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COMPETITION INDICES OF TOURNAMENTS

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Abstract. For a positive integer m and a digraph D, the m-step competition graph $C^m(D)$ of D has the same set of vertices as D and an edge between vertices u and v if and only if there is a vertex x in D such that there are directed walks of length m from u to x and from v to x. Cho and Kim [6] introduced notions of competition index and competition period of D for a strongly connected digraph D. In this paper, we extend these notions to a general digraph D. In addition, we study competition indices of tournaments.

1. Preliminaries and notations

In 1968, Cohen [8] introduced the notion of competition graph in connection with a problem in ecology. The *competition graph* of a digraph D, denoted by $C(D)$, has the same vertex set of D, and there is an edge between vertices x and y in $C(D)$ if and only if there is a vertex z such that (x, z) and (y, z) are arcs of D. Since the notion of competition graphs was introduced, there has been a very large literature on competition graphs. For surveys on the literature of competition graphs, see [11]. In addition to ecology, their various applications include applications to channel assignments, coding, and modeling of complex economic and energy systems.

Recently, Cho et al. [7] introduced the m-step competition graph, a generalization of the competition graph. Let D be a digraph (with or without loops) with vertex set $\{v_1, v_2, \ldots, v_n\}$. Given a positive integer m, we say that a vertex v_k of D is an *m-step common prey* for v_i and v_j if there are two directed walks of length m, one of which is from v_i to v_k, and the other from v_j to v_k. Then the *m-step competition graph of D*, denoted by $C^m(D)$, has the same vertex set as D, and there is an edge between vertices v_i and v_j ($v_i \neq v_j$) if and only if v_i and v_j have an m-step common prey in D. The m-step digraph of D, denoted by D^m, has the same vertex set as D and an arc (v_i, v_j) if and only if there is a directed walk of length m from v_i to v_j. Then we have the following proposition.

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Proposition 1.1 (Cho et al. [7]). *For any digraph D (possibly with loops) and positive integer m,*

$$C^m(D) = C(D^m).$$

The concept of m-step digraph is not new, and some asymptotic behavior of the digraph sequence $D, D^2, D^3, \ldots, D^m, \ldots$ is well known (see [1, 12, 17] and for undefined graph terminology see [4, 7]). This motivated us to study some asymptotic behavior of the competition graph sequence $C^1(D)$, $C^2(D)$, $C^3(D)$, ..., $C^m(D)$, ... Note that this graph sequence is the same as $C(D^1)$, $C(D^2)$, $C(D^3)$, ..., $C(D^m)$, ... by Proposition 1.1.

For the two-element Boolean algebra $\mathcal{B} = \{0, 1\}$, $\mathcal{B}_n$ denotes the set of all $n \times n$ (Boolean) matrices over $\mathcal{B}$. Under the Boolean operations, we can define matrix addition and multiplication in $\mathcal{B}_n$. Let D be a digraph with vertex set $\{v_1, v_2, \ldots, v_n\}$, and $A = [a_{ij}]$ be the (Boolean) adjacency matrix of D such that a_{ij} is one if and only if (v_i, v_j) is an arc in D. Notice that for a positive integer m, the (Boolean) m-th power $A^m = [b_{ij}]$ of A is a Boolean matrix such that b_{ij} is one if and only if there is a directed walk of length m from v_i to v_j in D. Thus two rows i and i' of A^m have non-zero entry in the j-th column if and only if vertex v_j is an m-step common prey of vertices v_i and $v_{i'}$ in D.

For a Boolean matrix A, the *row graph* $\mathcal{R}(A)$ of A is the graph whose vertices are the rows of A, and two vertices in $\mathcal{R}(A)$ are adjacent if and only if their corresponding rows have a non-zero entry in the same column of A. This notion was studied by Greenberg et al. [9]. From the definition of row graphs and m-step competition graphs, the following proposition follows immediately.

Proposition 1.2 (Cho et al. [7]). *A graph G with n vertices is an m-step competition graph if and only if there is a Boolean matrix A in $\mathcal{B}_n$ such that G is the row graph of A^m.*

Note that for a digraph D and its adjacency matrix A, the graph sequence $C^1(D), C^2(D), C^3(D), \ldots, C^m(D), \ldots$ is equivalent to the row graph sequence $\mathcal{R}(A)$, $\mathcal{R}(A^2)$, $\mathcal{R}(A^3)$, ..., $\mathcal{R}(A^m)$, ... by Proposition 1.2.

2. Competition indices and competition periods of digraphs

For a digraph D of order n and its $n \times n$ adjacency matrix A, consider a digraph sequence $D, D^2, D^3, \ldots, D^m, \ldots$ that is equivalent to a matrix sequence $A, A^2, A^3, \ldots, A^m, \ldots$. Since the cardinality of the (Boolean) matrix set $\mathcal{B}_n$ is equal to a finite number 2^{n^2}, there is a smallest positive integer q such that $A^q = A^{q+r}$ (equivalently $D^q = D^{q+r}$) for some positive integer r. Such an integer q is called the *index* of D and is denoted by index(D). Then, there is also a smallest positive integer p such that $A^q = A^{q+p}$ (equivalently $D^q = D^{q+p}$) where $q =$ index(D), and such an integer p is called the *period* of D and is denoted by period(D).

Cho and Kim [6] defined the competition index of D, denoted by cindex(D) for a strongly connected digraph D as follows: For a strongly connected digraph

D, the competition index of D is a smallest positive integer q such that $C^q(D) = C^{q+r}(D)$ for some positive integer r. But this definition is not suitable for a general digraph. Consider the following digraph D_5 shown in Figure 1 which is not strongly connected. We have

$$C^1(D_5) = C^2(D_5) = C^3(D_5) \neq C^4(D_5) = C^5(D_5) = \cdots.$$

If we define the competition index q such that $C^q(D) = C^{q+r}(D)$ for some positive integer r, we have the competition index of D_5 is 1 since $C^1(D_5) = C^2(D_5)$. But this graph sequence $C^1(D_5), C^2(D_5), \ldots, C^m(D_5), \ldots$ is stable when $m \geq 4$.

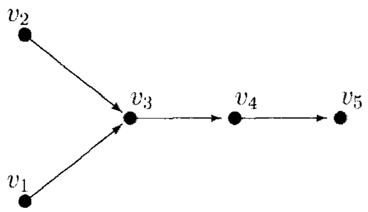

FIGURE 1. D_5

Considering this limitation on the definition of a competition index for a strongly connected digraph, we extend the notion of cindex(D) to a general digraph. Consider the competition graph sequence $C^1(D)$, $C^2(D)$, $C^3(D), \ldots,$ $C^m(D), \ldots$. By Proposition 1.1 and 1.2, this graph sequence is equivalent to the row graph sequence $\mathcal{R}(A)$, $\mathcal{R}(A^2)$, $\mathcal{R}(A^3), \ldots,$ $\mathcal{R}(A^m), \ldots$. Thus there is a smallest positive integer q such that $C^{q+i}(D) = C^{q+r+i}(D)$ (equivalently $\mathcal{R}(A^{q+i}) = \mathcal{R}(A^{q+r+i})$) for some positive integer r and all nonnegative integer i. Such an integer q is called the *competition index* of D and is denoted by cindex(D). For $q =$ cindex(D), there is also a smallest positive integer p such that $C^q(D) = C^{q+p}(D)$ (equivalently $\mathcal{R}(A^q) = \mathcal{R}(A^{q+p})$). Such an integer p is called the *competition period* of D and is denoted by cperiod(D). From the definition of index(D) and cindex(D), the next proposition immediately follows.

Proposition 2.1. *For any digraph D, we have* index$(D) \geq$ cindex(D).

For a strongly connected digraph D, the greatest common divisor of the lengths of cycles of D is called the *index of imprimitivity* of D, and is denoted by $k(D)$. (If $n = 1$ and D does not contain a loop, $k(D)$ is undefined.) If $k(D) = 1$, we say D is *primitive*. For a primitive digraph D and its adjacency matrix A, the exponent exp(D) of D has been defined to be the smallest positive integer q such that A^k is a positive matrix for all integers $k \geq q$. Note that for a primitive digraph D, exp$(D) =$ index(D) and cindex(D) is the smallest integer q such that the row graph of A^k is a complete graph for any $k \geq q$.

It is well known that the index of imprimitivity of D is equal to period(D) for a strongly connected digraph D. (For surveys of the literature of exponent (index) and period, see [5, 13, 14, 16, 17].) The competition period of a strongly connected digraph is 1 by the following theorem.

Theorem 2.2 (Cho and Kim [6]). *If there is no vertex whose outdegree is zero in a digraph D, we have*

$$\mathrm{cperiod}(D) = 1.$$

It follows from Theorem 2.2 that cperiod(D) = 1 for a strongly connected digraph D.

Consider the following digraph D_4 shown in Figure 2. The vertex u has only loop in D_4 and D_4 is not strongly connected since the outdegree of w is zero. Vertices v_1 and u have only $(2m-1)$-step common prey w for any positive integer m. In addition, v_2 and u have only $2m$-step common prey w for any positive integer m. The other pair of vertices do not have an m-step common prey for any positive integer m. Therefore we have

$$C^1(D_4) = C^3(D_4) = \cdots = C^{2m+1}(D_4) = \cdots,$$
$$C^2(D_4) = C^4(D_4) = \cdots = C^{2m}(D_4) = \cdots.$$

Since $C^1(D_4) \neq C^2(D_4)$, we have cindex(D_4) = 1 and cperiod(D_4) = 2.

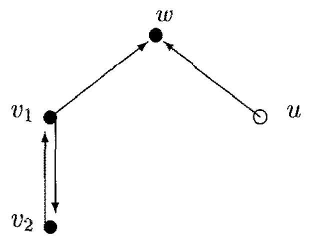

FIGURE 2. D_4

The upper bound of competition index of a strongly connected digraph was studied in [6]. For a positive integer n, let $w_1 = w_2 = 1$ and $w_n = \left\lceil \frac{n^2-2n+3}{2} \right\rceil$ when $n \geq 3$. For each integer $n (n \geq 3)$, any digraph isomorphic to the following digraph shown in Figure 3 is called a Wielandt digraph of order n, and is denoted by W_n. For $n \geq 3$, we have cindex(W_n) = w_n in [6]. The next theorem shows that the maximum on competition indices of strongly connected digraphs is w_n.

Theorem 2.3 (Cho and Kim [6]). *Let D be a strongly connected digraph D of order n. Then we have* cindex(D) $\leq w_n$. *The equality holds only when D is a Wielandt digraph when $n \geq 3$.*

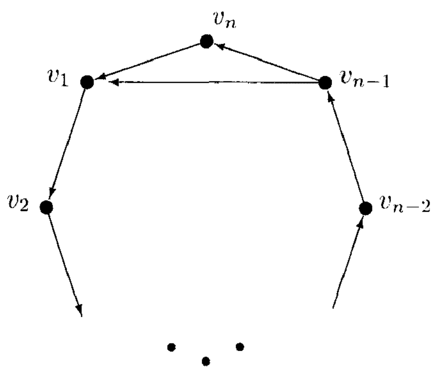

FIGURE 3. Wielandt digraph W_n of order n

We do not know a good upper bound of competition indices of general digraphs. But we do know that the maximum on competition indices is not attained by a Wielandt digraph whose index (exponent) is the maximum on indices of digraphs having same order.

3. Competition indices of strongly connected tournaments

An *n-tournament* T_n is a digraph with n vertices in which every pair of vertices is joined by exactly one arc. Assigning an orientation to each edge of complete graph results in a tournament. Let the score $s(v)$ be outdegree of v, i.e., the number of arcs from v. A vertex z is called a ***sink*** if $s(z) = 0$ in a tournament. There is at most one sink in a tournament. Example 1 shows competition indices of tournaments of order 3.

Example 1. We have only two 3-tournaments C and C' in Figure 4. And we have that $\text{cindex}(C) = 2$ and $\text{cindex}(C') = 1$.

FIGURE 4. All tournaments of order 3.

Theorem 3.1 (Moon and Pullman [13]). *An n-tournament T_n is primitive if and only if T_n is irreducible (strongly connected) and $n > 3$.*

Theorem 3.2 (Moon and Pullman [13]). *Each vertex of a strongly connected tournament T_n, when $n \geq 3$, is contained in at least one simple cycle of each length between 3 and n, inclusive.*

Theorem 3.3. *For any strongly connected n-tournament T_n, we have*

$$\text{cindex}(T_n) \leq 4.$$

Proof. It is trivial if $n \leq 2$. The digraph C' in Figure 4 is the only strongly connected 3-tournament. Thus if $n = 3$, we have

$$\text{cindex}(T_n) = 1.$$

Suppose $n > 3$. Then T_n is primitive by Theorem 3.1. Take two vertices x and y in T_n, there exists an arc between x and y. Without loss of generality, we may suppose that there is an arc (x, y). By Theorem 3.2, there exist a 3-cycle containing x and a 4-cycle containing y. Therefore, x and y have a 4-step common prey y. Vertices x and y have an m-step common prey when $m \geq 4$ since $s(y) \geq 1$. Thus competition index of T_n is at most 4. □

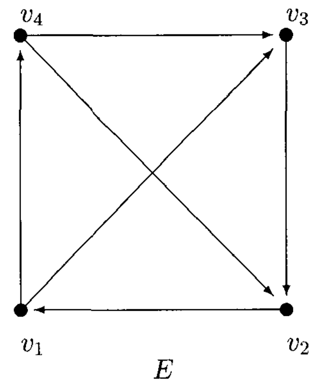

FIGURE 5. The competition index of E is 4.

This bound is sharp as it can be achieved by the strongly connected 4-tournament E given in Figure 5 as vertices v_1 and v_3 have a 4-step common prey v_3 while they do not have a 3-step common prey. Thus we have $\text{cindex}(E) = 4$.

Theorem 3.4. *For any strongly connected n-tournament T_n where $n \geq 5$, we have*

$$\text{cindex}(T_n) \leq 3$$

and the bound is sharp for all $n \geq 5$.

Proof. Since T_n is strongly connected, there is no sink. Thus, if two vertices have an l-step common prey, then they have an m-step common prey for $m \geq l$. Therefore, it suffices to show that every pair of vertices has a p-step common prey for some $p \in \{1, 2, 3\}$.

Take two vertices x and y. Without loss of generality, we may assume that there is an arc (x, y). Since T_n is strongly connected, $s(y) \geq 1$. We take an out-neighbor of y and denote it by z. If (x, z) is an arc of T_n, then z is a 1-step common prey of x and y. Thus we may assume that (z, x) is an arc of T_n. Suppose that $s(y) \geq 2$. Then there exists another out-neighbor $w(\neq z)$ of y. Since T_n is a tournament, either (z, w) or (w, z) is an arc of T_n. If (z, w) is an arc of T_n, then w is a 2-step common prey of x and y. If (w, z) is an arc of T_n, then z is a 2-step common prey of x and y. Thus, if $s(y) \geq 2$, then x and y have a 2-step common prey. Now suppose that z is the only out-neighbor of y. Suppose that $s(z) \geq 2$. Then there is an out-neighbor w of z other than x. Since $s(y) = 1$ and T_n is a tournament, there is an arc from w to y. Since T_n is a tournament, (x, w) or (w, x) is an arc of T_n. If (x, w) ia an arc of T_n, then w is a 3-step common prey of x and y. If (w, x) is an arc of T_n, then x is a 3-step common prey of x and y. Now it remains to consider the case where $s(z) = 1$, i.e., x is the only out-neighbor of z. By Theorem 3.2, there is a 5-cycle containing y since $n \geq 5$. Since z (resp. x) is the only out-neighbor of y (resp. z), every 5-cycle contains vertex sequence yzx. Thus a 5-cycle containing y has vertex sequence $yzxuvy$ for some vertices u, v distinct from x, y, z. Then there are two directed walks $xuvy$ and $yzxy$ and so y is a 3-step common prey of x and y. Therefore we have

$$\text{cindex}(T_n) \leq 3.$$

To show that the bound is sharp, define an n-tournament $F = (V, A)$ for $n \geq 5$ as follows:

$$\begin{aligned} V &= \{v_1, v_2, \ldots, v_n\}, \\ A &= \{(v_i, v_j) \mid i < j\} \backslash \{(v_1, v_n)\} \cup \{(v_n, v_1)\}. \end{aligned}$$

Since there is an n-cycle in F, F is strongly connected. And v_{n-1} and v_n have a 3-step common prey but do not have a 2-step common prey. Therefore, $\text{cindex}(F) = 3$. □

The out-neighborhood of v, denoted by $N^+(v)$, is the set of all vertices outgoing from v.

Example 2. Define an n-tournament $F' = (V, A)$ for $n \geq 5$ as follows:

$$\begin{aligned} V &= \{v_1, v_2, \ldots, v_n\}, \\ A &= \{(v_i, v_j) \mid i > j\} \backslash [\{(v_{i+1}, v_i) \mid i \leq n-1\} \cup \{(v_{i+2}, v_i) \mid i \leq n-2\}] \\ &\quad \cup \{(v_i, v_{i+1}) \mid i \leq n-1\} \cup \{(v_i, v_{i+2}) \mid i \leq n-2\}. \end{aligned}$$

Then F' is strongly connected since there is an n-cycle in F'. Take two vertices x and y in F'. We may assume that there exist an arc (x, y). Then we know $s(y) \geq 2$ in F'. Take two vertices $z, w \in N^+(y)$. Since F' is a tournament, either (z, w) or (w, z) is an arc of F'. If (z, w) is an arc, then w is a 2-step common prey of x and y. If (w, z) is an arc, then z is a 2-step common prey of

x and y. Thus, $\text{cindex}(F') \leq 2$. Since v_1 and v_3 have no 1-step common prey, we have $\text{cindex}(F') = 2$.

4. Competition indices of tournaments

In this section, we study competition indices of tournaments which are not strongly connected. If there is no sink in a tournament T_n, we have $\text{cperiod}(T_n) = 1$ by Theorem 2.2.

Theorem 4.1. *For any tournament T_n without sink, we have*

$$\text{cindex}(T_n) \leq 4,$$

and the bound is sharp for all $n \geq 4$.

Proof. Since there is no sink, it is true that if two vertices have an m-step common prey, then they have a $(m+1)$-step common prey. Therefore it is suffices to show that every pair of vertices has a p-step common prey for some $p \in \{1, 2, 3, 4\}$, or has no m-step common prey for any positive integer m. Suppose x and y be vertices that have an m-step common prey for some positive integer m. If x and y are in the same strong component, x and y have at least 4-step common prey by Theorem 3.3. In case where x and y are in different strong components, we may assume that there is an arc (x, y). Since there is no sink, there exists some vertex z such that (y, z). There is no arc (z, x) since we assume that x and y are in different strong components. Thus there is an arc (x, z) and so z is a 1-step common prey of vertices x and y. Therefore, we have

$$\text{cindex}(T_n) \leq 4.$$

Let us construct an n-tournament T such that $\text{cindex}(T) = 4$ for $n \geq 4$. Let $\{v_1, v_2, v_3, v_4, \ldots, v_n\}$ be the vertex set of T. Let E denote the subtournament induced by $\{v_1, v_2, v_3, v_4\}$ (see Figure 5) and other arcs (v_i, v_j) if and only if $i > j$ for $i \geq 5$. Then v_k and v_l have an m-step common prey for any positive integer m where $k \geq 5$. But v_1 and v_3 have a 4-step common prey but do not have a 3-step common prey. Thus we have $\text{cindex}(T) = 4$ and the given bound is sharp for all $n \geq 4$. □

Corollary 4.2. *There exists an n-tournament T_n $(n \geq 5)$ without sink such that* $\text{cindex}(T_n) = r$ *for $1 \leq r \leq 4$.*

Proof. The case where $r = 2, 3, 4$ is taken care of by Example 2, Theorem 3.4, and Theorem 4.1, respectively.

Define an n-tournament $H = (V, A)$ for $n \geq 4$ as follows:

$$\begin{aligned} V &= \{v_1, v_2, v_3, \ldots, v_n\}, \\ A &= \{(v_i, v_j) \mid i > j\} \setminus \{(v_3, v_1)\} \cup \{(v_1, v_3)\}. \end{aligned}$$

If either $k \geq 4$ and $l \in \{1, 2, \ldots, n\} \setminus \{k\}$ or $k \in \{1, 2, 3\}$ and $l \in \{1, 2, \ldots, n\} \setminus \{k\}$, v_k and v_l have an m-step common prey for any positive integer m. For $k, l \in$

$\{1,2,3\}$ such that $k \neq l$, v_k and v_l do not have an m-step common prey for any positive integer m. Thus we have $\text{cindex}(H) = 1$. □

An n-tournament S_n is *transitive* if its vertices can be labeled as $v_1, v_2, \ldots, v_n$ in such a way that there is an arc (v_i, v_j) if and only if $i < j$.

Lemma 4.3. *If S_n is an n-transitive tournament with $n \geq 3$, then we have*

$$\text{cindex}(S_n) = n - 1.$$

Proof. Since S_n is transitive, there is a vertex labeling $v_1, v_2, \ldots, v_n$ in S_n in such a way that there is an arc (v_i, v_j) if and only if $i < j$. Thus each pair of vertices has a 1-step common prey v_n. If $i < j < n$, then v_i and v_j have an m-step common prey for $m \leq n - j$, but have no m-step common prey for $m > n - j$. We can check that v_1 and v_2 have an $(n-2)$-step common prey but do not have an $(n-1)$-step common prey. Therefore, we have $\text{cindex}(S_n) = n - 1$. □

Lemma 4.4. *Let T_n be a tournament with a sink for an integer $n \geq 5$. Let (x, y) be an arc in T_n where y is not sink and S be the subtournament of T_n induced by $N^+(y) \cup \{y\}$. Then the following are true:*

(i) *If x and y are in the same strong component, then x and y have an l-step common prey for any positive integer l.*
(ii) *If x and y are in different strong components and S contains a non-trivial strong component, then x and y have an l-step common prey for any positive integer l.*
(iii) *If x and y are in different strong components and S contains no non-trivial strong component, then x and y have an $(m-1)$-step common prey but do not have an l-step common prey for any $l \geq m$ where $m = s(y) + 1$.*

Proof. Let z be a sink in T_n. Take two vertices x and y. Since z is a sink, there are arcs (x, z) and (y, z). Thus, z is a 1-step common prey of x and y. If x and y are in the same strong component, then x and y have an l-step common prey z for any positive integer l. Assume that x and y are in different components. Suppose that S contains a nontrivial strong component T. Let w be a vertex in T. Then there are directed walks xyz and ywz and so z is a 2-step common prey of x and y. Since T is a strongly connected tournament with at least 3 vertices, w is contained in a 3-cycle in T by Theorem 3.2. Thus, z is an l-step common prey of x and y for any $l \geq 2$. Suppose that S contains no nontrivial strong component. Then S is transitive and its order is $m = s(y) + 1$. Thus x and y have an $(m-1)$-step common prey but do not have an l-step common prey for any $l \geq m$. □

Theorem 4.5. *For any tournament T_n where $n \geq 5$, we have*

$$\text{cindex}(T_n) \leq n - 1.$$

Proof. If T_n has no sink, it holds by Theorem 4.1. Suppose T_n has a sink z. Take two vertices x, y. We may assume that there is an arc (x, y). Then $x \neq z$. If $y = z$, then x and y do not have an m-step common prey for any positive integer m. Now assume that $y \neq z$. By Lemma 4.4, x and y have an l-step common prey for any positive integer l, or x and y have an $(m-1)$-step common prey but do not have an l-step common prey for any $l \geq m$ where $m = s(y) + 1$. Thus we have $\text{cindex}(T_n) \leq n - 1$ since $s(y) \leq n - 2$. Lemma 4.3 shows that the bound is sharp. □

We call I_n *competition index set of tournaments with order* n where

$$I_n = \{\text{cindex}(T_n) \mid T_n \textit{ is } n\textit{-tournament}\}.$$

Theorem 4.6. *There is no n-tournament whose competition index* $(n-2)$ *where* $n > 6$ *and this is the only gap in the competition index set of tournaments with order n.*

Proof. Suppose there exists an n-tournament T_n such that $\text{cindex}(T_n) = n-2$. If T_n has no sink, then $\text{cindex}(T_n) \leq 4 < n-2$ by Theorem 4.1. Therefore T_n has one sink z. If there is no cycle in T_n, then T_n is transitive and $\text{cindex}(T_n) = n-1$ by Lemma 4.3. We may assume that T_n has a cycle. Then T_n has a cycle of length at least 3 since there is no 2-cycle and 1-cycle in a tournament. Take two vertices x and y. We may assume there is an arc (x, y). Then $x \neq z$ and we may assume that $y \neq z$. In cases (i) and (ii) of Lemma 4.4, we have x and y have an l-step common prey for any positive integer l. In case (iii) of Lemma 4.4, we have $s(y) \leq n-4$ since S contains no cycle in this case. Thus we have $\text{cindex}(T_n) \leq n-3$. It contradicts that $\text{cindex}(T_n) = n-2$. Therefore there is no tournament whose competition index is $n-2$.

There is an n-tournament H such that $\text{cindex}(H) = 1$, see Corollary 4.2. We construct a n-tournament T_n such that $\text{cindex}(T_n) = r$, where $2 \leq r \leq n-3$. Define an $T_n = (V, A)$ for $n > 6$ as follows:

$$\begin{aligned} V &= \{v_1, v_2, \ldots, v_{n-r}, \ldots, v_n\}, \\ A &= \{(v_i, v_j) \mid i < j\} \backslash \{(v_1, v_{n-r})\} \cup \{(v_{n-r}, v_1)\}. \end{aligned}$$

Then v_n is a sink. Take two vertices x and y. For $x, y \in \{v_1, \ldots, v_{n-r}\}$, x and y have an l-step common prey for any positive integer l by Lemma 4.4 (i). If either $x, y \in \{v_{n-r+1}, \ldots, v_n\}$, or $x \in \{v_1, \ldots, v_{n-r}\}$ and $y \in \{v_{n-r+1}, \ldots, v_n\}$, x and y do not have an l-step common prey for any $l \geq r$ by Lemma 4.4 (iii). Note that v_1 and v_{n-r+1} have an $(r-1)$-step common prey. Then we have $\text{cindex}(T_n) = r$. The proof is completed. □

5. Closing remarks

In this paper, we generalize the concept of the competition index of a digraph in terms of the notion of m-step competition graph in [7]. In addition, we study competition indices of tournaments.

Recently, studies on indices of digraphs have many important results and variations such as generalized indices (see [2, 3, 12, 15, 18]). Also, since Cohen introduced the notion of competition graph in 1968, various variations such as competition common enemy graph, niche graph, p-competition graph have been defined and studied by many authors (see [11] for survey of literature of competition graphs). Since the concept of competition index lies between index theory and competition graph theory, it may be possible to generalize the notion of competition index in both ways.

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