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IMAGINARY BICYCLIC FUNCTION FIELDS WITH THE REAL CYCLIC SUBFIELD OF CLASS NUMBER ONE

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Abstract. Let $k = \mathbb{F}_q(T)$ and $\mathbb{A} = \mathbb{F}_q[T]$. Fix a prime divisor ℓ of $q-1$. In this paper, we consider a ℓ-cyclic real function field $k(\sqrt[\ell]{P})$ as a subfield of the imaginary bicyclic function field $K = k(\sqrt[\ell]{P}, \sqrt[\ell]{-Q})$, which is a composite field of $k(\sqrt[\ell]{P})$ with a ℓ-cyclic totally imaginary function field $k(\sqrt[\ell]{-Q})$ of class number one, and give various conditions for the class number of $k(\sqrt[\ell]{P})$ to be one by using invariants of the relatively cyclic unramified extensions K/F_i over ℓ-cyclic totally imaginary function field $F_i = k(\sqrt[\ell]{-P^iQ})$ for $1 \le i \le \ell - 1$.

1. Introduction

Gauss has conjectured that there exist infinitely many real quadratic fields of class number one, which is now still unsolved. In [9], in connection with this Gauss' conjecture, Yokoi considered a real quadratic field $\mathbb{Q}(\sqrt{p})$, where p is a prime number with $p \equiv 1 \bmod 4$, as a subfield of the imaginary biquadratic field $K = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$, which is a composite field of $\mathbb{Q}(\sqrt{p})$ with an imaginary quadratic field $\mathbb{Q}(\sqrt{-q})$ of class number one, and gave various conditions for the class number of $\mathbb{Q}(\sqrt{p})$ to be one by using invariants of the relatively cyclic unramified extension K/F over imaginary quadratic field $F = \mathbb{Q}(\sqrt{-pq})$. In this paper we extend Yokoi's result to the imaginary bicyclic function field case.

Let $k = \mathbb{F}_q(T)$, the rational function field with constant field $\mathbb{F}_q$, and $\mathbb{A} = \mathbb{F}_q[T]$. Fix a prime divisor ℓ of $q-1$. For a monic prime $P \in \mathbb{A}$, if $\ell | \deg P$, then $k(\sqrt[\ell]{P})$ is the unique ℓ-cyclic real subfield of K_P^+, where K_P^+ is the maximal real subfield of the P-th cyclotomic function field K_P. Otherwise $k(\sqrt[\ell]{-P})$ is the unique ℓ-cyclic totally imaginary subfield of K_P. In this paper, we consider a ℓ-cyclic real function field $k(\sqrt[\ell]{P})$ as a subfield of the imaginary bicyclic function field $K = k(\sqrt[\ell]{P}, \sqrt[\ell]{-Q})$, which is a composite field of $k(\sqrt[\ell]{P})$ with a ℓ-cyclic totally imaginary function field $k(\sqrt[\ell]{-Q})$ of class number one, and give various

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conditions for the class number of $k(\sqrt[\ell]{P})$ to be one by using invariants of the relatively cyclic unramified extensions K/F_i over ℓ-cyclic totally imaginary function field $F_i = k(\sqrt[\ell]{-P^iQ})$ for $1 \le i \le \ell - 1$. When $\ell = 2$ (q is odd) our result is a function field analog of Yokoi's result.

2. Notations and preliminary results

Let F be a global function field with constant field $\mathbb{F}_q$ and $S_\infty(F)$ be a fixed finite set of primes of F. For any finite separable extension K/F, we denote by $S_\infty(K)$ the set of primes of K which are extensions of primes in $S_\infty(F)$. Let $\mathcal{O}_K$ be the ring of functions in K whose poles are in $S_\infty(K)$. Let $\mathcal{I}_K$ be the group of nonzero fractional ideals of $\mathcal{O}_K$ and $\mathcal{P}_K$ be its subgroup of nonzero principal fractional ideals of $\mathcal{O}_K$. Set $\mathcal{C}l_K := \mathcal{I}_K/\mathcal{P}_K$, called the ideal class group of $\mathcal{O}_K$, and $h_K := |\mathcal{C}l_K|$, called the ideal class number of $\mathcal{O}_K$.

For a finite Galois extension K/F with Galois group $G := Gal(K/F)$, the following notations are used throughout this paper:

- $\mathcal{A}_{K/F}$: the group of ambiguous classes of $\mathcal{C}l_K$ with respect to K/F
- $a_{K/F} := |\mathcal{A}_{K/F}|$, the ambiguous class number of K/F
- $\mathcal{A}^0_{K/F}$: the group of ideal classes of $\mathcal{C}l_K$ represented by ambiguous ideals with respect to K/F, and $a^0_{K/F} := |\mathcal{A}^0_{K/F}|$
- $N_{K/F}$: the norm mapping with respect to K/F, and simultaneously the homomorphism from $\mathcal{C}l_K$ to $\mathcal{C}l_F$ induced by the norm mapping
- $j_{K/F}$: the homomorphism from $\mathcal{C}l_F$ to $\mathcal{C}l_K$ induced by extension of ideals
- $\widetilde{N}_{K/F} := j_{K/F} \circ N_{K/F}$, the endomorphism of $\mathcal{C}l_K$ defined as composition of $N_{K/F}$ and $j_{K/F}$
- $\mathcal{A}_F$: the group of classes of $\mathcal{C}l_K$ represented by ideals of F, and $a_F := |\mathcal{A}_F|$
- $\mathcal{C}l^0_F$: the group of those classes of $\mathcal{C}l_F$ whose ideals become principal in K, and $h_0 := |\mathcal{C}l^0_F|$.

For each prime $\mathfrak{p}$ of F,

- $e(\mathfrak{p}, K/F)$: the ramification index of $\mathfrak{p}$ in K/F
- $f(\mathfrak{p}, K/F)$: the inertia degree of $\mathfrak{p}$ in K/F
- $g(\mathfrak{p}, K/F) = \frac{[K:F]}{e(\mathfrak{p},K/F)f(\mathfrak{p},K/F)}$: the number of primes of K lying over $\mathfrak{p}$
- $d(\mathfrak{p}, K/F)$: the order of decomposition group of $\mathfrak{p}$ in K/F.

When K is fixed, we set $e(\mathfrak{p}) := e(\mathfrak{p}, K/F)$, $f(\mathfrak{p}) := f(\mathfrak{p}, K/F)$, $g(\mathfrak{p}) := g(\mathfrak{p}, K/F)$ and $d(\mathfrak{p}) := d(\mathfrak{p}, K/F)$ for simplicity.

Some results of Yokoi in [8] are translated into function field case by Kang and Lee in [4, §1] when $|S_\infty(F)| = 1$. It is easy to show that all statements in [4, §1] also hold with some modification even though $|S_\infty(F)| > 1$. Thus we will only refer their results when we need them.

Lemma 2.1. *Let K/F be a finite Galois extension with $G = Gal(K/F)$. Then*

(i) $a^0_{K/F} = h_F \cdot \dfrac{\prod_{\mathfrak{p}} e(\mathfrak{p})}{|H^1(G, E_K)|}$, *where* $\mathfrak{p}$ *runs through all finite primes of* F.
(ii) $H^1(G, E_K) \simeq \mathcal{P}^G_K/\mathcal{P}_F$ *and* $|H^1(G, E_K)| \equiv 0 \bmod h_0$.

Proof. (i) See Proposition 1.1 in [4].
(ii) See the proof of [5, Theorem 1] for the first one. The second one follows immediately from Lemma 1.2 (iii) in [4]. □

Let K/F be a finite cyclic extension with $G = Gal(K/F)$. Let B be any abelian group on which G acts. We denote by $Q(B)$ the Herbrand quotient of B with respect to G. It is well known that $Q(Cl_K) = 1$ and $Q(E_K) = (\prod_{\mathfrak{p}_\infty \in S_\infty(F)} d(\mathfrak{p}_\infty))/[K : F]$.

Lemma 2.2. *Let* K/F *be a finite cyclic extension with* $G = Gal(K/F)$. *Then*
(i) $a_{K/F} = h_F \cdot \frac{\prod_{\mathfrak{p}_\infty \in S_\infty(F)} d(\mathfrak{p}_\infty) \prod_{\mathfrak{p} \notin S_\infty(F)} e(\mathfrak{p})}{[K:F][E_F : E_F \cap N_{K/F}(K^*)]} = |\widetilde{N}_{K/F}(Cl_K)||H^0(G, Cl_K)|$.
(ii) $a_{K/F}/a^0_{K/F} = [E_F \cap N_{K/F}(K^*) : N_{K/F}(E_K)]$ *and*

$$a^0_{K/F}/a_F = \frac{h_0 \cdot \prod_{\mathfrak{p} \notin S_\infty(F)} e(\mathfrak{p})}{|H^1(G, E_K)|}.$$

(iii) $\prod_{\mathfrak{p}_\infty \in S_\infty(F)} d(\mathfrak{p}_\infty) \prod_{\mathfrak{p} \notin S_\infty(F)} e(\mathfrak{p}) \equiv 0 \bmod [E_F : E_F \cap N_{K/F}(K^*)]$.

Proof. (i) It follows directly from Theorem 1.5 in [4] and definition of $H^0(G, Cl_K)$.
(ii) See Lemma 1.4 and Corollary 2 in [4].
(iii) It follows from the fact that n_2 is an integer in [4, Theorem 1.5]. □

3. Finite unramified cyclic extension

By the *Hilbert class field* H_F of $\mathcal{O}_F$, we means the maximal unramified abelian extension of F in which each $\mathfrak{p}_\infty \in S_\infty(F)$ splits completely. The Galois group $Gal(H_F/F)$ is isomorphic to Cl_F via Artin automorphism, and so $[H_F : F] = h_F$. The *genus field* $G_{K/F}$ of K/F is the maximal extension of K in H_K which is the composite of K and some abelian extension of F. If K/F is a finite cyclic extension, then $Gal(G_{K/F}/K)$ is isomorphic to $Cl_K/Cl_K^{1-\sigma}$ (Proposition 2.4 in [2]), where σ is a generator of $Gal(K/F)$. Since $\mathcal{A}_{K/F}$ is the kernel of the multiplication by $1-\sigma$ on Cl_K, we have $a_{K/F} = |Cl_K/Cl_K^{1-\sigma}| = [G_{K/F} : K]$.

In the following, by a *finite unramified cyclic extension of* F, we always mean a finite cyclic extension of F which is contained in H_F.

Proposition 3.1. *Let* K/F *be a finite unramified cyclic extension with* $G = Gal(K/F)$. *Then we have*
(i) $a_{K/F} = h_F/[K : F]$, *i.e.*, $H_F = G_{K/F}$.
(ii) $h_0 = |H^1(G, E_K)| = [K : F][E_F \cap N_{K/F}(K^*) : N_{K/F}(E_K)]$.
(iii) $|H^0(G, Cl_K)| = |Cl^0_F \cap N_{K/F}(Cl_K)|$.

(iv) $|H^0(G, Cl_K)| \equiv 0 \bmod |H^0(G, E_K)|$ *and* $|H^0(G, Cl_K)| = |H^0(G, E_K)|$ *if and only if* $\mathcal{A}_F = \widetilde{N}_{K/F}(Cl_K)$.

(v) *Any ambiguous ideal class of* K/F *becomes principal in* H_F.

Proof. (i) In [4, Theorem 1.5], we have $\prod_{\mathfrak{p}_\infty \in S_\infty(F)} d(\mathfrak{p}_\infty) \prod_{\mathfrak{p} \notin S_\infty(F)} e(\mathfrak{p}) = 1$, and so $n_2 = [E_F : E_F \cap N_{K/F}(K^*)] = 1$. Thus, we have $a_{K/F} = h_F/[K : F] = [H_F : K]$, and so $H_F = G_{K/F}$.

(ii) By Lemma 2.2 (iii), we have $[E_F : E_F \cap N_{K/F}(K^*)] = 1$. We also have $\mathcal{I}_K^G = \mathcal{I}_F$ as in the proof of [5, Theorem 1]. Thus $|H^0(G, E_K)| = [E_F \cap N_{K/F}(K^*) : N_{K/F}(E_K)]$ and $h_0 = |H^1(G, E_K)| = [K : F]|H^0(G, E_K)|$ by Corollary 2 in [4]. Thus we get the result.

(iii) By definition, $|H^0(G, Cl_K)| = a_{K/F}/|\widetilde{N}_{K/F}(Cl_K)|$. By (i) and Class field theory, we have $|N_{K/F}(Cl_K)| = h_F/[K : F] = a_{K/F}$. From the exact sequence

$$1 \longrightarrow Cl_F^0 \cap N_{K/F}(Cl_K) \longrightarrow N_{K/F}(Cl_K) \overset{j_{K/F}}{\longrightarrow} \widetilde{N}_{K/F}(Cl_K) \longrightarrow 1,$$

we have $|\widetilde{N}_{K/F}(Cl_K)| = |N_{K/F}(Cl_K)|/|Cl_F^0 \cap N_{K/F}(Cl_K)|$. Thus we get the result.

(iv) By Lemma 2.2, we have $[\mathcal{A}_{K/F} : \mathcal{A}_{K/F}^0] = [E_F : N_{K/F}(E_K)] = |H^0(G, E_K)|$. Since $\mathcal{I}_K^G = \mathcal{I}_F$, we have $[\mathcal{A}_{K/F}^0 : \mathcal{A}_F] = 1$. Thus

$$\begin{aligned} |H^0(G, Cl_K)| &= [\mathcal{A}_{K/F} : \mathcal{A}_{K/F}^0][\mathcal{A}_F : \widetilde{N}_{K/F}(Cl_K)] \\ &= |H^0(G, E_K)|[\mathcal{A}_F : \widetilde{N}_{K/F}(Cl_K)]. \end{aligned}$$

Hence we get the result.

(v) It follows from Theorem 3.7 in [3] and (i). □

Proposition 3.2. *Let* K/F *be a finite unramified cyclic extension with* $G = Gal(K/F)$. *Then the following conditions are equivalent*:

(i) $a_{K/F} = a_{K/F}^0$, *i.e.*, $\mathcal{A}_{K/F} = \mathcal{A}_{K/F}^0$.

(ii) $[E_F \cap N_{K/F}(K^*) : N_{K/F}(E_K)] = 1$.

(iii) $H^0(G, E_K) = 1$.

(iv) $|H^1(G, E_K)| = h_0 = [K : F]$.

Proof. (i)⇔(ii) It follows immediately from Lemma 2.2 (ii).

(ii)⇔(iii) By Lemma 2.2 (iii), we have $[E_F : E_F \cap N_{K/F}(K^*)] = 1$, and so

$$|H^0(G, E_K)| = [E_F \cap N_{K/F}(K^*) : N_{K/F}(E_K)].$$

Thus we get the result.

(ii)⇔(iv) It follows immediately from Proposition 3.1 (ii). □

Proposition 3.3. *Let* K/F *be a finite unramified cyclic extension. Then the following conditions are equivalent*:

(i) $Cl_F = Cl_F^0 \times N_{K/F}(Cl_K)$.

(ii) $\mathrm{Ker}(\widetilde{N}_{K/F}) = \mathrm{Ker}(N_{K/F})$.

(iii) $H^0(G, Cl_K) = 1$.

Proof. (i)⇒(ii) It generally holds that $\mathrm{Ker}(N_{K/F}) \subseteq \mathrm{Ker}(\widetilde{N}_{K/F}) = N_{K/F}^{-1}(\mathcal{C}l_F^0)$. For any $\mathfrak{c} \in \mathrm{Ker}(\widetilde{N}_{K/F})$, $N_{K/F}(\mathfrak{c}) \in \mathcal{C}l_F^0 \cap N_{K/F}(\mathcal{C}l_K) = 1$. Thus $\mathrm{Ker}(\widetilde{N}_{K/F}) = \mathrm{Ker}(N_{K/F})$.

(ii)⇒(iii) Since $N_{K/F}^{-1}(\mathcal{C}l_F^0 \cap N_{K/F}(\mathcal{C}l_K)) = N_{K/F}^{-1}(\mathcal{C}l_F^0) = \mathrm{Ker}(\widetilde{N}_{K/F}) = \mathrm{Ker}(N_{K/F})$, we have $\mathcal{C}l_F^0 \cap N_{K/F}(\mathcal{C}l_K) = 1$. Thus $H^0(G, \mathcal{C}l_K) = 1$ by Proposition 3.1 (iii).

(iii)⇒(i) If $H^0(G, \mathcal{C}l_K) = 1$, then $\mathcal{C}l_F^0 \cap N_{K/F}(\mathcal{C}l_K) = 1$ by Proposition 3.1 (iii). Since $|\mathcal{C}l_F^0| \equiv 0 \bmod [K:F]$ (by Proposition 3.1 (ii)) and $|N_{K/F}(\mathcal{C}l_K)| = h_F/[K:F]$,

$$|\mathcal{C}l_F^0 \times N_{K/F}(\mathcal{C}l_K)| = |\mathcal{C}l_F^0||N_{K/F}(\mathcal{C}l_K)| = \frac{|\mathcal{C}l_F^0|}{[K:F]} \cdot h_F \geq |\mathcal{C}l_F|,$$

and so $\mathcal{C}l_F = \mathcal{C}l_F^0 \times N_{K/F}(\mathcal{C}l_K)$. □

4. Imaginary bicyclic function fields

Let $k := \mathbb{F}_q(T)$, the rational function field with constant field $\mathbb{F}_q$ and $\mathbb{A} := \mathbb{F}_q[T]$. We set $S_\infty(k) := \{\infty\}$, where ∞ is the prime divisor of k associated to $(1/T)$. For each monic $N \in \mathbb{A}$, let K_N be the N-th cyclotomic function field and K_N^+ be its maximal real subfield. Fix a prime divisor ℓ of $q-1$. For a monic prime $P \in \mathbb{A}$, if $\ell \mid \deg(P)$, then $k(\sqrt[\ell]{P})$ is the unique ℓ-cyclic real subfield of K_P^+. Otherwise $k(\sqrt[\ell]{-P})$ is the unique ℓ-cyclic totally imaginary subfield of K_P.

Let P and Q be fixed monic primes in $\mathbb{A}$ with $\ell | \deg P$ and $\ell \nmid \deg Q$. Set $K_0 := k(\sqrt[\ell]{-Q})$, $K_1 := k(\sqrt[\ell]{P})$, $K := k(\sqrt[\ell]{P}, \sqrt[\ell]{-Q})$ and $F_i := k(\sqrt[\ell]{-P^iQ})$ for $1 \leq i \leq \ell - 1$. We also set $G := Gal(K/k)$ and $G_i := Gal(K/F_i)$.

Lemma 4.1. *$H^0(\Delta, E_{K_1})$ is trivial, where $\Delta = Gal(K_1/k)$.*

Proof. At first we note that P is the only finite prime of k which is (totally) ramified in K_1. Then, by Theorem 1.5 in [4], we have $a_{K_1/k} = 1$ and so $a_{K_1/k}^0 = 1$. Thus $|H^1(\Delta, E_{K_1})| = \ell$ by Proposition 1.1 in [4]. Since the Herbrand quotient $Q(E_{K_1})$ with respect to Δ is $1/\ell$, we have $|H^0(\Delta, E_{K_1})| = 1$. □

Theorem 4.2. *Let K and F_i be as above. Then, for $1 \leq i \leq \ell - 1$, K/F_i is a finite unramified cyclic extension of degree ℓ, and moreover the followings hold:*

(i) $G_{K/F_i} = H_{F_i}$.

(ii) $h_{F_i} \equiv 0 \bmod \ell$ *for* $1 \leq i \leq \ell - 1$ *and* $h_K = h_{K_0} h_{K_1} \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell}$.

(iii) $H^0(G_i, E_K) = 1$.

(iv) $a_{K/F_i} = a_{K/F_i}^0$, *i.e.*, $\mathcal{A}_{K/F_i} = \mathcal{A}_{K/F_i}^0$.

(v) $h_0 = \ell$.

(vi) $H^0(G_i, \mathcal{C}l_K) = 1$ *or cyclic group of order* ℓ.

Proof. P and Q are the only ramified finite primes of k which are ramified on K, and $e(P, K/k) = e(P, F_i/k) = e(Q, F_i/k) = e(Q, K/k) = \ell$ by Abhyankar's Lemma ([6, III. 8.9 Proposition]). Thus K/F_i is unramified at all finite primes. Since ∞ is totally ramified in K_0 and splits completely in K_1, we have $e(\infty, K/k) = \ell$. Thus $e(\mathfrak{p}_\infty, K/F_i) = 1$ for any $\mathfrak{p}_\infty \in S_\infty(F_i)$. Hence $K \subseteq H_{F_i}$, and so K/F_i is a finite unramified cyclic extension of degree ℓ.

(i) follows immediately from Proposition 3.1.

(ii) By Proposition 3.1 (i), we have $h_{F_i} \equiv 0 \bmod \ell$. Since F_i and K_0 are totally imaginary extensions over k, $E_{F_i} = E_{K_0} = \mathbb{F}_q^*$. Thus, by the Main Theorem in [10], we have

$$h_K = \varrho\, h_{K_0} h_{K_1} \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell},$$

where $\varrho = [E_K : E_{K_1}]$. It remains to show that $\varrho = 1$. Since K_1 is the maximal real subfield of K, by Lemma 2.2 in [1], $\varrho = 1$ or ℓ. But any primes $\mathfrak{Q}$ of K_1 lying over Q is totally ramified in K. Thus $\varrho = 1$.

(iii) Since $E_K = E_{K_1}$, we have $H^0(G_i, E_K) \simeq H^0(\Delta, E_{K_1}) = 1$ by Lemma 4.1.

(iv), (v) are immediate consequences of Proposition 3.2 and (iii).

(vi) Since $|Cl^0_{F_i}| = \ell$, we have $|Cl^0_{F_i} \cap N_{K/F_i}(Cl_K)| = 1$ or ℓ.

Thus $|H^0(G_i, Cl_K)| = 1$ or ℓ by Proposition 3.1 (iii). Thus $H^0(G_i, Cl_K) = 1$ or cyclic group of order ℓ. □

Corollary 4.3. *Let K and F_i be as in Theorem 4.2. Then*

(i) $a_{K/F_i} = a^0_{K/F_i} = h_{F_i}/\ell$.

(ii) $H^1(G_i, E_K)$ *is a cyclic group of order* ℓ.

Proof. Both (i) and (ii) are immediate consequences of Proposition 3.2, Theorem 4.2 and Proposition 3.1. □

Let $N_K : Cl_K \to \prod_{i=1}^{\ell-1} Cl_{F_i}$ and $j_K : \prod_{i=1}^{\ell-1} Cl_{F_i} \to Cl_K$ be the homomorphisms defined by $N_K(\mathfrak{c}) = (N_{K/F_1}(\mathfrak{c}), \ldots, N_{K/F_{\ell-1}}(\mathfrak{c}))$ and $j_K(\mathfrak{c}_1, \ldots, \mathfrak{c}_{\ell-1}) = \prod_{i=1}^{\ell-1} j_{K/F_i}(\mathfrak{c}_i)$, respectively.

Theorem 4.4. *If $h_{K_0} = 1$, then the following conditions are equivalent:*

(i) $h_{K_1} = 1$.

(ii) $h_K = \prod_{i=1}^{\ell-1} a_{K/F_i}$, *i.e.*, $Cl_K = \bigoplus_{i=1}^{\ell-1} \mathcal{A}_{K/F_i}$.

(iii) N_K *is injective with* $N_K(Cl_K) = \prod_{i=1}^{\ell-1} N_{K/F_i}(Cl_K)$.

(iv) j_K *is surjective with kernel* $\prod_{i=1}^{\ell-1} Cl^0_{F_i}$.

Proof. (i) $\Leftrightarrow$ (ii) By Theorem 4.2 (ii) and Proposition 3.1 (i), $h_{K_1} = 1$ if and only if $h_K = \prod_{i=1}^{\ell-1} a_{K/F_i}$, i.e., $Cl_K = \bigoplus_{i=1}^{\ell-1} \mathcal{A}_{K/F_i}$.

(i) $\Leftrightarrow$ (iii) Note that $N_K(\mathcal{C}l_K) \subseteq \prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K)$. Thus

$$|\operatorname{Coker}(N_K)| \geq \prod_{i=1}^{\ell-1} [\mathcal{C}l_{F_i} : N_{K/F_i}(\mathcal{C}l_K)] = \ell^{\ell-1},$$

and so

$$h_K = \frac{|\operatorname{Ker}(N_K)|}{|\operatorname{Coker}(N_K)|} \prod_{i=1}^{\ell-1} h_{F_i} \leq |\operatorname{Ker}(N_K)| \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell}.$$

Hence $h_{K_1} = 1 \Leftrightarrow h_K = \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell} \Leftrightarrow \operatorname{Ker}(N_K) = 1$ and $|\operatorname{Coker}(N_K)| = \ell^{\ell-1}$ $\Leftrightarrow N_K$ is injective with $N_K(\mathcal{C}l_K) = \prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K)$.

(i) $\Leftrightarrow$ (iv) Note that $\prod_{i=1}^{\ell-1} \mathcal{C}l_{F_i}^0 \subseteq \operatorname{Ker}(j_K)$. Thus

$$\prod_{i=1}^{\ell-1} |\mathcal{C}l_{F_i}^0| = \ell^{\ell-1} \leq |\operatorname{Ker}(j_K)|,$$

and so

$$h_K = \frac{|\operatorname{Coker}(j_K)|}{|\operatorname{Ker}(j_K)|} \prod_{i=1}^{\ell-1} h_{F_i} \leq |\operatorname{Coker}(j_K)| \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell}.$$

Hence $h_{K_1} = 1 \Leftrightarrow h_K = \prod_{i=1}^{\ell-1} \frac{h_{F_i}}{\ell} \Leftrightarrow \operatorname{Coker}(j_K) = 1$ and $\operatorname{Ker}(j_K) = \prod_{i=1}^{\ell-1} \mathcal{C}l_{F_i}^0$ $\Leftrightarrow j_K$ is surjective with $\operatorname{Ker}(j_K) = \prod_{i=1}^{\ell-1} \mathcal{C}l_{F_i}^0$. □

Corollary 4.5. *Suppose that $h_{K_0} = 1$. If $h_{K_1} = 1$, then H_K is the compositum of the $H_{F_1}, \dots, H_{F_{\ell-1}}$, i.e., $\bigcap_{i=1}^{\ell-1} \mathcal{C}l_K^{1-\sigma_i} = 1$, where σ_i is a generator of $G_i = Gal(K/F_i)$. When $\ell = 2$ (q is odd), the converse also holds.*

Proof. At first, we show that $\operatorname{Ker}(N_K) = \bigcap_{i=1}^{\ell-1} \mathcal{C}l_K^{1-\sigma_i}$. Clearly, $\mathcal{C}l_K^{1-\sigma_i} \subseteq \operatorname{Ker}(N_{K/F_i})$ and $\operatorname{Ker}(N_K) = \bigcap_{i=1}^{\ell-1} \operatorname{Ker}(N_{K/F_i})$. We also have $|\operatorname{Coker}(N_{K/F_i})| = [K : F_i] = \ell$ by Class field theory. It follows from the exact sequence

$$1 \longrightarrow \operatorname{Ker}(N_{K/F_i}) \longrightarrow \mathcal{C}l_K \xrightarrow{N_{K/F_i}} \mathcal{C}l_{F_i} \longrightarrow \operatorname{Coker}(N_{K/F_i}) \longrightarrow 1$$

that $|\operatorname{Ker}(N_{K/F_i})| = \ell h_K / h_{F_i} = h_K / a_{K/F_i} = |\mathcal{C}l_K^{1-\sigma_i}|$. Thus $\operatorname{Ker}(N_{K/F_i}) = \mathcal{C}l_K^{1-\sigma_i}$, and so $\operatorname{Ker}(N_K) = \bigcap_{i=1}^{\ell-1} \mathcal{C}l_K^{1-\sigma_i}$. By Theorem 4.2 (i), $Gal(H_K/H_{F_i}) \simeq \mathcal{C}l_K^{1-\sigma_i}$. Thus it follows from Theorem 4.4 (iii) that H_K is the compositum of the $H_{F_1}, \dots, H_{F_{\ell-1}}$.

When $\ell = 2$ (q is odd), $H_K = H_{F_1}$ ($\Leftrightarrow \mathcal{C}l_K^{1-\sigma_1} = 1$) implies that $\mathcal{C}l_K = \mathcal{A}_{K/F_1}$. Thus $h_{K_1} = 1$ by Theorem 4.4. □

Proposition 4.6. *For $1 \leq i \leq \ell - 1$, the ℓ-rank of $\mathcal{A}_{K/F_i}$ is equal to ϱ_i or $\varrho_i + 1$, where $\varrho_i = \dim_{\mathbb{F}_\ell}(\mathcal{C}l_K^{1-\sigma_i} / \mathcal{C}l_K^{(1-\sigma_i)^2})$.*

Proof. At first, we note that

$$\mathcal{C}l_K^{1-\sigma_i} = \mathrm{Ker}(N_{K/F_i}) \subseteq \mathrm{Ker}(\widetilde{N}_{K/F_i}) = N_{K/F_i}^{-1}(\mathcal{C}l_{F_i}^0 \cap N_{K/F_i}(\mathcal{C}l_K))$$

and

$$H^0(G_i, \mathcal{A}_{K/F_i}) = \mathcal{A}_{K/F_i}/\mathcal{A}_{K/F_i}^{\ell} \simeq H^1(G_i, \mathcal{A}_{K/F_i}) = \mathrm{Ker}(\widetilde{N}_{K/F_i}) \cap \mathcal{A}_{K/F_i},$$
$$H^0(G_i, \mathcal{C}l_K^{1-\sigma_i}) = \mathrm{Ker}(N_{K/F_i}) \cap \mathcal{A}_{K/F_i} \simeq H^1(G_i, \mathcal{C}l_K^{1-\sigma_i}) = \mathcal{C}l_K^{1-\sigma_i}/\mathcal{C}l_K^{(1-\sigma_i)^2}$$

as $\mathbb{F}_\ell$-vector spaces. Thus, ℓ-rank of $\mathcal{A}_{K/F_i} = \dim_{\mathbb{F}_\ell}(\mathrm{Ker}(\widetilde{N}_{K/F_i}) \cap \mathcal{A}_{K/F_i})$. The morphism $N_{K/F_i} : \mathrm{Ker}(\widetilde{N}_{K/F_i}) \to \mathcal{C}l_{F_i}^0 \cap N_{K/F_i}(\mathcal{C}l_K)$ induces the following injective morphisms

$$\frac{\mathrm{Ker}(\widetilde{N}_{K/F_i}) \cap \mathcal{A}_{K/F_i}}{\mathrm{Ker}(N_{K/F_i}) \cap \mathcal{A}_{K/F_i}} \hookrightarrow \frac{\mathrm{Ker}(\widetilde{N}_{K/F_i})}{\mathrm{Ker}(N_{K/F_i})} \hookrightarrow \mathcal{C}l_{F_i}^0 \cap N_{K/F_i}(\mathcal{C}l_K).$$

By Proposition 3.1 (iii) and Theorem 4.2 (iv), $|\mathcal{C}l_{F_i}^0 \cap N_{K/F_i}(\mathcal{C}l_K)| = 1$ or ℓ. Thus $|(\mathrm{Ker}(\widetilde{N}_{K/F_i}) \cap \mathcal{A}_{K/F_i})/(\mathrm{Ker}(N_{K/F_i}) \cap \mathcal{A}_{K/F_i})| = 1$ or ℓ, and so ℓ-rank of $\mathcal{A}_{K/F_i}$ is equal to $\dim_{\mathbb{F}_\ell}(\mathcal{C}l_K^{1-\sigma_i}/\mathcal{C}l_K^{(1-\sigma_i)^2})$ or $\dim_{\mathbb{F}_\ell}(\mathcal{C}l_K^{1-\sigma_i}/\mathcal{C}l_K^{(1-\sigma_i)^2}) + 1$. □

Corollary 4.7. *Suppose that $h_{K_0} = 1$. If $h_{K_1} = 1$, then*

$$\sum_{i=1}^{\ell-1} \varrho_i \leq \ell\text{-rank of } \mathcal{C}l_K \leq \sum_{i=1}^{\ell-1} \varrho_i + (\ell - 1).$$

Proof. It follows immediately from Theorem 4.4 (ii) and Proposition 4.6. □

Remark 4.8. When $\ell = 2$ (q is odd), if $h_{K_0} = h_{K_1} = 1$, then $\mathcal{C}l_K = \mathcal{A}_{K/F_1}$ and $\mathcal{C}l_K^{1-\sigma_1} = 1$. Thus $\varrho_1 = 0$, and so the ℓ-rank of $\mathcal{C}l_K$ is equal to 0 or 1 (compare to Proposition 5 (ii) in [9]).

Let $\widetilde{N}_K = j_K \circ N_K$ be the endomorphism of $\mathcal{C}l_K$ defined as composition of N_K and j_K.

Proposition 4.9. *If the endomorphism $\widetilde{N}_K$ of $\mathcal{C}l_K$ is injective or surjective, then*

(i) $h_{K_0} = h_{K_1} = 1$.
(ii) $H^n(G_i, \mathcal{C}l_K) = 1$ *for any integer n and $1 \leq i \leq \ell - 1$.*
(iii) *ℓ-rank of $\mathcal{C}l_K$ is equal to $\sum_{i=1}^{\ell-1} \varrho_i$.*

Proof. Since $\mathcal{C}l_K$ is a finite abelian group, if $\widetilde{N}_K$ is injective or surjective, then it is an automorphism. The condition that $\widetilde{N}_K$ is an automorphism is equivalent to the condition that N_K is injective and j_K is surjective with $\prod_{i=1}^{\ell-1} \mathcal{C}l_{F_i} = N_K(\mathcal{C}l_K) \times \mathrm{Ker}(j_K)$. Then $N_K(\mathcal{C}l_K) \subseteq \prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K)$ and $|N_K(\mathcal{C}l_K)| =$

h_K. Thus

$$\begin{aligned}
&\prod_{i=1}^{\ell-1} h_{F_i}\\
&= \Big[\prod_{i=1}^{\ell-1} \mathcal{C}l_{F_i} : \prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K)\Big]\Big[\prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K) : N_K(\mathcal{C}l_K)\Big][N_K(\mathcal{C}l_K) : 1]\\
&= \ell^{\ell-1}\Big[\prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K) : N_K(\mathcal{C}l_K)\Big] h_K.
\end{aligned}$$

By Theorem 4.2 (ii), we have $h_{K_0} = h_{K_1} = 1$ and $N_K(\mathcal{C}l_K) = \prod_{i=1}^{\ell-1} N_{K/F_i}(\mathcal{C}l_K)$. Then $\mathcal{C}l_K = \widetilde{N}_K(\mathcal{C}l_K) = \prod_{i=1}^{\ell-1} \widetilde{N}_{K/F_i}(\mathcal{C}l_K)$. Since $\widetilde{N}_{K/F_i}(\mathcal{C}l_K) \subseteq \mathcal{A}_{K/F_i}$, by Theorem 4.4 (ii), we have $\widetilde{N}_{K/F_i}(\mathcal{C}l_K) = \mathcal{A}_{K/F_i}$ for all $1 \le i \le \ell - 1$. Thus $H^0(G_i, \mathcal{C}l_K) = 1$, and so $H^n(G_i, \mathcal{C}l_K) = 1$ for any integer n. By Proposition 3.1 (iii), we also have $\mathcal{C}l^0_{F_i} \cap N_{K/F_i}(\mathcal{C}l_K) = 1$. In the proof of Proposition 4.6, $|\operatorname{Ker}(\widetilde{N}_{K/F_i}) \cap \mathcal{A}_{K/F_i}| = |\operatorname{Ker}(N_{K/F_i}) \cap \mathcal{A}_{K/F_i}|$ and so the ℓ-rank of $\mathcal{A}_{K/F_i}$ is equal to ϱ_i. Thus, by Theorem 4.4 (ii), the ℓ-rank of $\mathcal{C}l_K$ is equal to $\sum_{i=1}^{\ell-1} \varrho_i$. □

Remark 4.10. When $\ell = 2$ (q is odd), if the endomorphism $\widetilde{N}_K$ of $\mathcal{C}l_K$ is injective or surjective (i.e., automorphism), then the ℓ-rank of $\mathcal{C}l_K$ is equal to 0 (compare to Proposition 7 (iii) in [9]).

Now we consider the case $\ell = 2$ (q is odd). Then P and Q are monic primes in $\mathbb{A}$ of even and odd degree, respectively. Set $F := k(\sqrt{-PQ})$ and

$$\lambda_i := \dim_{\mathbb{F}_2}\left(\mathcal{C}l_{F,2}^{(1-\tau)^{i-1}} / \mathcal{C}l_{F,2}^{(1-\tau)^i}\right) \text{ for } i \ge 1,$$

where $\mathcal{C}l_{F,2}$ denotes the Sylow 2-subgroup of $\mathcal{C}l_F$ and τ is the generator of $Gal(F/k)$. Then $\lambda_1 = 1$ (Theorem 2.1 in [7]), and by section 3, part (ii) in [7], we have

$$\lambda_2 = \begin{cases} 1 & \text{if } (Q/P) = 1, \\ 0 & \text{if } (Q/P) = -1, \end{cases}$$

where $(/)$ is the Legendre symbol. Thus, $h_F \not\equiv 0 \bmod 4$ (i.e., $2||h_F$) if and only if $(Q/P) = -1$. We follow exactly the same process as in [9, Proposition 6] to get the following.

Proposition 4.11. *If $h_{K_0} = h_{K_1} = 1$, then the following conditions are equivalent:*

(i) $(Q/P) = -1$.
(ii) $h_F \not\equiv 0 \bmod 4$, *i.e.*, $2||h_F$.
(iii) *2-rank of $\mathcal{C}l_K$ is equal to* 0, *i.e.*, $(2, h_K) = 1$.
(iv) $H^n(G_1, \mathcal{C}l_K) = 1$ *for any integer n, where $G_1 = Gal(K/F)$.*

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