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SOME ALGEBRA FOR PEARSON TYPE VII RANDOM VARIABLES

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Abstract. The distributions of products and ratios of random variables are of interest in many areas of the sciences. In this paper, the exact distributions of the product $|XY|$ and the ratio $|X/Y|$ are derived when X and Y are independent Pearson type VII random variables.

1. Introduction

For given random variables X and Y, the distributions of the product XY and the ratio X/Y are of interest in many areas of the sciences.

In traditional portfolio selection models certain cases involve the product of random variables. The best examples of this are in the case of investment in a number of different overseas markets. In portfolio diversification models (see, for example, Grubel [6]) not only are prices of shares in local markets uncertain but also the exchange rates are uncertain so that the value of the portfolio in domestic currency is related to a product of random variables. Similarly in models of diversified production by multinationals (see, for example, Rugman [20]) there is local production uncertainty and exchange rate uncertainty so that profits in home currency are again related to a product of random variables. An entirely different example is drawn from the econometric literature. In making a forecast from an estimated equation Feldstein [4] pointed out that both the parameter and the value of the exogenous variable in the forecast period could be considered as random variables. Hence the forecast was proportional to a product of random variables.

An important example of ratios of random variables is the stress–strength model in the context of reliability. It describes the life of a component which has a random strength Y and is subjected to random stress X. The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever $Y > X$. Thus, $\Pr(X < Y)$ is a measure of component reliability. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static

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fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

The distributions of XY and $X/(X+Y)$ have been studied by several authors especially when X and Y are independent random variables and come from the same family. With respect to products of random variables, see Sakamoto [21] for uniform family, Harter [7] and Wallgren [27] for Student's t family, Springer and Thompson [23] for normal family, Stuart [25] and Podolski [15] for gamma family, Steece [24], Bhargava and Khatri [3] and Tang and Gupta [26] for beta family, AbuSalih [1] for power function family, and Malik and Trudel [11] for exponential family (see also Rathie and Rohrer [19] for a comprehensive review of known results). With respect to ratios of random variables, see Marsaglia [12] and Korhonen and Narula [9] for normal family, Press [16] for Student's t family, Basu and Lochner [2] for Weibull family, Shcolnick [22] for stable family, Hawkins and Han [8] for non-central chi-squared family, Provost [17] for gamma family, and Pham-Gia [14] for beta family.

In this paper, we study the exact distributions of $|XY|$ and $|X/Y|$ when X and Y are independent random variables having the Pearson type VII distribution with the pdfs

$$f(x) = \frac{\Gamma(M-1/2)}{\sqrt{m\pi}\Gamma(M-1)} \left(1 + \frac{x^2}{m}\right)^{1/2-M} \tag{1}$$

and

$$f(y) = \frac{\Gamma(N-1/2)}{\sqrt{n\pi}\Gamma(N-1)} \left(1 + \frac{y^2}{n}\right)^{1/2-N}, \tag{2}$$

respectively, for $-\infty < x < \infty$ and $-\infty < y < \infty$, where $m > 0$, $n > 0$, $M > 1$ and $N > 1$. These distributions are closely related to the well-known Student's t distributions: if $M = 1 + a/2$, $N = 1 + b/2$, and

$$(U, V) = \left(\sqrt{\frac{a}{m}}X, \sqrt{\frac{b}{n}}Y\right), \tag{3}$$

then U and V are Student's t random variables with degrees of freedom a and b, respectively. Note that the pdf of a Student's t random variable with degrees of freedom ν is given by

$$f(x) = \frac{1}{\sqrt{\nu}B(\nu/2, 1/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(1+\nu)/2} \tag{4}$$

for $-\infty < x < \infty$. Nadarajah and Kotz [13] have shown that the cdf corresponding to (4) can be expressed as

(5)

$$F(x) = \begin{cases} \frac{1}{2} + \frac{1}{\pi} \arctan\left(\frac{x}{\sqrt{\nu}}\right) + \frac{1}{2\pi} \sum_{l=1}^{(\nu-1)/2} B\left(l, \frac{1}{2}\right) \frac{\nu^{l-1/2} x}{\left(\nu + x^2\right)^l}, & \text{if } \nu \text{ is odd}, \\ \frac{1}{2} + \frac{1}{2\pi} \sum_{l=1}^{\nu/2} B\left(l - \frac{1}{2}, \frac{1}{2}\right) \frac{\nu^{l-1} x}{\left(\nu + x^2\right)^{l-1/2}}, & \text{if } \nu \text{ is even}. \end{cases}$$

This result will be crucial for the calculations of this note. The calculations involve the complete elliptical integral of the first kind defined by

$$\mathrm{K}(a) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-a^2x^2}} dx,$$

the complete elliptical integral of the second kind defined by

$$\mathrm{E}(a) = \int_0^1 \frac{\sqrt{1-a^2x^2}}{\sqrt{1-x^2}} dx,$$

and the Gauss hypergeometric function defined by

$$G(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

where $(e)_k = e(e+1) \cdots (e+k-1)$ denotes the ascending factorial. We also need the following important lemma.

Lemma 1 (Equation (3.197.9), Gradshteyn and Ryzhik [5]). *For $\mu > \lambda > 0$,*

$$\int_0^{\infty} x^{\lambda-1} (1+x)^{\nu-\lambda} (x+\beta)^{-\nu} dx = B(\mu-\lambda, \lambda) G(\nu, \mu-\lambda; \mu; 1-\beta).$$

Further properties of the above special functions can be found in Prudnikov *et al.* [18] and Gradshteyn and Ryzhik [5].

2. Product

Theorem 1 derives an explicit expression for the cdf of $\mid XY \mid$ in terms of the hypergeometric function when the degrees of freedom $2(M-1)$ is an odd integer.

Theorem 1. *Suppose X and Y are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. If $a = 2(M-1)$ is an odd integer and $b = 2(N-1)$, then the cdf of $Z = |XY|$ can be expressed as*

(6)

$$F(z) = I(b) + \frac{r}{\pi\sqrt{ab} B(b/2, 1/2)} \sum_{k=1}^{(a-1)/2} B\left(\frac{1+b}{2}, k\right) B\left(k, \frac{1}{2}\right) G\left(k, \frac{1+b}{2}; k + \frac{1+b}{2}; 1 - \frac{r^2}{ab}\right),$$

where $r = \sqrt{ab/(mn)}z$ *and*

$$(7) \quad I(b) = \frac{4}{\pi\sqrt{b}B\left(b/2, 1/2\right)} \int_0^\infty \arctan\left(\frac{r}{\sqrt{a}y}\right)\left(1+\frac{y^2}{b}\right)^{-(1+b)/2} dy.$$

Proof. Using the relationship (3), one can write the cdf as $\Pr(|XY| \leq z) = \Pr(|UV| \leq r)$, which can be expressed as

(8)

$$\begin{aligned} F(r) &= \frac{1}{\sqrt{b}B\left(b/2, 1/2\right)} \int_{-\infty}^\infty \left\{F\left(\frac{r}{\mid y\mid}\right) - F\left(-\frac{r}{\mid y\mid}\right)\right\}\left(1+\frac{y^2}{b}\right)^{-(1+b)/2} dy \\ &= \frac{2}{\sqrt{b}B\left(b/2, 1/2\right)} \int_0^\infty \left\{F\left(\frac{r}{y}\right) - F\left(-\frac{r}{y}\right)\right\}\left(1+\frac{y^2}{b}\right)^{-(1+b)/2} dy, \end{aligned}$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's t random variable with degrees of freedom a. Substituting the form for F given by (5) for odd degrees of freedom, (8) can be reduced to

$$(9) \qquad F(r) = I(b) + \frac{2r}{\pi\sqrt{ab}B\left(b/2, 1/2\right)} \sum_{k=1}^{(a-1)/2} B\left(k, \frac{1}{2}\right) J(k),$$

where $J(k)$ denotes the integral

$$(10) \qquad J(k) = \int_0^\infty \frac{y^{2k-1}}{\left(y^2 + r^2/a\right)^k \left(1 + y^2/b\right)^{(1+b)/2}} dy.$$

Substituting $w = r^2/b$, (10) can be reduced to

$$(11) \qquad \begin{aligned} J(k) &= \frac{1}{2}\int_0^\infty \frac{w^{k-1}}{\left\{w + r^2/(ab)\right\}^k (1+w)^{(1+b)/2}} dw \\ &= \frac{1}{2} B\left(\frac{1+b}{2}, k\right) G\left(k, \frac{1+b}{2}; k + \frac{1+b}{2}; 1 - \frac{r^2}{ab}\right), \end{aligned}$$

where the last step follows by direct application of Lemma 1. The result of the theorem follows by substituting (11) into (9). □

Theorem 2 is the analogue of Theorem 1 for the case when the degrees of freedom $2(M-1)$ is an even integer.

Theorem 2. *Suppose* X *and* Y *are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. If* $a = 2(M-1)$ *is an even integer and* $b = 2(N-1)$, *then the cdf of* $Z = |XY|$ *can be expressed as*

(12)

$$F(z) = \frac{r}{\pi\sqrt{ab}B\left(b/2, 1/2\right)} \sum_{k=1}^{a/2} B\left(k - \frac{1}{2}, \frac{1}{2}\right) B\left(\frac{b+1}{2}, k - \frac{1}{2}\right) G\left(k - \frac{1}{2}, \frac{1+b}{2}; k + \frac{b}{2}; 1 - \frac{r^2}{ab}\right),$$

where $r = \sqrt{ab/(mn)}z$.

Proof. Substituting the form for F given by (5) for even degrees of freedom, (8) can be reduced to

$$F(z) = \frac{2r}{\pi\sqrt{ab}B\left(b/2,1/2\right)} \sum_{k=1}^{a/2} B\left(k-\frac{1}{2},\frac{1}{2}\right) J(k), \tag{13}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y^{2k-2}}{\left(y^2+r^2/a\right)^{k-1/2}\left(1+y^2/b\right)^{(1+b)/2}} dy. \tag{14}$$

Substituting $w = r^2/b$, (14) can be reduced to

$$\begin{aligned} J(k) &= \frac{1}{2}\int_0^\infty \frac{w^{k-3/2}}{\left\{w+r^2/(ab)\right\}^{k-1/2}(1+w)^{(1+b)/2}} dw \\ &= \frac{1}{2}B\left(\frac{1+b}{2},k-\frac{1}{2}\right) G\left(k-\frac{1}{2},\frac{1+b}{2};k+\frac{b}{2};1-\frac{r^2}{ab}\right), \end{aligned} \tag{15}$$

where the last step again follows by direct application of Lemma 1. The result of the theorem follows by substituting (15) into (13). □

Figure 1 illustrates possible shapes of the pdf of $\mid XY \mid$ for a range of values of M and N. Note that the shapes are unimodal and that the densities appear to shrink with increasing values of M and decreasing values of N.

3. Particular cases

Here, we derive particular forms of (6) and (12) for $2(M-1) = 2,3,4,5$ and $2(N-1) = 1,2,3,4,5$. In our calculations, we have used various special properties of the Gauss hypergeometric function (see, for example, Section 7.3 in volume 3 of Prudnikov *et al.* [18]). When $2(M-1)$ is odd the expressions for the cdf involve the integral $I(\cdot)$ in (7) which should be computed numerically. When both $2(M-1)$ and $2(N-1)$ are even the expressions involve the complete elliptical integrals of the first and second kind. In all other cases, the resulting expressions for the cdf are elementary.

Corollary 1. *Suppose X and Y are independent Pearson type VII random variables with pdfs (1) and (2), respectively. For $2(M-1)=2$ and $2(N-1) = 1,2,3,4,5$, the cdf of $Z = \mid XY \mid$ can be expressed as*

$$F(z) = \frac{2r\operatorname{arctanh}\left(1/2\sqrt{4-2r^2}\right)}{\pi\sqrt{2-r^2}},$$

$$F(z) = \frac{-4\mathrm{K}\left(\frac{\sqrt{-4+r^2}}{r}\right) + r^2\mathrm{E}\left(\frac{\sqrt{-4+r^2}}{r}\right)}{-4+r^2},$$

$$F(z) = \frac{2r\left\{\arctan\left(1/6\sqrt{-36+6r^2}\right)r^2 + \sqrt{-36+6r^2} - 12\arctan\left(1/6\sqrt{-36+6r^2}\right)\right\}}{\pi\left(-6+r^2\right)^{3/2}},$$

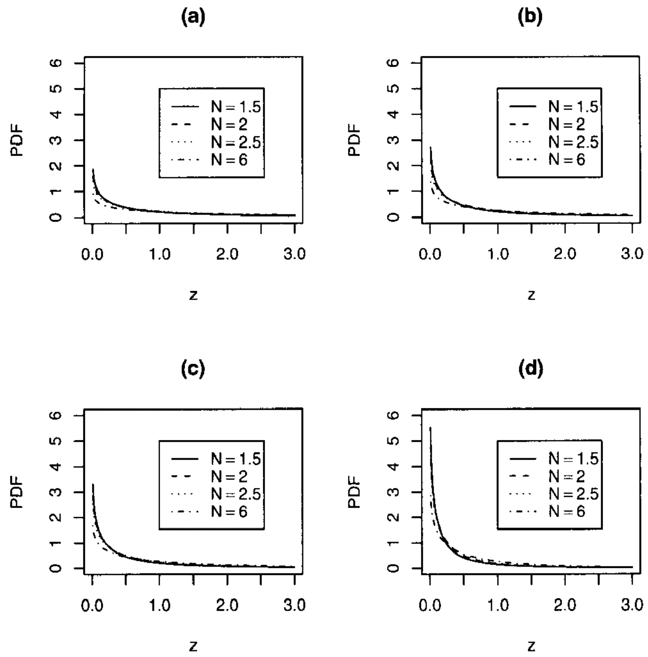

FIGURE 1. Plots of the pdf given by (6) and (12) for $m = 1$, $n = 1$ and (a) $M = 1.5$ and $N = 1.5, 2, 2.5, 6$; (b) $M = 2$ and $N = 1.5, 2, 2.5, 6$; (c) $M = 2.5$ and $N = 1.5, 2, 2.5, 6$; and, (d) $M = 6$ and $N = 1.5, 2, 2.5, 6$.

$$F(z) = \frac{96\mathrm{K}\left(\frac{\sqrt{-16+2r^2}\sqrt{2}}{2r}\right) - 4\mathrm{K}\left(\frac{\sqrt{-16+2r^2}\sqrt{2}}{2r}\right) r^2 - 16r^2\mathrm{E}\left(\frac{\sqrt{-16+2r^2}\sqrt{2}}{2r}\right) + r^4\mathrm{E}\left(\frac{\sqrt{-16+2r^2}\sqrt{2}}{2r}\right)}{\left(8 - r^2\right)^2},$$

$$\begin{aligned} F(z) = & \frac{2r\left\{3\arctan\left(1/10\sqrt{-100+10r^2}\right)r^4 + 3\sqrt{-100+10r^2}r^2\right\}}{3\pi\left(-10+r^2\right)^{5/2}} \\ & + \frac{2r\left\{-80\arctan\left(1/10\sqrt{-100+10r^2}\right)r^2 - 60\sqrt{-100+10r^2}\right\}}{3\pi\left(-10+r^2\right)^{5/2}} \\ & + \frac{2r\left\{800\arctan\left(1/10\sqrt{-100+10r^2}\right)\right\}}{3\pi\left(-10+r^2\right)^{5/2}}, \end{aligned}$$

where $r = \sqrt{ab/(mn)}z$.

Corollary 2. *Suppose X and Y are independent Pearson type VII random variables with pdfs (1) and (2), respectively. For $2(M-1)=3$ and $2(N-1)=1,2,3,4,5$, the cdf of $Z=\mid XY \mid$ can be expressed as*

$$F(z)=I(1)+\frac{2\sqrt{3}r\left(\log 3-2\log r\right)}{\pi^2\left(3-r^2\right)},$$

$$F(z)=I(2)+\frac{2\sqrt{6}r\left\{\sqrt{36-6r^2}\operatorname{arctanh}\left(1/6\sqrt{36-6r^2}\right)-6+r^2\right\}}{\pi\left(6-r^2\right)^2},$$

$$F(z)=I(3)+\frac{12r\left(18\log 3-18\log r-9+r^2\right)}{\pi^2\left(9-r^2\right)^2},$$

$$F(z)=I(4)+\frac{2\sqrt{3}r\left\{-72\sqrt{36-3r^2}\operatorname{arctanh}\left(1/6\sqrt{36-3r^2}\right)+576-60r^2+r^4\right\}}{\pi\left(-12+r^2\right)^3},$$

$$F(z)=I(5)+\frac{8\sqrt{15}r\left(-450\log 3-450\log 5+900\log r+675-60r^2+r^4\right)}{3\pi^2\left(-15+r^2\right)^3},$$

where $r=\sqrt{ab/(mn)}z$.

Corollary 3. *Suppose X and Y are independent Pearson type VII random variables with pdfs (1) and (2), respectively. For $2(M-1)=4$ and $2(N-1)=1,2,3,4,5$, the cdf of $Z=\mid XY \mid$ can be expressed as*

$$F(z)=\frac{2r\left\{6\sqrt{4-r^2}\operatorname{arctanh}\left(1/2\sqrt{4-r^2}\right)-r^2\sqrt{4-r^2}\operatorname{arctanh}\left(1/2\sqrt{4-r^2}\right)-4+r^2\right\}}{\pi\left(4-r^2\right)^2},$$

$$F(z)=\frac{96\mathrm{K}\left(\frac{\sqrt{-16+2r^2\sqrt{2}}}{2r}\right)-4\mathrm{K}\left(\frac{\sqrt{-16+2r^2\sqrt{2}}}{2r}\right)r^2-16r^2\mathrm{E}\left(\frac{\sqrt{-16+2r^2\sqrt{2}}}{2r}\right)+r^4\mathrm{E}\left(\frac{\sqrt{-16+2r^2\sqrt{2}}}{2r}\right)}{\left(8-r^2\right)^2},$$

$$F(z)=\frac{2r\left\{\arctan\left(1/6\sqrt{-36+3r^2}\right)r^4+2\sqrt{-36+3r^2}r^2-30\arctan\left(1/6\sqrt{-36+3r^2}\right)\right\}}{\pi\left(-12+r^2\right)^{5/2}}$$
$$+\frac{2r\left\{r^2-60\sqrt{-36+3r^2}+432\arctan\left(1/6\sqrt{-36+3r^2}\right)\right\}}{\pi\left(-12+r^2\right)^{5/2}},$$

$$F(z)=\frac{-9216\mathrm{K}\left(\frac{\sqrt{-16+r^2}}{r}\right)+192\mathrm{K}\left(\frac{\sqrt{-16+r^2}}{r}\right)r^2-8\mathrm{K}\left(\frac{\sqrt{-16+r^2}}{r}\right)r^4}{\left(-16+r^2\right)^3}$$
$$+\frac{960r^2\mathrm{E}\left(\frac{\sqrt{-16+r^2}}{r}\right)-44r^4\mathrm{E}\left(\frac{\sqrt{-16+r^2}}{r}\right)+r^6\mathrm{E}\left(\frac{\sqrt{-16+r^2}}{r}\right)}{\left(-16+r^2\right)^3}$$

$$F(z) = \frac{2r\left\{3\arctan\left(1/10\sqrt{-100+5r^2}\right)r^6 + 6\sqrt{-100+5r^2}r^4\right\}}{3\pi\left(-20+r^2\right)^{7/2}}$$
$$+\frac{2r\left\{-210\arctan\left(1/10\sqrt{-100+5r^2}\right)r^4 - 340\sqrt{-100+5r^2}r^2\right\}}{3\pi\left(-20+r^2\right)^{7/2}}$$
$$+\frac{2r\left\{4800\arctan\left(1/10\sqrt{-100+5r^2}\right)r^2 - 96000\arctan\left(1/10\sqrt{-100+5r^2}\right)\right\}}{3\pi\left(-20+r^2\right)^{7/2}}$$
$$+\frac{2r\left\{10400\sqrt{-100+5r^2}\right\}}{3\pi\left(-20+r^2\right)^{7/2}},$$

where $r = \sqrt{ab/(mn)}z$.

Corollary 4. *Suppose* X *and* Y *are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. For* $2(M-1) = 5$ *and* $2(N-1) = 1, 2, 3, 4, 5$, *the cdf of* $Z = \mid XY \mid$ *can be expressed as*

$$F(z) = I(1) + \frac{2\sqrt{5}r\left(25\log 5 - 3\log 5r^2 - 50\log r + 6\log r r^2 - 10 + 2r^2\right)}{3\pi^2\left(5-r^2\right)^2},$$

$$F(z) = I(2) + \frac{2r\left\{3\sqrt{-100+10r^2}r^2 + 10\arctan\left(1/10\sqrt{-100+10r^2}\right)r^2\right\}}{3\pi\left(-10+r^2\right)^{5/2}}$$
$$+\frac{2r\left\{3\sqrt{-100+10r^2}\sqrt{100-10r^2}\operatorname{arctanh}\left(1/10\sqrt{100-10r^2}\right)\right\}}{3\pi\left(-10+r^2\right)^{5/2}}$$
$$+\frac{2r\left\{-60\sqrt{-100+10r^2} + 200\arctan\left(1/10\sqrt{-100+10r^2}\right)\right\}}{3\pi\left(-10+r^2\right)^{5/2}},$$

$$F(z) = I(3) + \frac{4\sqrt{15}r\left(-375\log 3 + 5\log 3r^2 - 375\log 5 + 5\log 5r^2\right)}{\pi^2\left(-15+r^2\right)^3}$$
$$+\frac{4\sqrt{15}r\left(750\log r - 10\log r r^2 + 525 - 50r^2 + r^4\right)}{\pi^2\left(-15+r^2\right)^3},$$

$$F(z) = I(4) + \frac{2r\left\{3\sqrt{-100+5r^2}r^4 - 220\sqrt{-100+5r^2}r^2 - 1800\arctan\left(1/10\sqrt{-100+5r^2}\right)r^2\right\}}{3\pi\left(-20+r^2\right)^{7/2}}$$
$$+\frac{2r\left\{-360\sqrt{-100+5r^2}\sqrt{100-5r^2}\operatorname{arctanh}\left(1/10\sqrt{100-5r^2}\right)\right\}}{3\pi\left(-20+r^2\right)^{7/2}}$$

$$+\frac{2r\left\{-24000\arctan\left(1/10\sqrt{-100+5r^2}\right)+9200\sqrt{-100+5r^2}\right\}}{3\pi\left(-20+r^2\right)^{7/2}},$$

$$F(z)=I(5)+\frac{40r\left\{312500\log 5+2500\log 5r^2-312500\log r\right\}}{9\pi^2\left(25-r^2\right)^4}$$

$$+\frac{40r\left\{-2500\log r r^2-296875+18125r^2-325r^4+3r^6\right\}}{9\pi^2\left(25-r^2\right)^4},$$

where $r=\sqrt{ab/(mn)}z$.

4. Ratio

Theorem 3 derives an explicit expression for the cdf of $|X/Y|$ in terms of the hypergeometric function when the degrees of freedom $2(M-1)$ is an odd integer.

Theorem 3. *Suppose* X *and* Y *are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. If* $a=2(M-1)$ *is an odd integer and* $b=2(N-1)$, *then the cdf of* $Z=|X/Y|$ *can be expressed as*

(16)

$$F(z)=I(b)+\frac{2\sqrt{b}r}{\pi\sqrt{a}B(b/2,1/2)}\sum_{k=1}^{(a-1)/2}\frac{a^kB(k,1/2)}{r^{2k}b^k(2k+b-1)}G\left(k,k+\frac{b-1}{2};k+\frac{1+b}{2};1-\frac{a}{br^2}\right),$$

where $r=\sqrt{na/(mb)}z$ *and*

$$I(b)=\frac{4}{\pi\sqrt{b}B(b/2,1/2)}\int_0^\infty\arctan\left(\frac{ry}{\sqrt{a}}\right)\left(1+\frac{y^2}{b}\right)^{-(1+b)/2}dy. \tag{17}$$

Proof. Using the relationship (3), one can write the cdf as $\Pr(|X/Y|\leq z)=\Pr(|U/V|\leq r)$, which can be expressed as

(18)

$$F(r)=\frac{1}{\sqrt{b}B(b/2,1/2)}\int_{-\infty}^{\infty}\{F(r\mid y\mid)-F(-r\mid y\mid)\}\left(1+\frac{y^2}{b}\right)^{-(1+b)/2}dy$$

$$=\frac{2}{\sqrt{b}B(b/2,1/2)}\int_0^\infty\{F(ry)-F(-ry)\}\left(1+\frac{y^2}{b}\right)^{-(1+b)/2}dy,$$

where $F(\cdot)$ inside the integrals denotes the cdf of a Student's t random variable with degrees of freedom a. Substituting the form for F given by (5) for odd degrees of freedom, (18) can be reduced to

$$F(r)=I(b)+\frac{2r}{\pi\sqrt{b}B(b/2,1/2)}\sum_{k=1}^{(a-1)/2}a^{k-1/2}r^{-2k}B\left(k,\frac{1}{2}\right)J(k), \tag{19}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y}{\left(y^2 + a/r^2\right)^k \left(1 + y^2/b\right)^{(1+b)/2}} dy. \tag{20}$$

Substituting $w = 1/(1 + r^2/b)$, (20) can be reduced to

$$\begin{aligned} J(k) &= \frac{1}{2b^{k-1}} \int_0^\infty \frac{w^{k+(1+b)/2-2}}{\left[1 + \left\{a/(br^2) - 1\right\} w\right]^k} dw \\ &= \frac{1}{b^{k-1}(2k + b - 1)} G\left(k, k + \frac{1+b}{2} - 1; k + \frac{1+b}{2}; 1 - \frac{a}{br^2}\right), \end{aligned} \tag{21}$$

where the last step follows by direct application of Lemma 1. The result of the theorem follows by substituting (21) into (19). □

Theorem 4 is the analogue of Theorem 3 for the case when the degrees of freedom $2(M - 1)$ is an even integer.

Theorem 4. *Suppose X and Y are independent Pearson type VII random variables with pdfs* (1) *and* (2)*, respectively. If $a = 2(M - 1)$ is an even integer and $b = 2(N - 1)$, then the cdf of $Z =\mid X/Y \mid$ can be expressed as*

$$F(z) = \frac{2\sqrt{b}r}{\pi\sqrt{a}B(b/2, 1/2)} \sum_{k=1}^{a/2} \frac{a^{k-1/2} B(k - 1/2, 1/2)}{r^{2k-1} b^{k-1/2} (2k - 2 + b)} G\left(k - \frac{1}{2}, k + \frac{b}{2} - 1; k + \frac{b}{2}; 1 - \frac{a}{br^2}\right), \tag{22}$$

where $r = \sqrt{na/(mb)}z$.

Proof. Substituting the form for F given by (5) for even degrees of freedom, (18) can be reduced to

$$F(r) = \frac{2r}{\pi\sqrt{b}B(b/2, 1/2)} \sum_{k=1}^{a/2} a^{k-1} r^{1-2k} B\left(k - \frac{1}{2}, \frac{1}{2}\right) J(k), \tag{23}$$

where $J(k)$ denotes the integral

$$J(k) = \int_0^\infty \frac{y}{\left(y^2 + a/r^2\right)^{k-1/2} \left(1 + y^2/b\right)^{(1+b)/2}} dy. \tag{24}$$

Substituting $w = 1/(1 + r^2/b)$, (24) can be reduced to

$$\begin{aligned} J(k) &= \frac{1}{2b^{k-3/2}} \int_0^\infty \frac{w^{k+b/2-2}}{\left[1 + \left\{a/(br^2) - 1\right\} w\right]^{k-1/2}} dw \\ &= \frac{1}{b^{k-3/2}(2k + b - 2)} G\left(k - \frac{1}{2}, k + \frac{b}{2} - 1; k + \frac{b}{2}; 1 - \frac{a}{br^2}\right), \end{aligned} \tag{25}$$

where the last step again follows by direct application of Lemma 1. The result of the theorem follows by substituting (25) into (23). □

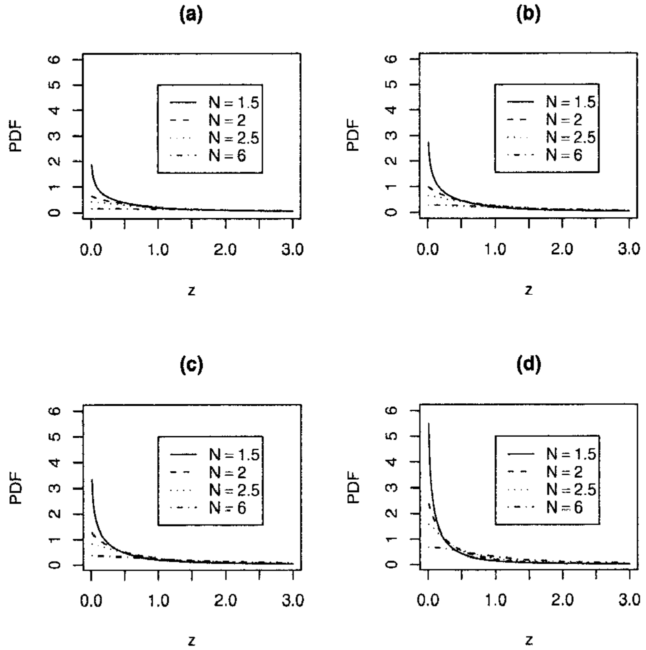

FIGURE 2. Plots of the pdf given by (16) and (22) for $m = 1$, $n = 1$ and (a) $M = 1.5$ and $N = 1.5, 2, 2.5, 6$; (b) $M = 2$ and $N = 1.5, 2, 2.5, 6$; (c) $M = 2.5$ and $N = 1.5, 2, 2.5, 6$; and, (d) $M = 6$ and $N = 1.5, 2, 2.5, 6$.

Figure 2 illustrates possible shapes of the pdf of $|X/Y|$ for a range of values of M and N. Note that the shapes are unimodal and that the densities appear to shrink with increasing values of N and decreasing values of M.

5. Particular cases

Here, we derive particular forms of (16) and (22) for $2(M - 1) = 2, 3, 4, 5$ and $2(N - 1) = 1, 2, 3, 4, 5$. In our calculations, we have used various special properties of the Gauss hypergeometric function (see, for example, Section 7.3 in volume 3 of Prudnikov *et al.* [18]). When $2(M-1)$ is odd the expressions for the cdf involve the integral $I(\cdot)$ in (17) which should be computed numerically. All the remaining terms in the resulting expressions for the cdf are elementary.

Corollary 5. *Suppose X and Y are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. For $2(M-1) = 2$ and $2(N-1) = 1, 2, 3, 4, 5$, the cdf of $Z =| X/Y |$ can be expressed as*

$$F(z) = \frac{2 \arcsin(y)}{\pi y},$$

$$F(z) = \frac{r}{r+1},$$

$$F(z) = \frac{2\sqrt{3}\left\{3r\arcsin\left(y/\sqrt{3}\right) - \sqrt{2}y\right\}}{\pi y^3 r},$$

$$F(z) = \frac{r\left(8r^3 - 6\sqrt{2}r^2 + \sqrt{2}\right)}{2\left(2r^2-1\right)^2},$$

$$F(z) = \frac{2\sqrt{5}\left\{75r^3\arcsin\left(y/\sqrt{5}\right) - 25\sqrt{2}r^2 y + 4\sqrt{2}y\right\}}{3\pi y^5 r^3},$$

where $r = \sqrt{na/(mb)}z$ *and* $y = \sqrt{(br^2-2)/r^2}$.

Corollary 6. *Suppose X and Y are independent Pearson type VII random variables with pdfs (1) and (2), respectively. For $2(M-1) = 3$ and $2(N-1) = 1,2,3,4,5$, the cdf of $Z = \mid X/Y \mid$ can be expressed as*

$$F(z) = I(1) + \frac{2\sqrt{3}\left(\log 3 - 2\log r\right)}{\pi^2\left(r^2-3\right)},$$

$$F(z) = I(2) + \frac{2\sqrt{6}\left(\sqrt{2}yr^2\operatorname{arctanh}\left(y/\sqrt{2}\right) - 2r^2 + 3\right)}{\pi r^3 y^4},$$

$$F(z) = I(3) + \frac{4r\left(r^2 - 1 - 2r^2\log r\right)}{\pi^2\left(r^2-1\right)^2},$$

$$F(z) = I(4) + \frac{2\sqrt{3}\left(24yr^4\operatorname{arctanh}\left(y/2\right) - 64r^4 + 60r^2 - 9\right)}{\pi r^5 y^6},$$

$$F(z) = I(5) - \frac{8\sqrt{15}r\left(50\log 3r^4 - 50\log 5r^4 - 100\log r r^4 + 75r^4 - 60r^2 + 9\right)}{3\pi^2\left(5r^2-3\right)^3},$$

where $r = \sqrt{na/(mb)}z$ *and* $y = \sqrt{(br^2-3)/r^2}$.

Corollary 7. *Suppose X and Y are independent Pearson type VII random variables with pdfs (1) and (2), respectively. For $2(M-1) = 4$ and $2(N-1) = 1,2,3,4,5$, the cdf of $Z = \mid X/Y \mid$ can be expressed as*

$$F(z) = \frac{2\left\{\left(r^2-6\right)\arcsin\left(y\right) + ry\right\}}{\pi y^3 r^2},$$

$$F(z) = \frac{r\left(\sqrt{2}r + 3\right)}{\sqrt{2}\left(r + \sqrt{2}\right)^2},$$

$$F(z) = \frac{6\sqrt{3}r\left\{r\left(3r^2 - 10\right)\arcsin\left(y/\sqrt{3}\right) + 4y\right\}}{\pi y^5 r^3},$$

$$F(z) = \frac{r\left(4r^5 - 12r^3 - 3r^4 + 14r^2 - 3\right)}{4\left(r^2 - 1\right)^3},$$

$$F(z) = \frac{2\sqrt{5}r\left\{75r^3\left(5r^2 - 14\right)\arcsin\left(y/\sqrt{5}\right) - \left(50r^4 - 460r^2 + 96\right)y\right\}}{3\pi y^7 r^5},$$

where $r = \sqrt{na/(mb)}z$ *and* $y = \sqrt{(br^2 - 4)/r^2}$.

Corollary 8. *Suppose* X *and* Y *are independent Pearson type VII random variables with pdfs* (1) *and* (2), *respectively. For* $2(M-1) = 5$ *and* $2(N-1) = 1, 2, 3, 4, 5$, *the cdf of* $Z = \mid X/Y \mid$ *can be expressed as*

$$F(z) = I(1) - \frac{2\sqrt{5}r\left\{\left(3\log 5 - 2\right)r^2 - 6\log r r^2 + 50\log r + 10 - 25\log 5\right\}}{3\pi^2\left(r^2 - 5\right)^2},$$

$$F(z) = I(2) + \frac{2\sqrt{10}\left\{6\sqrt{2}\left(r^2 - 5\right)\operatorname{arctanh}\left(y/\sqrt{2}\right) + \left(25 - 4r^2\right)y\right\}}{3\pi r^3 y^5},$$

$$F(z) = I(3) - \frac{4\sqrt{15}r\left\{27\log(5/3)r^4 + 104\log(3/5)r^2 - 54\log r r^4\right\}}{3\pi^2\left(3r^2 - 5\right)^3}$$

$$- \frac{4\sqrt{15}r\left\{210\log r r^2 + 9r^4 - 9r^2 + 125\right\}}{3\pi^2\left(3r^2 - 5\right)^3},$$

$$F(z) = I(4) + \frac{2\sqrt{5}\left\{96r^2\left(3r^2 - 10\right)\operatorname{arctanh}\left(y/2\right) - \left(144r^4 - 580r^2 + 125\right)y\right\}}{3\pi r^5 y^7},$$

$$F(z) = I(5) - \frac{8r\left(5r^6 - 27r^4 + 27r^2 - 5 - 12\log r r^6 + 36\log r r^4\right)}{9\pi^2\left(r^2 - 1\right)^4},$$

where $r = \sqrt{na/(mb)}z$ *and* $y = \sqrt{(br^2 - 5)/r^2}$.

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