\tau)]'' + a(t)f(x(t-\sigma)) = 0, \quad t \geq t_0. \tag{1.5}$$

It was assumed that $r(t), a(t) \in C([0,\infty),[0,\infty))$ and $0 \leq r(t) \leq r < 1$, f is nondecreasing, satisfying $x\ f(x) > 0$ for $x \neq 0$ and $\sigma > \tau$. Then every solution of equation (1.5) is oscillatory if

$$\int^{\infty} a(t)dt = \infty. \tag{1.6}$$

Note that condition (1.6) can not be applied for the second order neutral equation

$$[x(t) - r(t)x(\tau(t))]'' + \frac{\gamma}{(t-h)^2}x^{\lambda}(t-h) = 0, \tag{1.7}$$

where $\gamma > 0$ and $0 \leq \lambda = \frac{p}{q} < 1$ with p, q are odd integers, for other results see [9], [12].

If $\mathbb{T} = \mathbb{Z}$, we have $\sigma(t) = t+1$, $\mu(t) = 1$, $f^{\Delta} = \Delta f$, and (A & B) become the second-order neutral delay difference equations

$$\Delta^2[x(t) \pm r(t)x(\tau(t))] + H(t, x(h_1(t)), \triangle x(h_2(t))) = 0. \tag{1.8}$$

If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, we have $\sigma(t) = t+h$, $\mu(t) = h$, $f^{\Delta} = \Delta_h f = \frac{f(t+h)-f(t)}{h}$ and (A & B) become the second-order neutral delay difference equations

$$\Delta_h^2[y(t) \pm r(t)x(\tau(t))] + H(t, x(h_1(t)), \triangle_h x(h_2(t))) = 0. \tag{1.9}$$

If $\mathbb{T}=q^{\mathbb{N}} = \{t : t = q^n,\ n \in \mathbb{N},\ q > 1\}$, we have $\sigma(t) = qt$, $\mu(t) = (q-1)t$, $x_q^{\Delta}(t) = \frac{x(qt)-x(t)}{(q-1)t}$, and (A & B) become the second order $q-$neutral delay difference equation

$$\Delta_q^2[y(t) \pm r(t)x(\tau(t))] + H(t, x(h_1(t)), x_q^{\Delta}(h_2(t))) = 0. \tag{1.10}$$

The paper is organized as follows: In Section 2 we present some preliminaries on time scales. In Section 3, we establish some new sufficient conditions for oscillation of (A & B). In Section 4, we present some illustrative examples to show that our results not only new but also improved most previous results.

2. Some preliminaries on time scales

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. On any time scale $\mathbb{T}$, we defined the *forward* and *backward jump* operators by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}. \tag{2.1}$$

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$ is said to be *left-dense* if $\rho(t) = t$, *right-dense* if $t > \sup \mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$ and *right-scattered* if $\sigma(t) >

t. The graininess function $\mu : \mathbb{T} \to [0,\infty)$ is defined by $\mu(t) := \sigma(t) - t$. For the function $f : \mathbb{T} \to \mathbb{R}$ the (*delta*) derivative is defined by

$$f^{\triangle}(t) := \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{2.2}$$

f is said to be differentiable if its derivative exists. A useful formula is

$$f^{\sigma} := f(\sigma(t)) = f(t) + \mu(t) f^{\triangle}(t). \tag{2.3}$$

If f, g are differentiable, then fg and the quotient $\frac{f}{g}$ (where $gg^{\sigma} \neq 0$) are differentiable with

$$(fg)^{\triangle} = f^{\triangle} g + f^{\sigma} g^{\triangle} = f g^{\triangle} + f^{\triangle} g^{\sigma}, \tag{2.4}$$

and

$$\left(\frac{f}{g}\right)^{\triangle} := \frac{f^{\triangle} g - f g^{\triangle}}{g g^{\sigma}}. \tag{2.5}$$

If $f^{\triangle}(t) \geq 0$, then f is nondecreasing.

A function $f : [a,b] \to \mathbb{R}$ is said to be *right-dense continuous* if it right continuous at each right-dense point and there exists a finite left limit at all left-dense points. A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive, if $1 + \mu(t) f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all functions $f : \mathbb{T} \to \mathbb{R}$ which are regressive and rd-continuous will be denoted by C_r. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by $\mathcal{R}^+ = \{f \in \mathcal{R} : 1 + \mu(t) f(t) \neq 0,\ t \in \mathbb{T}\}$. A function F with $F^{\triangle} = f$ is called an antiderivative of f and then we define

$$\int_a^b f(t) \triangle t = F(b) - F(a), \tag{2.6}$$

where $a, b \in \mathbb{T}$. It is well known that rd-continuous functions possess antiderivatives. A simple consequence of formula (2.3) is

$$\int_t^{\sigma(t)} f(s) \triangle s = \mu(t) f(t), \tag{2.7}$$

and infinite integrals are defined as

$$\int_a^{\infty} f(t) \triangle t = \lim_{b \to \infty} \int_a^b f(t) \triangle t. \tag{2.8}$$

3. Main results

In this section, we establish some sufficient conditions for the oscillation of equations (A & B). For the remainder of the paper we assume that $\delta^{-1}(t)$ is the inverse of the function $\delta(t)$ exists and satisfies $\delta^{-(n+1)}(t) = t + n\delta$.

I. Oscillatory behavior of solutions of equation (A)

Theorem 3.1. *Assume that $H_1 - H_3$ hold. Then every solution of equation* (A) *oscillates, if*

$$\int_{t_5}^{\infty} \{\alpha(s)(k[1-r(h_1(s))]h_1(s))^{\lambda} + \frac{1}{2^{\lambda}}\beta(s)\}\Delta s = \infty. \tag{3.1}$$

Proof. Suppose to the contrary that equation (A) has a nonoscillatory solution $x(t)$. We may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ where $\delta = \min\{h_1, h_2\}$ for all $t > t_1$.

Set

$$y(t) = x(t) + r(t)x(\tau(t)). \tag{3.2}$$

Then, equation (A) takes the form

$$y^{\Delta\Delta}(t) + H(t, x(h_1(t)), x^{\Delta}(h_2(t))) = 0.$$

and $y(t) > 0$. Hence, from equation (3.2) it follows that $y(t) > x(t)$ for all $t > t_1$. Assume that $x^{\Delta}(\delta(t)) < 0$ for all $t > t_1$.Then (H_2) implies that $y^{\Delta\Delta} > 0$ and then either $y^{\Delta} < 0$ or $y^{\Delta} > 0$. If $y^{\Delta} < 0$ then $y \to -\infty$ which contradicts $y > 0$. Also, from nonincreasing of $r(t)$ and $x^{\Delta}(t) < 0$ implies that $y^{\Delta} < 0$, which contradicts $y^{\Delta} > 0$. Now, it follows that $x^{\Delta}(t) > 0$ and then from (H_2) implies that, $y^{\Delta\Delta} < 0$ for all $t > t_1$. Thus $y^{\Delta}(t)$ is strictly decreasing. We prove that $y^{\Delta}(t) > 0$ on the interval $[t_1, \infty)_{\mathbb{T}}$. Assume not. Then there exists $t_2 \geq t_1$ such that $y^{\Delta}(t_2) = C < 0$. Then, since $y^{\Delta\Delta}(t) < 0$, we have

$$y^{\Delta}(t) \leq y^{\Delta}(t_2) = C, \quad \text{for } t \geq t_2, \tag{3.3}$$

and therefore

$$y^{\Delta}(t) \leq C \quad \text{for all} \quad t \geq t_2. \tag{3.4}$$

Integrating inequality (3.4) from t_2 to t, we obtain

$$y(t) = y(t_2) + \int_{t_2}^{t} y^{\Delta}(s)\Delta s \leq y(t_2) + C(t - t_2), \tag{3.5}$$

and consequently $y(t) \to -\infty$ as $t \to \infty$ which contradicts $y > 0$. Hence $y^{\Delta}(t) > 0$ and consequently, $y(t)$ is strictly increasing, then from (3.2) we have, $y(t) - r(t)y(\tau(t)) = x(t) - r(t)r(\tau(t))x(\tau(\tau(t))) < x(t)$, therefore

$$x(t) > (1 - r(t))y(t). \tag{3.6}$$

Since $y^{\Delta\Delta}(t) < 0$ and $y(t) > 0$, then

$$\begin{aligned} y(t) &= y(t_4) + \int_{t_4}^{t} y^{\Delta}(s)\Delta s \\ &> (t - t_4)y^{\Delta}(t) \\ &> kty^{\Delta}(t) \quad \text{for } t > \frac{t_1}{(1-k)} := t_2, 0 < k < 1. \end{aligned} \tag{3.7}$$

Since $r(t)$ is nonincreasing, $x > 0$ and $x^{\triangle} > 0$, it follows that

$$\begin{aligned} y^{\triangle}(t) &< x^{\triangle}(t) + r(\sigma(t)(x(\tau(t)))^{\triangle} \\ &< x^{\triangle}(t) + (x(\tau(t)))^{\triangle} \end{aligned}$$

and then,

$$y^{\triangle}(t) < \begin{cases} 2x^{\triangle}(t) & \text{if } x^{\triangle}(t) \geq (x(\tau(t)))^{\triangle} \\ 2(x(\tau(t)))^{\triangle} & \text{if } x^{\triangle}(t) \leq (x(\tau(t)))^{\triangle}. \end{cases} \tag{3.8}$$

If $x^{\triangle}(t) \geq (x(\tau(t)))^{\triangle}$, then $x^{\triangle}(t) > \frac{1}{2}y^{\triangle}(t)$ which implies that

$$x^{\triangle}(\tau(t)) > \frac{1}{2}y^{\triangle}(\tau(t)) > \frac{1}{2}y^{\triangle}(t).$$

On the other side, if $x^{\triangle}(t) \leq (x(\tau(t)))^{\triangle}$, then $x^{\triangle}(\tau(t)) > \frac{1}{2}y^{\triangle}(t)$. Thus, we have

$$x^{\triangle}(\tau(t)) > \frac{1}{2}y^{\triangle}(t). \tag{3.9}$$

Since $\tau(h(t)) < h(t) < t$ and from nonincreasing of $y^{\triangle}(t)$, we get

$$y^{\triangle}(\tau(h(t))) \geq y^{\triangle}(h(t)) \geq y^{\triangle}(t). \tag{3.10}$$

Since $x > 0, x^{\triangle} > 0$ and $H(t,u,v)$ nondecreasing in u,v, then, by using $(H_1),(H_3)$, (3.6), (3.7) and (3.9), we get

$$\begin{aligned} 0 &= y^{\triangle\triangle}(t) + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) \\ &\geq y^{\triangle\triangle}(t) + H(t, [1-r(h_1(t))]y(h_1(t)), \frac{1}{2}y^{\triangle}(\tau^{-1}(h_2(t)))) \\ &\geq y^{\triangle\triangle}(t) + H(t, k[1-r(h_1(t))]h_1(t)y^{\triangle}(h_1(t)), \frac{1}{2}y^{\triangle}(t)) \\ &\geq y^{\triangle\triangle}(t) + \alpha(t)(k[1-r(h_1(t))]h_1(t)y^{\triangle}(t))^{\lambda} + \beta(t)(\frac{1}{2}y^{\triangle}(t))^{\lambda} \\ &\geq y^{\triangle\triangle}(t) + [\alpha(t)(k[1-r(h_1(t))]h_1(t))^{\lambda} + \frac{1}{2^{\lambda}}\beta(t)]y^{\triangle}(t))^{\lambda}. \end{aligned}$$

Hence,

$$-\frac{y^{\triangle\triangle}(t)}{y^{\triangle}(t))^{\lambda}} \geq \alpha(t)(k[1-r(h_1(t))]h_1(t))^{\lambda} + \frac{1}{2^{\lambda}}\beta(t).$$

Integrating the above inequality from t_2 to ∞, we get

$$\int_{t_5}^{\infty} \{\alpha(s)(k[1-r(h_1(s))]h_1(s))^{\lambda} + \frac{1}{2^{\lambda}}\beta(s)\}\triangle s$$

$$\begin{aligned} &\leq -\int_{t_5}^{\infty} \frac{y^{\triangle\triangle}(s)}{(y^{\triangle}(s))^{\lambda}}\triangle s. \\ &= -\lim_{t\to\infty}\int_{y^{\triangle}(t_5)}^{y^{\triangle}(t)} \frac{\triangle s}{s^{\lambda}} \\ &= -\int_{y^{\triangle}(t_5)}^{0} \frac{\triangle s}{s^{\lambda}} < \infty. \end{aligned}$$

which contradicts (3.1), and consequently, equation (A) has no eventually positive solution. Similarly, by using the same technique we can prove that equation (A) has no eventually negative solution. Thus equation (A) is oscillatory. The proof is complete. □

II. Oscillatory behavior of solutions of equation (B)

Theorem 3.2. *Assume that* $H_1 - H_3$ *hold with* $0 \leq r(t) = r < 1$. *Then every solution of equation* (B) *oscillates, if*

$$\int_{t_5}^{\infty} \{(kr)^{\lambda}\alpha(s)(\tau(h_1(s))^{\lambda} + r^{\lambda}\beta(s)\}\triangle s = \infty. \tag{3.11}$$

Proof. Suppose to the contrary that equation (B) has a nonoscillatory solution $x(t)$. We may assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ where $\delta = \min\{h_1, h_2\}$ for all $t > t_1$. Set

$$z(t) = x(t) - rx(\tau(t)). \tag{3.12}$$

Then, equation (B) takes the form

$$z^{\triangle\triangle}(t) + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) = 0, \tag{3.13}$$

and from (3.12), we get $z(t) < x(t)$. Assume that $x^{\triangle}(\delta(t)) < 0$ for all $t > t_1$. Then $x(t)$ is nonincreasing and consequently (H_2) implies that,

$$H(t, x(h_1(t)), x^{\triangle}(h_2(t))) < 0.$$

Thus from (3.13) it follows that $z^{\triangle\triangle} > 0$ and then $z^{\triangle}$ is strictly increasing. So, we may have either $z^{\triangle} < 0$ or $z^{\triangle} > 0$. Assume that $z^{\triangle} < 0$ then from (3.12) we have $x^{\triangle}(t) < r(x(\tau(t)))^{\triangle}$ and consequently,

$$x^{\triangle}(t) < r(x(\tau(t)))^{\triangle} < r^2(x(\tau^2(t)))^{\triangle} < \cdots < r^k(x(\tau^k(t)))^{\triangle} = r^k x^{\triangle}(t_0)$$

for sufficiently large k such that $t_0 = \tau^k(t)$. Hence, $x(t) < x(t_0) + r^k x^{\triangle}(t_0)(t - t_0)$ which implies that $x(t) < 0$ which contradicts $x(t) > 0$. So $z^{\triangle} < 0$ is impossible and then $z^{\triangle} > 0$.

Also, since $x > 0$ and $x^{\triangle} < 0$, then $0 \leq \lim_{t\to\infty} x(t) = p < \infty$ and consequently, from (3.12) it follows that $\lim_{t\to\infty} z(t) = (1-r)p < \infty$. But, since $z^{\triangle\triangle} > 0$ then $z^{\triangle}(t)$ is an increasing. So, $0 < z(t_1) < z(t)$ for all $t > t_1$. Then $z(t) = z(t_1) + \int_{t_1}^{t} z^{\triangle}(s)\triangle s > z(t_1) + z^{\triangle}(t_1)(t - t_1)$ and therefore $z(t) \to \infty$ as $t \to \infty$ which is a contradiction. This shows that $x^{\Delta} < 0$ is impossible, and consequently $x^{\Delta} > 0$.

Now $x > 0, x^{\Delta} > 0$. Then (H_2) implies that $H(t, x(h_1(t)), x^{\Delta}(h_2(t))) > 0$ and then $z^{\Delta\Delta} < 0$. Thus $z^{\Delta}(t)$ is strictly decreasing. We prove that $z^{\Delta}(t) > 0$ on the interval $[t_3, \infty)_{\mathbb{T}}$. Assume not. Then there exists $t_4 \geq t_3$ such that $z^{\Delta}(t_4) = C < 0$. Then, since $z^{\Delta\Delta}(t) < 0$, we have

$$z^{\Delta}(t) \leq z^{\Delta}(t_4) = C \quad \text{for} \quad t \geq t_4, \tag{3.14}$$

and therefore

$$z^{\Delta}(t) \leq C \quad \text{for all} \quad t \geq t_4. \tag{3.15}$$

Integrating the last inequality from t_2 to t, we obtain

$$z(t) = z(t_2) + \int_{t_4}^{t} z^{\Delta}(s)\Delta s \leq z(t_4) + C(t - t_4), \tag{3.16}$$

and consequently $z(t) \to -\infty$ as $t \to \infty$ which implies that there exists $c > 0$ and $t_5 \geq t_4$ such that $z(t) < -c$ for $t \geq t_5$. Then, we have from (3.2) that

$$x(t) < -c + rx(\tau(t)) \quad \text{for} \quad t \geq t_5, \tag{3.17}$$

which implies that $x(\delta^{-1}(t_5)) < -c + rx(t_5)$. Thus

$$x(\delta^{-(n+1)}(t_5)) \leq -c\sum_{i=0}^{n} r^i + r^{n+1}x(t_5) \leq -c + r^{n+1}x(t_5), \tag{3.18}$$

and so $x(\delta^{-(n+1)}(t_5)) < 0$ for large n, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Hence $z^{\Delta}(t) > 0$ and this implies that $z(t)$ is strictly increasing on $[t_1, \infty)$. We prove now that $z(t) > 0$ for $t \geq t_2$ where t_2 is large enough. Suppose not. Then there exists a $t_3 \geq t_1$ with $z(t_3) < 0$. Now, since $z(t)$ is strictly increasing then $z(t) > 0$ for $t \geq t_3$ (for if there exists a $t_4 > t_3$ with $z(t_4) > 0$, then $z(t) > 0$ for $t \geq t_4$, but we are assuming that $z(t) > 0$ for t large enough is not true). Then from (3.2) that $x(t) < rx(\tau(t))$, for $t \geq t_3$. Thus $x(\tau^{-1}(t)) \leq rx(t)$ and this implies after iteration that $x(\delta^{-(n+1)}(t)) \leq r^{n+1}x(t) \to 0$ for large n, since $0 \leq r < 1$ and so $x(\delta^{-(n+1)}(t)) < 0$ again, which contradicts the fact that $x(t) > 0$ for all $t \geq t_1$. Then, we have

$$z(t) > 0,\ z^{\Delta}(t) > 0,\ z^{\Delta\Delta}(t) < 0 \ \text{ for } \ t \geq t_1. \tag{3.19}$$

Since $z^{\Delta\Delta}(t) < 0$ and $z(t) > 0$, then

$$\begin{aligned} z(t) &= z(t_4) + \int_{t_4}^{t} z^{\Delta}(s)\Delta s \\ &> (t - t_4)z^{\Delta}(t) \\ &> ktz^{\Delta}(t) \ \text{ for } t > \frac{t_4}{(1-k)} := t_5, 0 < k < 1. \end{aligned} \tag{3.20}$$

Since $z^{\triangle}(t) > 0, x^{\triangle}(t) > 0$, $0 \le r < 1$, and $x(t) > z(t) > ktz^{\triangle}(t)$ then we have

$$\begin{aligned} z^{\triangle}(t) &= x^{\triangle}(t) - r(x(\tau(t)))^{\triangle} \\ x^{\triangle}(t) &= z^{\triangle}(t) + r(x(\tau(t))^{\triangle} \ge z^{\triangle}(t). \end{aligned} \tag{3.21}$$

Now $z(t) > 0$ implies that $x(t) > rx(\tau(t))$. Since $z(t) < x(t), 0 < z^{\triangle}(t)$ is nonincreasing and $H(t,u,v)$ nondecreasing in u and v. Then, using (3.20) and (3.21) we get

$$\begin{aligned} 0 &= z^{\triangle\triangle}(t) + H(t, x(h_1(t)), x^{\triangle}(h_2(t))) \\ &\ge z^{\triangle\triangle}(t) + H(t, rx(\tau(h_1(t))), rx^{\triangle}(\tau(h_2(t)))) \\ &\ge z^{\triangle\triangle}(t) + H(t, rz(\tau(h_1(t))), rz^{\triangle}(\tau(h_2(t))))) \\ &\ge z^{\triangle\triangle}(t) + \alpha(t)(rz(\tau(h_1(t))))^{\lambda} + \beta(t)(rz^{\triangle}(\tau(h_2(t))))^{\lambda} \\ &\ge z^{\triangle\triangle}(t) + \alpha(t)(kr\tau(h_1(t))z^{\triangle}(\tau(h_1(t))))^{\lambda} \\ &\quad + r^{\lambda}\beta(t)(z^{\triangle}(t))^{\lambda} \\ &\ge z^{\triangle\triangle}(t) + ((kr)^{\lambda}\alpha(t)(\tau(h_1(t))^{\lambda} + r^{\lambda}\beta(t))(z^{\triangle}(t))^{\lambda}. \end{aligned}$$

Then

$$((kr)^{\lambda}\alpha(t)(\tau(h_1(t))^{\lambda} + r^{\lambda}\beta(t)) \le -\frac{z^{\triangle\triangle}(t)}{(z^{\triangle}(t))^{\lambda}}. \tag{3.22}$$

Integrating inequality (3.22) from t_5 to ∞, we get

$$\begin{aligned} &\int_{t_5}^{\infty} \{(kr)^{\lambda}\alpha(s)(\tau(h_1(s))^{\lambda} + r^{\lambda}\beta(s)\}\triangle s \\ &\le -\int_{t_5}^{\infty} \frac{z^{\triangle\triangle}(s)}{(z^{\triangle}(s))^{\lambda}}\triangle s. \\ &= -\lim_{t\to\infty} \int_{z^{\triangle}(t_5)}^{z^{\triangle}(t)} \frac{\triangle s}{s^{\lambda}} \\ &= -\int_{y^{\triangle}(t_5)}^{0} \frac{\triangle s}{s^{\lambda}} < \infty, 0 \le \lambda < 1. \end{aligned}$$

Hence,

$$\int_{t_5}^{\infty} \{(kr)^{\lambda}\alpha(s)(\tau(h_1(s))^{\lambda} + r^{\lambda}\beta(s)\}\triangle s < \infty,$$

which contradicts (3.11). So, equation (B) has no eventually positive solution. Similarly, by using the same procedure we can prove that equation (B) has no eventually negative solution. Thus equation (B) is oscillatory. □

From Theorems (3.1) and (3.2), we have the following results:

Theorem 3.3. *Assume that $H_1 - H_3$ hold. If*

$$\int_{t_5}^{\infty} \alpha(s)(k[1 - r(h_1(s))]h_1(s))^{\lambda}\triangle s = \infty. \tag{3.23}$$

Then every solution of the equation

$$(x(t)+r(t)x(\tau(t)))^{\triangle\triangle}+H(t,x(h_1(t)))=0 \tag{3.24}$$

oscillates.

Theorem 3.4. *Assume that H_1-H_3 hold with $0\leq r(t)=r<1$. Then every solution of the equation*

$$(x(t)-r(t)x(\tau(t)))^{\triangle\triangle}+H(t,x(h_1(t)))=0 \tag{3.25}$$

oscillates, if

$$\int_{t_5}^{\infty}\alpha(s)(\tau(h_1(s))^{\lambda}\triangle s=\infty. \tag{3.26}$$

Theorem 3.5. *Assume that H_1-H_2 hold. Then every solution of* (A) *oscillates, if the inequality*

$$z^{\triangle}(t)+[\alpha(t)(k[1-r(h_1(t))]h_1(t))^{\lambda}+\frac{1}{2^{\lambda}}\beta(t)]z^{\lambda}(\tau(t))\leq 0, 0<k<1 \tag{3.27}$$

has no eventually positive solution.

Theorem 3.6. *Assume that H_1-H_2 hold. Then every solution of* (B) *oscillates, if the inequality*

$$z^{\triangle}(t)+[(kr)^{\lambda}\alpha(t)(\tau(h_1(t))^{\lambda}+r^{\lambda}\beta(t)]z^{\lambda}(\tau(t))\leq 0, 0<k<1 \tag{3.28}$$

has no eventually positive solution.

Theorems 3.5 and 3.6 reduce the question of oscillation of equation (A) and (B) to that of the absence of eventually positive solutions of the dynamic inequality (3.27) and (3.28) respectively.

The following result concerning with oscillation of second-order nonlinear delay dynamic equations on time scales. In [17], it was proved the equivalence of the oscillation of nonlinear dynamic equations

$$x^{\triangle\triangle}(t)+a(t)f(x(t-\tau))=0 \tag{3.29}$$

and

$$x^{\triangle\triangle}(t)+a(t)(f\circ x^{\sigma})=0. \tag{3.30}$$

It was proved ([17], Corollary 2.2) that equation (3.29) is oscillatory if

$$\int^{\infty}a(s)\triangle s=\infty, \tag{3.31}$$

which is the same result obtained by M. Bohner et. al. (see [5], Theorem 3.2). Also, E. Akin et. al. [4] considered the equation

$$x^{\triangle\triangle}(t)+a(t)(x^{\sigma})^{\lambda}=0, \tag{3.32}$$

and proved that, if (3.31) satisfied then equation (3.32) is oscillatory. This is equivalent to oscillation of

$$x^{\triangle\triangle}(t)+a(t)x^{\lambda}(t-\tau)=0. \tag{3.33}$$

Note that condition (3.31) can not be applied for the second order neutral equation

$$x^{\Delta\Delta}(t) + \frac{\gamma}{(t-h)^2} x^{\lambda}(t-h) = 0, \tag{3.34}$$

where $\gamma > 0$ and $0 \leq \lambda = \frac{p}{q} < 1$ with p, q are odd integers.

Theorem 3.7. *Assume that $H_1 - H_3$ hold with $r(t) = 0$ and $\beta(t) = 0$. If*

$$\int_{t_5}^{\infty} \alpha(s)(h_1(s))^{\lambda} \Delta s = \infty. \tag{3.35}$$

Then every solution of the equation

$$x^{\Delta\Delta}(t) + a(t)x^{\lambda}(h_1(t)) = 0 \tag{3.36}$$

oscillates.

4. Examples

In this section, we give some examples to illustrate our main results. In fact, Example 4.1 and Example 4.2 are not discussed before and there is no previous theorems determine the oscillatory behavior of such equations. But Example 4.3 shows that our results improve some previous results.

Example 4.1. Consider the following second order neutral delay dynamic equation

(4.1)

$$\begin{aligned}&(x(t) + e^{-\frac{1}{\lambda}(t-\tau)}x(t-\tau))^{\Delta\Delta} + \frac{1}{t}(|x(t-h_1)|^{\lambda} + \frac{1}{e^{-t}+1}\left|x^{\Delta}(t-h_2)\right|^{\lambda}) \\ &\times (\text{sgn } x(t-h_1))(\text{sgn } x^{\Delta}(t-h_2)) = 0,\end{aligned}$$

where $0 \leq \lambda = \frac{p}{q} < 1$, p, q are odd integers, τ, h_1, $h_2 > 0$, $r(t) = e^{-\frac{1}{\lambda}(t-\tau)}$, $\tau(t) = t - \tau$, $h_1(t) = t - h_1$, $h_2(t) = t - h_2$, $H(t, x(h_1(t)), x^{\Delta}(h_2(t)) = (|x(t-h_1)|^{\lambda} + \left|x^{\Delta}(t-h_2)\right|^{\lambda})$ (sgn $x(t-h_1)$) (sgn $x^{\Delta}(t-h_2)$) (i.e., $\alpha(t) = \frac{1}{t}$ and $\beta = 1 + e^{-t}$). Then we have

$$\begin{aligned}&\int_{t_5}^{\infty} \{\alpha(s)(k[1 - r(h_1(s))]h_1(s))^{\lambda} + \frac{1}{2^{\lambda}}\beta(s)\}\Delta s \\ = \;& k^{\lambda}\int_{t_5}^{\infty} \frac{1}{s}[1 - e^{-\frac{1}{\lambda}(s-\tau-h_1)}](s-h_1)^{\lambda}\Delta s + \frac{1}{2^{\lambda}}\int_{t_5}^{\infty}(1+e^{-s})\Delta s \\ \geq \;& k^{\lambda}\int_{t_5}^{\infty} \frac{(s-h_1)^{\lambda}}{s}\Delta s + \frac{1}{2^{\lambda}}\int_{t_5}^{\infty}\Delta s = \infty.\end{aligned}$$

Therefore, by Theorem (3.1), equation (4.1) is oscillatory.

Example 4.2. Consider the following second order neutral delay dynamic equation

$$(4.2)\qquad \begin{aligned} &(x(t)-\frac{1}{c}x(\gamma_1 t))^{\Delta\Delta}+\{\frac{(2+\sin t)}{t^{\alpha_1}}\left|x(\gamma_2 t)\right|^{\lambda}+\frac{(3+\cos t)}{t^{\alpha_2}}\left|x^{\Delta}(\gamma_3 t)\right|^{\lambda}\}\\ &\times(\mathrm{sgn}\ x(\gamma_2 t))\ (\mathrm{sgn}\ x^{\Delta}(\gamma_3 t))\ =0, \end{aligned}$$

where $c>1$, $0\le\lambda=\frac{p}{q}<1$, p, q are odd integers, α_1, $\alpha_2\in[\lambda,\ \lambda+1)$ and γ_1, γ_2, $\gamma_3\in(0,1)$. In equation (B) $r(t)=\frac{1}{c}$, $\tau(t)=\gamma_1 t$, $h_1(t)=\gamma_2 t$, $h_2(t)=\gamma_2 t$, $H(t,x(h_1(t))$,

$$\begin{aligned} x^{\Delta}(h_2(t)) = \{&\frac{(2+\sin t)}{t^{\alpha_1}}\left|x(\gamma_2 t)\right|^{\lambda}+\frac{(3+\cos t)}{t^{\beta}}\left|(x^{\Delta}(\gamma_3 t))\right|^{\lambda}\}\\ &\times(\mathrm{sgn}\ x(\gamma_2 t))(\mathrm{sgn}\ x^{\Delta}(\gamma_3 t)). \end{aligned}$$

(i.e., $\alpha(t)=\frac{(2+\sin t)}{t^{\alpha_1}}$ and $\beta=\frac{(3+\cos t)}{t^{\alpha_2}}$). Then, we have

$$\begin{aligned} &\int_{t_5}^{\infty}\{(kr)^{\lambda}\alpha(s)(\tau(h_1(s))^{\lambda}+r^{\lambda}\beta(s)\}\Delta s\\ =\ &(kr)^{\lambda}\int_{t_5}^{\infty}\{\frac{(2+\sin s)}{s^{\alpha_1}}(\gamma_1\gamma_2 s)^{\lambda}\}\Delta s+r^{\lambda}\int_{t_5}^{\infty}\frac{(3+\cos s)}{s^{\alpha_2}}\Delta s\\ \ge\ &(kr)^{\lambda}\int_{t_5}^{\infty}\{\frac{1}{s^{\alpha_1}}(\gamma_1\gamma_2 s)^{\lambda}\}\Delta s+r^{\lambda}\int_{t_5}^{\infty}\frac{1}{s^{\alpha 2}}\Delta s\\ =\ &(\frac{k\gamma_1\gamma_2}{c})^{\lambda}\int_{t_5}^{\infty}\frac{\Delta s}{s^{\alpha_1-\lambda}}+(\frac{1}{c})^{\lambda}\int_{t_5}^{\infty}\frac{\Delta s}{s^{\alpha_2}}=\infty\ \text{ for } \alpha_1,\alpha_2\in[\lambda,\lambda+1). \end{aligned}$$

Hence, by Theorem (3.2) every solution of equation (4.2) oscillates.

Example 4.3. Consider the following specific second order neutral delay dynamic equation

$$(4.3)\qquad (x(t)-\frac{1}{2}x(t-\tau))''+\frac{\gamma}{(t-h)^2}x^{\lambda}(t-h)=0, t\in\mathbb{T},$$

where $\mathbb{T}$ is a time scale, and

$$\tau,h>0,r(t)=\frac{1}{2},\tau(t)=t-\tau,h_1(t)=t-h,H(t,x(h_1(t)),x'(h_2(t))=x(h(t))$$

(i.e., $\alpha(t)=\frac{\gamma}{(t-h)^2}$, $\gamma>0$ and $\beta(t)=0$). Then X. Lin's result [13] fail to determine oscillatory behavior of this equation, since

$$\int^{\infty}a(t)dt<\infty$$

But, according to Theorem 3.3 when $\mathbb{T} = \mathbb{R}$ we have

$$\begin{aligned} &\int_{t_5}^{\infty} \alpha(s)(\tau(h_1(s))^{\lambda} ds \\ = &\int_{t_5}^{\infty} \frac{\gamma(s-\tau-h)}{2(s-h)^2} ds \\ = &\frac{\gamma}{2}\int_{t_5}^{\infty} \frac{1}{s-h}(1-\frac{\tau}{s-h})ds = \infty. \end{aligned}$$

Hence, every solution of equation (4.3) is oscillatory.

Example 4.4. Consider the following specific sublinear second order delay dynamic equation

$$x^{\Delta\Delta}(t) + \frac{1}{t(h_1(t))^{\frac{1}{3}}} x^{\frac{1}{3}}(h_1(t)) = 0. \tag{4.4}$$

Here $a(t) = \frac{1}{t(h_1(t))^{\frac{1}{3}}}$. Then ([4], Corollary 5.3), ([5], Theorem 3.2) and ([17], Corollary 2.2) results fail to determine oscillatory behavior of this equation, since

$$\int^{\infty} a(t)dt < \infty.$$

But

$$\int_{t_5}^{\infty} \alpha(s)(h_1(s))^{\lambda} \Delta s = \int_{t_5}^{\infty} \frac{\Delta s}{s} = \infty.$$

Hence, by Theorem 3.7, equation (4.4) is oscillatory.

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