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SASAKIAN MANIFOLDS WITH QUASI-CONFORMAL CURVATURE TENSOR

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Abstract. The object of the paper is to study a Sasakian manifold with quasi-conformal curvature tensor.

1. Introduction

The notion of the quasi-conformal curvature tensor was given by Yano and Sawaki [11]. According to them a quasi-conformal curvature tensor $\tilde{C}$ is defined by

$$\begin{aligned}\tilde{C}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] - \frac{r}{n}[\frac{a}{n-1} + 2b][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1.1}$$

where a and b are constants and R, S, Q and r are the Riemannian curvature tensor of type (1,3), the Ricci tensor of type (0,2), the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and the scalar curvature of the manifold respectively. If $a = 1$ and $b = -\frac{1}{n-2}$, then (1.1) takes the form

$$\begin{aligned}\tilde{C}(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \\ &= C(X,Y)Z,\end{aligned}$$

where C is the conformal curvature tensor [4]. Thus the conformal curvature tensor C is a particular case of the tensor $\tilde{C}$. For this reason $\tilde{C}$ is called the quasi-conformal curvature tensor. A manifold $(M^n, g)(n > 3)$ shall be called quasi-conformally flat if the quasi-conformal curvature tensor $\tilde{C} = 0$. It is known [1] that the quasi-conformally flat manifold is either conformally flat if $a \neq 0$ or, Einstein if $a = 0$ and $b \neq 0$. Since, they give no restrictions for manifolds if $a = 0$ and $b = 0$, it is essential for us to consider the case of $a \neq 0$ or $b \neq 0$.

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An almost contact metric manifold is said to be an η-Einstein manifold if the Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{1.2}$$

where a, b are certain scalars. It is known [10] that in a Sasakian manifold a, b are constants. A Riemannian or a semi-Riemannian manifold is said to be semi-symmetric [8] if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y) \cdot \tilde{C} = 0$, where $\tilde{C}$ is the quasi-conformal curvature tensor, then the manifold is said to be quasi-conformally semi-symmetric manifold.

It is known [5] that a conformally flat Sasakian manifold is of constant curvature and a Weyl semi-symmetric Sasakian manifold is locally isometric with the unit sphere $S^n(1)$ [3]. In the present paper we have studied quasi-conformally flat and quasi-conformally semi-symmetric Sasakian manifolds. At first we prove that a Sasakian manifold is quasi-conformally flat if and only if it is locally isometric with the unit sphere $S^n(1)$. Also it is proved that a compact orientable quasi-conformally flat Sasakian manifold can not admit a non-isometric conformal transformation. Finally, we have shown that a Sasakian manifold is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.

2. Preliminaries

Let S and r denote respectively the Ricci tensor of type (0,2) and the scalar curvature in a Sasakian manifold (M^n, g). It is known that in a Sasakian manifold M^n, the following relations hold [6], [2], [7]:

$$\phi(\xi) = 0 \tag{2.1}$$

$$\eta(\xi) = 1 \tag{2.2}$$

$$\phi^2 X = -X + \eta(X)\xi \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.4}$$

$$g(\xi, X) = \eta(X) \tag{2.5}$$

$$\nabla_X \xi = -\phi X \tag{2.6}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{2.7}$$

$$g(R(\xi,X)Y,\xi) = g(X,Y) - \eta(X)\eta(Y) \tag{2.8}$$

$$R(\xi,X)\xi = -X + \eta(X)\xi \tag{2.9}$$

$$g(R(X,Y)\xi, Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X) \tag{2.10}$$

$$\text{and} \quad (\nabla_X \phi)(Y) = R(\xi,X)Y \tag{2.11}$$

for any vector fields X, Y.

The above results will be used in the next section.

3. η-Einstein Sasakian manifold

Let l^2 be the square of the length of the Ricci tensor, then

$$l^2 = \sum_{i=1}^{n} S(Qe_i, e_i), \tag{3.1}$$

where Q is the symmetric endomorphism of the tangent space at a point corresponding to the Ricci tensor S and $\{e_i\}$, $i = 1, 2, \ldots, n$, is an orthonormal basis of the tangent space at any point. Now putting $X = Y = \{e_i\}$ in (1.2), and taking summation over $i, 1 \leq i \leq n$, we get

$$r = na + b, \tag{3.2}$$

where r is the scalar curvature. Again from (1.2) we obtain

$$S(\xi, \xi) = a + b. \tag{3.3}$$

Now we get from (1.2) with the help of (3.1), (3.2) and (3.3)

$$l^2 = (n-1)a^2 + (a+b)^2. \tag{3.4}$$

Since the scalars a and b are constants of an η-Einstein Sasakian manifold, it follows from (3.2) that r is constant and so also is the length of the Ricci tensor. Next we suppose that the manifold under consideration admits a non-isometric conformal motion generated by a vector field X. Since l^2 is constant, it follows that

$$L_X l^2 = 0, \tag{3.5}$$

where L_X denotes Lie-differentiation with respect to X. Now it is known [9] that if a compact Reimannian manifold $M^n (n > 2)$ with constant scalar curvature admits an infinitesimal nonisometric conformal transformation X such that $L_X l^2 = 0$, then M is isometric to a sphere. But a sphere is an Einstein manifold. Hence we can state the following:

Theorem 3.1. *A compact orientable η-Einstein Sasakian manifold does not admit a nonisometric conformal transformation.*

4. Quasi-conformally flat Sasakian manifold

If the manifold under consideration is quasi-conformally flat, then we have from (1.1)

(4.1)

$$\begin{aligned} 'R(X,Y,Z,W) = {} & \frac{b}{a}[S(X,Z)g(Y,W) - S(Y,Z)g(X,W) \\ & + S(Y,W)g(X,Z) - S(X,W)g(Y,Z)] \\ & + \frac{r}{na}[\frac{a}{n-1} + 2b][g(Y,Z)g(X,W) - g(X,Z)g(Y,W)], \end{aligned}$$

where a and b are constants and $'R(X,Y,Z,W) = g(R(X,Y)Z,W)$.

Now putting $Z = \xi$ in (4.1) and using (2.2), (2.5), (2.7) and (2.10) we get

$$\begin{aligned} &g(X,W)\eta(Y) - g(Y,W)\eta(X) \\ &= \frac{b}{a}[(n-1)g(Y,W)\eta(X) - (n-1)g(X,W)\eta(Y) \\ &+ S(Y,W)\eta(X) - S(X,W)\eta(Y)] + \frac{r}{na}[\frac{a}{n-1} \\ &+ 2b][g(X,W)\eta(Y) - g(Y,W)\eta(X)]. \end{aligned} \tag{4.2}$$

Again putting $X = \xi$ in (4.2) and using (2.2), (2.5) and (2.7) it follows that

$$S(Y,W) = Ag(Y,W) + B\eta(Y)\eta(W), \tag{4.3}$$

where

$$A = [-(n-1) + \frac{r}{nb}(\frac{a}{n-1} + 2b) - \frac{a}{b}] \tag{4.4}$$

and

$$B = [2(n-1) - \frac{r}{nb}(\frac{a}{n-1} + 2b) + \frac{a}{b}]. \tag{4.5}$$

Here $A + B = (n-1)$. This leads to the following theorem:

Theorem 4.1. *A quasi-conformally flat Sasakian manifold is an η-Einstein manifold.*

Now from Theorem 3.1 we can state the following:

Corollary 4.1. *A compact orientable quasi-conformally flat Sasakian manifold can not admit a nonisometric conformal transformation.*

Putting $Y = W = \{e_i\}$ in (4.3) and taking summation over i, $1 \leq i \leq n$, we get

$$r = nA + B. \tag{4.6}$$

Now with the help of (4.4) and (4.5) the equation (4.6) gives

$$[(n-2) + \frac{a}{b}][\frac{r}{n} + (1-n)] = 0. \tag{4.7}$$

Hence either

$$b = \frac{a}{2-n}, \tag{4.8}$$

or,

$$r = n(n-1). \tag{4.9}$$

If $b = \frac{a}{(2-n)}$ then putting it into (1.1) we get

$$\tilde{C}(X,Y)Z = aC(X,Y)Z, \tag{4.10}$$

where $C(X,Y)Z$ denotes the Weyl conformal curvature tensor. So the quasi-conformally flatness and conformally flatness are equivalent in this case. A conformally flat Sasakian manifold $(M^n, g)(n \geq 5)$ is of constant curvature. But a manifold of constant curvature is conformally flat. Hence a Sasakian manifold is conformally flat if and only if it is locally isometric with a unit sphere $S^n(1)$. So in this case M^n is locally isometric to the unit sphere.

If $r = n(n-1)$, then from (4.3), (4.4) and (4.5) we obtain

$$S(Y,W) = (n-1)g(Y,W). \tag{4.11}$$

This implies that M^n is an Einstein manifold. So putting (4.8), (4.9) and (4.11) into (4.1) we obtain

$$R(X,Y,Z,W) = g(X,W)g(Y,Z) - g(X,Z)g(Y,W).$$

Then M^n is of constant curvature $+1$. Hence it is locally isometric with the unit sphere $S^n(1)$. If M^n is locally isometric to the unit sphere $S^n(1)$ then it is easy to see that M^n is quasi-conformally flat. This leads to the following theorem:

Theorem 4.2. *Let $(M^n,g)(n \geq 5)$ be a Sasakian manifold. Then M^n is quasi-conformally flat if and only if M^n is locally isometric to the unit sphere $S^n(1)$.*

5. Sasakian manifolds satisfying $R(X,Y)\cdot\tilde{C} = 0$

In this section we consider a Sasakian manifold M^n satisfying the condition

$$R(X,Y)\cdot\tilde{C} = 0. \tag{5.1}$$

Then we obtain from (1.1) by using (2.5), (2.7) and (2.10)

$$\begin{aligned}\eta(\tilde{C}(X,Y)Z) = {} & [a + b(n-1) - \frac{r}{n}(\frac{a}{n-1} + 2b)][g(Y,Z)\eta(X) \\ & - g(X,Z)\eta(Y)] + b[S(Y,Z)\eta(X) - S(X,Z)\eta(Y)].\end{aligned} \tag{5.2}$$

For $Z = \xi$, we get from (5.2)

$$\eta(\tilde{C}(X,Y)\xi) = 0. \tag{5.3}$$

Again putting $X = \xi$ in (5.2) we get

$$\begin{aligned}\eta(\tilde{C}(\xi,Y)Z) = {} & [a + b(n-1) - \frac{r}{n}(\frac{a}{n-1} + 2b)][g(Y,Z) \\ & - \eta(Y)\eta(Z)] + b[S(Y,Z) - (n-1)\eta(Y)\eta(Z)].\end{aligned} \tag{5.4}$$

In virtue of (5.1) we get

$$\begin{aligned}& R(X,Y)\tilde{C}(U,V)W - \tilde{C}(R(X,Y)U,V)W \\ & - \tilde{C}(U,R(X,Y)V)W - \tilde{C}(U,V)R(X,Y)W = 0,\end{aligned} \tag{5.5}$$

which implies that

$$\begin{aligned}& {}'\tilde{C}(U,V,W,Y) - \eta(Y)\eta(\tilde{C}(U,V)W) + \eta(U)\eta(\tilde{C}(Y,V)W) \\ & + \eta(V)\eta(\tilde{C}(U,Y)W) + \eta(W)\eta(\tilde{C}(U,V)Y) - g(Y,U)\eta(\tilde{C}(\xi,V)W) \\ & - g(Y,V)\eta(\tilde{C}(U,\xi)W) - g(Y,W)\eta(\tilde{C}(U,V)\xi) = 0,\end{aligned} \tag{5.6}$$

where ${}'\tilde{C}(U,V,W,Y) = g(\tilde{C}(U,V)W,Y)$.

Putting $U = Y$ in (5.6) and with the help of (5.2) and (5.3) we get

$$\begin{aligned}&{}'\tilde{C}(U,V,W,U) + \eta(W)\eta(\tilde{C}(U,V)U)\\&- g(U,U)\eta(\tilde{C}(\xi,V)W) - g(U,V)\eta(\tilde{C}(U,\xi)W) = 0.\end{aligned} \tag{5.7}$$

Now putting $U = \{e_i\}$, where $\{e_i\}, i = 1,2,\dots,n$, be an orthonormal basis of the tangent space at each point of the manifold, in (5.7) and taking the summation over i, $1 \le i \le n$, and using (5.2), (5.4) we get

$$S(V,W) = \lambda g(V,W) + \mu\eta(V)\eta(W), \tag{5.8}$$

where

$$\lambda = \frac{-br + (n-1)^2 b + (n-1)a}{a-b} \tag{5.9}$$

and

$$\mu = \frac{b[r - n(n-1)]}{a-b}. \tag{5.10}$$

Hence (5.8) leads to the following theorem:

Theorem 5.1. *A quasi-conformally semi-symmetric Sasakian manifold is an η-Einstein manifold.*

Now contracting (5.8) we get

$$r = n\lambda + \mu. \tag{5.11}$$

By (5.9) and (5.10) the equation (5.11) gives

$$[a + (n-2)b][r - n(n-1)] = 0.$$

Therefore, either

$$b = \frac{a}{2-n} \quad \text{or} \quad r = n(n-1). \tag{5.12}$$

From (5.9) and (5.12) we obtain

$$\lambda = (n-1). \tag{5.13}$$

By (5.10) and (5.12) we get

$$\mu = 0. \tag{5.14}$$

So, from (5.8), (5.13) and (5.14) we have

$$S(V,W) = (n-1)g(V,W). \tag{5.15}$$

Therefore, M^n is an Einstein manifold. Now with the help of (5.12) and (5.15) the equations (5.2) and (5.4) imply that

$$\eta(\tilde{C}(U,V)W) = 0 \tag{5.16}$$

and

$$\eta(\tilde{C}(\xi,U)V) = 0, \tag{5.17}$$

respectively. So using (5.16), (5.17) and (5.3) into the equation (5.6) we get

$$'\tilde{C}(U,V,W,Y) = 0. \tag{5.18}$$

Therefore, M^n is quasi-conformally flat. Then it is trivially quasi-conformally semi-symmetric. So we have the following result:

Theorem 5.2. *Let $(M^n, g)(n > 3)$ be a Sasakian manifold. Then M^n is quasi-conformally flat if and only if it is quasi-conformally semi-symmetric.*

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