ON FACTORIZATION OF THE SUBGROUPS OF SELF-HOMOTOPY EQUIVALENCES

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ABSTRACT. For a pointed space $X$, the subgroups of self-homotopy equivalences $\text{Aut}_N(X), \text{Aut}_\Omega(X), \text{Aut}_*(X)$ and $\text{Aut}_{\Sigma}(X)$ are considered, where $\text{Aut}_N(X)$ is the group of all self-homotopy classes $f$ of $X$ such that $f_i = \text{id} : \pi_i(X) \to \pi_i(X)$ for all $i \leq N \leq \infty$, $\text{Aut}_\Omega(X)$ is the group of all the above $f$ such that $\Omega f = \text{id}$; $\text{Aut}_*(X)$ is the group of all self-homotopy classes $g$ of $X$ such that $g_i = \text{id} : H_i(X) \to H_i(X)$ for all $i \leq \infty$, $\text{Aut}_{\Sigma}(X)$ is the group of all the above $g$ such that $\Sigma g = \text{id}$. We will prove that $\text{Aut}_\Omega(X_1 \times \cdots \times X_n)$ has two factorizations similar to those of $\text{Aut}_N(X_1 \times \cdots \times X_n)$ in reference [10], and that $\text{Aut}_{\Sigma}(X_1 \vee \cdots \vee X_n)$, $\text{Aut}_*(X_1 \vee \cdots \vee X_n)$ also have factorizations being dual to the former two cases respectively.

1. Introduction

For a pointed space $X$, let $\text{Aut}(X)$ denote the set of homotopy classes of pointed self-maps of $X$ that are homotopy equivalences. This set is a group, called the group of self-homotopy equivalences, with respect to the operation induced by the composition of maps. For a survey of the literature about $\text{Aut}(X)$ and related concepts, see [1] or [13]. In this paper, we consider the subgroups of the group of self-homotopy equivalences.

For a pointed space $X$ and an integer $N$ with $\text{dim}X \leq N \leq \infty$, we define the subgroups $\text{Aut}_N(X)$ and $\text{Aut}_\Omega(X)$ of $\text{Aut}(X)$ by

$$\text{Aut}_N(X) = \{f \in \text{Aut}(X)|f_i = \text{id} : \pi_i(X) \to \pi_i(X) \text{ for all } i \leq N\}$$

and

$$\text{Aut}_\Omega(X) = \{f \in \text{Aut}(X)|\Omega f = \text{id}\},$$

where $f_i$ is the homomorphism on homotopy group induced by $f$ and $\Omega$ is the loop functor. Since the homomorphisms induced by $\Omega f$ on the homotopy groups of $\Omega X$ are the same (after a shift in dimension) as those induced by $f$ on the homotopy groups of $X$, $\text{Aut}_\Omega(X)$ is a subgroup of $\text{Aut}_N(X)$. The
group \( \text{Aut}_{\Omega}(X) \) has been studied by many authors, see [2, 3, 4, 5, 8, 10]. For example, in [5], Farjoun and Zabrodsky proved that the group \( \text{Aut}_{\Omega}(X) \) is nilpotent whenever \( X \) is a finite-dimensional CW-complex; in [8], Maruyama proved that, under the same assumption and given a set of primes \( P \), the natural map \( \text{Aut}_{\Omega}(X) \to \text{Aut}_{\Omega}(X_P) \) is the \( P \)-localization homomorphism of the nilpotent group \( \text{Aut}_{\Omega}(X) \). In particular, in [10], Pavešić proved that, for two pointed CW-complexes \( X \) and \( Y \), the self-equivalences in \( \text{Aut}_{\Omega}(X \times Y) \) are always reducible (see Section 2), hence \( \text{Aut}_{\Omega}(X \times Y) \) can be decomposed as the product of its two natural subgroups. This paves the way to decompose \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \) as the \( n \)-fold product of its certain subgroups. Furthermore, Pavešić developed another method named as LU factorization by which \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \) can be decomposed as the product of its only two subgroups. In Section 2, we will prove that the self-equivalences in \( \text{Aut}_{\Omega}(X \times Y) \) are also always reducible, so that \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \) has factorizations similar to those of \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \).

For a pointed space \( X \), we can also define the subgroups \( \text{Aut}_\ast(X) \) and \( \text{Aut}_\Sigma(X) \) of \( \text{Aut}(X) \) by

\[
\text{Aut}_\ast(X) = \{ g \in \text{Aut}(X) | g_\ast = id : H_i(X) \to H_i(X) \text{ for all } i \geq 0 \}
\]

and

\[
\text{Aut}_\Sigma(X) = \{ g \in \text{Aut}(X) | \Sigma g = id \},
\]

where \( g_\ast \) is the homomorphism on homology group induced by \( g \) and \( \Sigma \) is the suspension functor. Similar to \( \text{Aut}_{\Omega}(X) \) and \( \text{Aut}_{\Omega}(X) \), we can see that \( \text{Aut}_{\Sigma}(X) \) is a subgroup of \( \text{Aut}_\ast(X) \). In [5], Farjoun and Zabrodsky also proved that \( \text{Aut}_\ast(X) \) is nilpotent whenever \( X \) is a finite-dimensional CW-complex; in [9], Maruyama proved that the natural map \( \text{Aut}_\ast(X) \to \text{Aut}_\ast(X_P) \) is a \( P \)-localization homomorphism. In Section 3, we will prove that, for any pointed simply-connected CW-complexes \( X \) and \( Y \), the self-equivalences in \( \text{Aut}_\ast(X \vee Y) \) are always reducible (see Section 3), and for pointed simply-connected CW-complexes \( X_1, \ldots, X_n \), \( \text{Aut}_\ast(X_1 \vee \cdots \vee X_n) \) has factorizations dual to those of \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \). In Section 4, we consider the group \( \text{Aut}_\Sigma(X) \) and show that \( \text{Aut}_\Sigma(X) \) acts dually to \( \text{Aut}_{\Omega}(X) \).

2. \( \text{Aut}_{\Omega}(X_1 \times \cdots \times X_n) \)

The group \( \text{Aut}_{\Omega}(X) \) was first introduced by Felix and Murillo in [6], where they showed that \( \text{Aut}_{\Omega}(X) \) and \( \text{Aut}_{\Omega}(X) \) are generally different. In [11], Pavešić constructed a spectral sequence converging to \( \text{Aut}_{\Omega}(X) \) by Cartan-Eilenberg system. He proved that, if \( X \) is a \( \text{Co-H} \)-space, then \( \text{Aut}_{\Omega}(X) \) is trivial, and that if \( X \) is a Postnikov piece, then for any set of primes \( P \), the natural map \( \text{Aut}_{\Omega}(X) \to \text{Aut}_{\Omega}(X_P) \) is the \( P \)-localization.

In order to state our results in this section, the following notations and notions are needed. In this section, all the spaces are pointed connected CW-complexes. Let \( i_X \) and \( i_Y \) denote the inclusions of \( X \) and \( Y \) in \( X \times Y \), and
let \( p_X \) and \( p_Y \) be the projections of \( X \times Y \) on \( X \) and \( Y \). Given a self-map \( f : X \times Y \to X \times Y \) and \( I, J \in \{X, Y\} \), write \( f_I : X \times Y \to I \) for the composition \( f_I = p_I f \) (so that \( f \) is represented component-wise as \( f = (f_X, f_Y) \) by the universal property of product spaces), and write \( f_{IJ} : J \to I \) for the composition \( f_{IJ} = p_I f_{ij} \). A self-homotopy equivalence \( f \) of \( X \times Y \) is said to be reducible if \( f_{XX} \) and \( f_{YY} \) are self-homotopy equivalences of \( X \) and \( Y \) respectively.

Let \( \text{Aut}_X(X \times Y) = \{ f \in \text{Aut}(X \times Y) | p_X f = p_X \} \) and \( \text{Aut}_Y(X \times Y) = \{ g \in \text{Aut}(X \times Y) | p_Y g = p_Y \} \). In [12], Pavesić proved that \( \text{Aut}_X(X \times Y) \) and \( \text{Aut}_Y(X \times Y) \) are subgroups of \( \text{Aut}(X \times Y) \), and that if all the self-equivalences of \( X \times Y \) are reducible, then \( \text{Aut}(X \times Y) \) is the product of \( \text{Aut}_X(X \times Y) \) and \( \text{Aut}_Y(X \times Y) \), i.e.,

\[
\text{Aut}(X \times Y) = \text{Aut}_X(X \times Y) \text{Aut}_Y(X \times Y).
\]

However, when considered the group \( \text{Aut}_{2N}(X \times Y) \) in [10], Pavesić found that all the self-equivalences in \( \text{Aut}_{2N}(X \times Y) \) are always reducible without any restriction on \( X \) and \( Y \), hence there is a corresponding decomposition of \( \text{Aut}_{2N}(X \times Y) \), i.e.,

\[
\text{Aut}_{2N}(X \times Y) = \text{Aut}_{X_{2N}}(X \times Y) \text{Aut}_{Y_{2N}}(X \times Y),
\]

where \( \text{Aut}_{X_{2N}}(X \times Y) = \text{Aut}_X(X \times Y) \cap \text{Aut}_{2N}(X \times Y) \) and \( \text{Aut}_{Y_{2N}}(X \times Y) = \text{Aut}_Y(X \times Y) \cap \text{Aut}_{2N}(X \times Y) \). This paves the way for a generation of our approach to factorization of self-equivalences of products of more than two CW-complexes, i.e.,

\[
\text{Aut}_{2N}(X_1 \times \cdots \times X_n) = \prod_{i=1}^{n} \text{Aut}_{1_{2N}}(X_1 \times \cdots \times X_n),
\]

where \( \prod_i \) denotes the subproduct of \( X_1 \times \cdots \times X_n \) obtained by omitting \( X_i \), i.e., \( \prod_i = X_1 \times \cdots \times \hat{X}_i \times \cdots \times X_n \) (refer to [10]).

For the group \( \text{Aut}_\Omega(X \times Y) \), we will also prove that all its self-equivalences are always reducible, that is,

**Lemma 2.1.** For any \( f \in \text{Aut}_\Omega(X \times Y) \), we have \( f_{XX} \in \text{Aut}_\Omega(X) \) and \( f_{YY} \in \text{Aut}_\Omega(Y) \).

**Proof.** Since \( \text{Aut}_\Omega(X \times Y) \subseteq \text{Aut}_{2N}(X \times Y) \) and all the self-equivalences in \( \text{Aut}_{2N}(X \times Y) \) are reducible by Lemma 2.1 of [10], we have \( f_{XX} \in \text{Aut}_{2N}(X) \) and \( f_{YY} \in \text{Aut}_{2N}(Y) \). Since

\[
\Omega f_{XX} = \Omega(p_X f_i X) = (\Omega p_X)(\Omega f)(\Omega i_X) = \Omega(p_X i_X) = id,
\]

we can get \( f_{XX} \in \text{Aut}_\Omega(X) \) and similarly \( f_{YY} \in \text{Aut}_\Omega(Y) \). This shows the result. \( \square \)

Now we can derive the following factorization from Theorem 2.5 of [12].
Theorem 2.2.

\[ \text{Aut}_\Omega(X \times Y) = \text{Aut}_{X\Omega}(X \times Y) \text{Aut}_{Y\Omega}(X \times Y), \]

where \( \text{Aut}_{X\Omega}(X \times Y) = \text{Aut}_X(X \times Y) \cap \text{Aut}_{\Omega}(X \times Y) \) and \( \text{Aut}_{Y\Omega}(X \times Y) = \text{Aut}_Y(X \times Y) \cap \text{Aut}_{\Omega}(X \times Y) \) are the subgroups of \( \text{Aut}_{\Omega}(X \times Y) \).

Proof. Given any \( h \in \text{Aut}_{\Omega}(X \times Y) \subseteq \text{Aut}(X \times Y) \), since \( h \) is reducible, then by Theorem 2.5 of [12], \( h \) can be decomposed as

\[ h = (p_X, f)(g, p_Y) = (g, f(g, p_Y)), \]

where \((p_X, f) \in \text{Aut}_X(X \times Y) \) and \((g, p_Y) \in \text{Aut}_Y(X \times Y) \). And since

\[ \Omega(g, p_Y) = \Omega(h, p_Y) = (\Omega(p_X, h), \Omega p_Y) = (\Omega(p_X), \Omega p_Y) \]

by \( \Omega h = id \), this implies that \( \Omega(g, p_Y) \in \text{Aut}_{Y\Omega}(X \times Y) \) and then follows \((p_X, f) \in \text{Aut}_{X\Omega}(X \times Y) \). As \( \text{Aut}_{X\Omega}(X \times Y) \) and \( \text{Aut}_{Y\Omega}(X \times Y) \) have trivial intersection, the above factorization is unique and then follows the desired theorem. \( \square \)

By applying Theorem 2.2 and a completely same proof as that of Theorem 2.4 in [10], we can get the generalization of Theorem 2.2 as follows.

Theorem 2.3. \( \text{Aut}_\Omega(X_1 \times \cdots \times X_n) = \prod_{i=1}^n \text{Aut}_{\Omega_i}(X_1 \times \cdots \times X_n) \).

Also we can decompose \( \text{Aut}_\Omega(X_1 \times \cdots \times X_n) \) as the product of its only two subgroups as follows.

Theorem 2.4.

\[ \text{Aut}_\Omega(X_1 \times \cdots \times X_n) = L(X_1 \times \cdots \times X_n)U(X_1 \times \cdots \times X_n), \]

where \( L(X_1 \times \cdots \times X_n) = \{ f \in \text{Aut}_\Omega(X_1 \times \cdots \times X_n) | f_{x_k} = f_{x_k}l_k, k = 1, \ldots, n \} \), \( U(X_1 \times \cdots \times X_n) = \{ f \in \text{Aut}_\Omega(X_1 \times \cdots \times X_n) | f_{x_k} = f_{x_k}u_k, f_{x_k}x_k = id, k = 1, \ldots, n \} \); \( l_k \) and \( u_k \) are self-maps of \( X_1 \times \cdots \times X_n \) defined by \( l_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k, *, \ldots, *) \) and \( u_k(x_1, \ldots, x_n) = (*, \ldots, *, x_k, \ldots, x_n) \) respectively.

The above factorization is called LU factorization as in Theorem 3.2 of [10] and their proofs are similar.

3. \( \text{Aut}_* (X_1 \vee \cdots \vee X_n) \)

In this section, we will prove that for any pointed simply-connected CW-complexes \( X_1, \ldots, X_n \), \( \text{Aut}_* (X_1 \vee \cdots \vee X_n) \) has two factorizations dual to those of \( \text{Aut}_{\text{H}(N)(X_1 \times \cdots \times X_n)} \) as in [10].

Before stating our main results, we first fix some notions and notations. In this section, all the spaces are pointed simply-connected CW-complexes. For two spaces \( X \) and \( Y \), let \( i_X : X \to X \vee Y \) and \( i_Y : Y \to X \vee Y \) be inclusions; \( p_X : X \vee Y \to X \) and \( p_Y : X \vee Y \to Y \) be projections. Given a self-map \( f : X \vee Y \to X \vee Y \) and \( I, J \in \{X, Y\} \), we denote \( fi_I \) by \( fi_I \) (so that we have
$f = (f_X, f_Y)$ by the universal property of wedge spaces) and $p_I f_i$ by $f_{II}$. We say $f \in \text{Aut}(X \vee Y)$ is reducible if $f_{XX} \in \text{Aut}(X)$ and $f_{YY} \in \text{Aut}(Y)$.

We must mention that the above notations have the same forms as those defined in Section 2, but they have different senses in this section.

Let $\text{Aut}^X(X \vee Y) = \{ f \in \text{Aut}(X \vee Y) | f_X = i_X \}$ and $\text{Aut}^Y(X \vee Y) = \{ g \in \text{Aut}(X \vee Y) | g_Y = i_Y \}$. In [14], H. B. Yu and W. H. Shen proved the following theorem:

**Theorem 3.1** ([14]). $\text{Aut}^X(X \vee Y)$ and $\text{Aut}^Y(X \vee Y)$ are subgroups of $\text{Aut}(X \vee Y)$, and that if all the self-equivalences in $\text{Aut}(X \vee Y)$ are reducible, then $\text{Aut}(X \vee Y) = \text{Aut}^X(X \vee Y) \text{Aut}^Y(X \vee Y)$.

In the follows, we will prove that all the self-equivalences in $\text{Aut}_*(X \vee Y)$ are always reducible, and then we use this result to derive factorizations of $\text{Aut}_*(X \vee Y)$ and their generalizations to $\text{Aut}_*(X_1 \vee \cdots \vee X_n)$.

**Lemma 3.2.** For any $f \in \text{Aut}_*(X \vee Y)$, we have $f_{XX} \in \text{Aut}_*(X)$ and $f_{YY} \in \text{Aut}_*(Y)$, which implies that all the self-equivalences in $\text{Aut}_*(X \vee Y)$ are reducible.

**Proof.** Given any $f \in \text{Aut}_*(X \vee Y)$, its induced endomorphism on $H_i(X) \oplus H_i(Y)$ is $H_i(f) : H_i(X) \oplus H_i(Y) \to H_i(X) \oplus H_i(Y)$ which can be represented by the following $2 \times 2$-matrix according to [14]:

$$H_i(f) = \begin{pmatrix} H_i(f_{XX}) & H_i(f_{XY}) \\ H_i(f_{YX}) & H_i(f_{YY}) \end{pmatrix}.$$

Since $f$ is in $\text{Aut}_*(X \vee Y)$, then we have $H_i(f) = 1_{H_i(X) \oplus H_i(Y)}$. It follows that $H_i(f_{XX}) = 1_{H_i(X)}$ and $H_i(f_{YY}) = 1_{H_i(Y)}$. By Whitehead theorem, we know that $f_{XX} \in \text{Aut}_*(X)$ and $f_{YY} \in \text{Aut}_*(Y)$.

Let $\text{Aut}^X(X \vee Y) = \text{Aut}_*(X \vee Y) \cap \text{Aut}^X(X \vee Y)$. Then we have

**Lemma 3.3.** For any $f \in \text{Aut}_*(X \vee Y)$, we have $(i_X, f_Y) \in \text{Aut}^X(X \vee Y)$.

**Proof.** The endomorphism on $H_i(X) \oplus H_i(Y)$ induced by $(i_X, f_Y)$ is $H_i(i_X, f_Y) : H_i(X) \oplus H_i(Y) \to H_i(X) \oplus H_i(Y)$ which can be represented by the following $2 \times 2$-matrix according to [14]:

$$H_i(i_X, f_Y) = \begin{pmatrix} 1_{H_i(X)} & H_i(f_{XY}) \\ 0 & H_i(f_{YY}) \end{pmatrix}.$$

Since $f \in \text{Aut}_*(X \vee Y)$ is reducible by Lemma 3.2, we have $H_i(f_{XX}) = 0$ and $H_i(f_{YY}) = 1_{H_i(Y)}$ which means that $H_i(i_X, f_Y) = 1_{H_i(X) \oplus H_i(Y)}$. By Whitehead theorem, we know that $i_X, f_Y \in \text{Aut}_*(X \vee Y)$.

Similarly, we let $\text{Aut}^Y(X \vee Y) = \text{Aut}_*(X \vee Y) \cap \text{Aut}^Y(X \vee Y)$, then we can derive a factorization of $\text{Aut}_*(X \vee Y)$ dual to that of $\text{Aut}^X_*(X \vee Y)$.

**Theorem 3.4.** $\text{Aut}_*(X \vee Y) = \text{Aut}^X_*(X \vee Y) \text{Aut}^Y_*(X \vee Y)$.
Proof. For any \( h \in \text{Aut}_*(X \vee Y) \), \( h \) is reducible according to Lemma 3.2. By the Theorem 3.1, \( h \) has a unique factorization \( h = (i_X, f)(g, i_Y) = ((i_X, f)g, f) \), where \( (i_X, f) \in \text{Aut}_X^*(X \vee Y) \) and \( (g, i_Y) \in \text{Aut}_Y^*(X \vee Y) \). Then by Lemma 3.3, we have \( (i_X, h_Y) = (i_X, f) \in \text{Aut}_X^*(X \vee Y) \). Since \( h \) and \( (i_X, f) \) are both in \( \text{Aut}_*(X \vee Y) \), we have \( (g, i_Y) \in \text{Aut}_*(X \vee Y) \), i.e., \( (g, i_Y) \in \text{Aut}_Y^*(X \vee Y) \). This shows the result. \( \square \)

Actually, the above \( \text{Aut}_X^*(X \vee Y) \) and \( \text{Aut}_Y^*(X \vee Y) \) can be further decomposed. If we let \( \text{Aut}_X^Y(X \vee Y) = \{ f \in \text{Aut}_X^*(X \vee Y) | p_Y f = p_Y \} \) and \( \text{Aut}_Y^X(X \vee Y) = \{ f \in \text{Aut}_Y^*(X \vee Y) | p_X f = p_X \} \), then we have the following result:

**Proposition 3.5.** \( \text{Aut}_*^X(X \vee Y) \) is the semi-direct product of \( \text{Aut}_*(Y) \) and \( \text{Aut}_*^X(X \vee Y) \).

Proof. Define a map \( \phi_Y : \text{Aut}_*^X(X \vee Y) \to \text{Aut}(Y) \) by \( \phi_Y(i_X, f_Y) = p_Y f_Y = f_Y Y \), it is easy to verify that \( \text{Ker} \phi_Y = \text{Aut}_*^X(X \vee Y) \). We begin to prove that \( \phi_Y \) is a homomorphism.

Since
\[
p_Y f_Y p_Y = (p_Y f_Y p_Y i_X, p_Y f_Y p_Y i_Y) = (p_Y i_X, p_Y f_Y) = p_Y (i_X, f_Y),
\]
then for any \((i_X, f_Y), (i_X, g_Y) \in \text{Aut}_*^X(X \vee Y)\), we have
\[
\phi_Y((i_X, f_Y)(i_X, g_Y)) = p_Y (i_X, f_Y)(i_X, g_Y)i_Y = p_Y f_Y p_Y g_Y = \phi_Y(i_X, f_Y) \phi_Y(i_X, g_Y).
\]

It follows that \( \phi_Y \) is a homomorphism which has a right inverse
\[
\psi_Y : \text{Aut}(Y) \to \text{Aut}_*^X(X \vee Y)
\]
given by \( \psi_Y(f) = (i_X, i_Y f) \), where \( f \in \text{Aut}(Y) \). Then we have the following split short exact sequence:
\[
0 \to \text{Aut}_*^X(X \vee Y) \xrightarrow{\phi_Y} \text{Aut}_*^X(X \vee Y) \to \text{Aut}_*(Y) \to 0.
\]
It means that \( \text{Aut}_*^X(X \vee Y) \) is the semi-direct product of \( \text{Aut}_*(Y) \) and \( \text{Aut}_*^X(X \vee Y) \). \( \square \)

Similarly, we can prove that \( \text{Aut}_*^Y(X \vee Y) \) is also the semi-direct product of \( \text{Aut}_*(X) \) and \( \text{Aut}_*^Y(X \vee Y) \).

We can now apply inductively Theorem 3.4 to obtain a factorization of \( \text{Aut}_*(X_1 \vee \cdots \vee X_n) \). First, we need the following lemma:

**Lemma 3.6.** \( \text{Aut}_*^X(X \vee Y \vee Z) = \text{Aut}_*^{XY}(X \vee Y \vee Z) \text{Aut}_*^{X \vee Z}(X \vee Y \vee Z) \).
Proof. Since Aut\(^{X \vee Y}(X \vee Y \vee Z) \cap Aut\(^{X \vee Z}(X \vee Y \vee Z) = id\), it is sufficient to prove that for any \( f \in Aut\(^{X}(X \vee Y \vee Z) \), we have \( f = gh \), where \( g \in Aut\(^{X \vee Y}(X \vee Y \vee Z) \) and \( h \in Aut\(^{X \vee Z}(X \vee Y \vee Z) \).

Since \( f \in Aut\(_{\ast}(X \vee Y \vee Z) \) can be represented by \((i_{X}, f_{Y}, f_{Z})\), by Lemma 3.3 we have \((i_{X}, i_{Y}, f_{Z}) \in Aut\(_{\ast}(X \vee Y \vee Z), i.e., \((i_{X}, i_{Y}, f_{Z}) \in Aut\(_{\ast}(X \vee Y \vee Z) \) and \((i_{X}, f_{Y}, i_{Z}) \in Aut\(_{\ast}(X \vee Y \vee Z). \) It follows that

\[
(i_{X}, f_{Y}, f_{Z}) = (i_{X}, i_{Y}, f_{Z})(i_{X}, (i_{X}, i_{Y}, f_{Z})^{-1}f_{Y}, i_{Z}).
\]

Since \((i_{X}, i_{Y}, f_{Z})^{-1} (i_{X}, f_{Y}, i_{Z}) = (i_{X}, (i_{X}, i_{Y}, f_{Z})^{-1}f_{Y}, (i_{X}, i_{Y}, f_{Z})^{-1}i_{Z})\) is also in \( Aut\(_{\ast}(X \vee Y \vee Z), \) also by Lemma 3.3 we get that \((i_{X}, (i_{X}, i_{Y}, f_{Z})^{-1}f_{Y}, i_{Z}) \in Aut\(_{\ast}(X \vee Y \vee Z), i.e.,

\[
(i_{X}, (i_{X}, i_{Y}, f_{Z})^{-1}f_{Y}, i_{Z}) \in Aut\(_{\ast}(X \vee Y \vee Z).
\]

This shows the result. \( \square \)

For pointed simply-connected CW complexes \( X_{1}, \ldots, X_{n} \), let \( \vee \) denote \( X_{1} \vee \ldots \vee X_{i} \vee \ldots \vee X_{n} \), where \( \check{X}_{i} \) means that \( X_{i} \) is omitted. Then we have the generalization of Theorem 3.4 as follows.

**Theorem 3.7.** \( Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}) = \prod_{i=1}^{n} Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}). \)

**Proof.** By Lemma 3.6, for \( k = 2, 3, \ldots, n \), we have

\[
Aut\(_{\ast}^{X_{1} \vee \ldots \vee X_{k-1}}(X_{1} \vee \ldots \vee X_{n}) = Aut\(_{\ast}^{X_{1} \vee \ldots \vee X_{k}}(X_{1} \vee \ldots \vee X_{n}) Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}).
\]

Then by Theorem 3.4, we get

\[
Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n})
\]

\[
= Aut\(_{\ast}^{X_{1}}(X_{1} \vee \ldots \vee X_{n}) Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n})
\]

\[
= Aut\(_{\ast}^{X_{1}}(X_{1} \vee \ldots \vee X_{n}) Aut\(_{\ast}(X_{2} \vee \ldots \vee X_{n}) Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n})
\]

\[
= \ldots
\]

\[
= Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}) \cdots Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n})
\]

\[
= \prod_{i=1}^{n} Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}).
\]

This shows our result. \( \square \)

Similar to Proposition 3.5, every \( Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}) \) can also be further decomposed as follows.

**Proposition 3.8.** \( Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}) \) is the semi-direct product of \( Aut\(_{\ast}(X_{1}) \) and \( Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}). \)

We now turn to give another factorization of \( Aut\(_{\ast}(X_{1} \vee \ldots \vee X_{n}) \) which is dual to that of \( Aut\(_{\ast}(X_{1} \times \ldots \times X_{n}) \) named as LU factorization.

We first fix some notations. For a self-map \( f : X_{1} \vee \ldots \vee X_{n} \to X_{1} \vee \ldots \vee X_{n} \), let \( f_{X_{k}} = fi_{X_{k}} \) and \( f_{X_{k}X_{k}} = p_{X_{k}}fi_{X_{k}} \). Moreover, let \( \phi_{k} \) and \( \psi_{k} \) be two self-maps of \( X_{1} \vee \ldots \vee X_{n} \) defined by
\[
\phi_k(*, \ldots, *, x_i, *, \ldots, *) = \begin{cases} 
(*, \ldots, *, x_i, *, \ldots, *), & i \leq k \\
* & i > k
\end{cases}
\]
and
\[
\psi_k(*, \ldots, *, x_i, *, \ldots, *) = \begin{cases} 
(*, \ldots, *, x_i, *, \ldots, *), & i \geq k \\
* & i < k
\end{cases}
\]
respectively, where \(x_i \in X_i\). It is easy to verify that \(\phi_k \psi_j = \psi_j \phi_k = \phi_{\min\{k,j\}}\) and \(\psi_k \psi_j = \psi_j \psi_k = \psi_{\max\{k,j\}}\).

We now define the factors of our new factorization. For any \(X_1, \ldots, X_n\), let
\[
\Phi(X_1, \ldots, X_n) = \{ f \in \text{Aut}_*(X_1 \vee \cdots \vee X_n) \mid f_k = \phi_k f_k, k = 1, \ldots, n \}
\]
and
\[
\Psi(X_1, \ldots, X_n) = \{ f \in \text{Aut}_*(X_1 \vee \cdots \vee X_n) \mid f_k = \psi_k f_k, f_{kk} = id, k = 1, \ldots, n \}.
\]
Defining relations for \(\Phi(X_1, \ldots, X_n)\) and \(\Psi(X_1, \ldots, X_n)\) are non-additive analogues of relations that define upper-triangular matrices and lower-triangular matrices respectively. Indeed, if we identify formally an element \(f \in \text{Aut}_*(X_1 \vee \cdots \vee X_n)\) with a \(n \times n\)-matrix \((f_{jk})\) with entries \(f_{jk}\), then the elements of \(\Phi(X_1, \ldots, X_n)\) yield upper-triangular matrices and those of \(\Psi(X_1, \ldots, X_n)\) yield lower-triangular matrices with identities on the diagonal entries.

**Proposition 3.9.** \(\Phi(X_1, \ldots, X_n) \subseteq \Phi(X_1 \vee X_2, \ldots, X_n) \subseteq \cdots \subseteq \Phi(X_1 \vee \cdots \vee X_n) = \text{Aut}_*(X_1 \vee \cdots \vee X_n)\) and \(\Psi(X_1, \ldots, X_n) \supseteq \Psi(X_1 \vee X_2, \ldots, X_n) \supseteq \cdots \supseteq \Psi(X_1 \vee \cdots \vee X_n) = \{ld\}\).

**Proof.** For any \(f \in \Phi(X_1 \vee \cdots \vee X_{k-1}, X_k, \ldots, X_n)\), \(k = 2, \ldots, n\), in order to prove that \(f \in \Phi(X_1 \vee \cdots \vee X_k, X_{k+1}, \ldots, X_n)\), it is obviously sufficient to prove that \(\phi_{X_1 \vee \cdots \vee X_k} f_i x_i \cdots x_k = f_i x_i \cdots x_k\), i.e., \(\phi_{X_1 \vee \cdots \vee X_k} (f_1, \ldots, f_k) = (f_1, \ldots, f_k)\), where \(f_j = f_i x_i\).

For \(j < k\), we have
\[
\phi_{X_1 \vee \cdots \vee X_k} (f_1, \ldots, f_k) i x_i = \phi_{X_1 \vee \cdots \vee X_k} f_j = \phi_{X_1 \vee \cdots \vee X_{k-1}} (f_1, \ldots, f_{k-1}) i x_i = (f_1, \ldots, f_{k-1}) i x_i
\]
and for \(j = k\), we have \(\phi_{X_1 \vee \cdots \vee X_k} (f_1, \ldots, f_k) i x_k = (f_1, \ldots, f_k) i x_k\) for \(f \in \Phi(X_1 \vee \cdots \vee X_{k-1}, X_k, \ldots, X_n)\). By the universal property of wedge spaces, we see that \(\phi_{X_1 \vee \cdots \vee X_k} (f_1, \ldots, f_k) = (f_1, \ldots, f_k)\). This finishes the proof of the first inclusions. The second inclusions can be proved similarly. \(\square\)

**Proposition 3.10.** \(\Phi(X_1, \ldots, X_n)\) and \(\Psi(X_1, \ldots, X_n)\) are both subgroups of \(\text{Aut}_*(X_1 \vee \cdots \vee X_n)\).

**Proof.** Let \(f, g \in \Phi(X_1, \ldots, X_n)\), then \(fi_j = f_j = \phi_j f_j\), where \(i_j = i x_j\). Therefore for \(j \leq k\), we have
\[
fi_j = \phi_j f_j = \phi_k \phi_j f_j = \phi_k f_i j.
\]
Since \( \phi_k i_j = i_j \), we have \( f \phi_k i_j = \phi_k f \phi_k i_j \) according to the above result. Then by the universal property of wedge spaces, we have \( f \phi_k = \phi_k f \phi_k \) and then
\[
fgi_k = f \phi_k g i_k = \phi_k f \phi_k g i_k = \phi_k f g i_k.
\]
It follows that \( fg \in \Phi(X_1, \ldots, X_n) \), hence \( \Phi(X_1, \ldots, X_n) \) is closed under multiplication.

In the follows, we prove inductively that for any \( f \in \Phi(X_1, \ldots, X_n) \), we have \( f^{-1} \in \Phi(X_1, \ldots, X_n) \).

When \( n = 2 \), \( f \in \Phi(X_1, X_2) \) can be decomposed as
\[
f = [f(i_1 f_{11}^{-1}, i_2)](i_1 f_{11}, i_2) = [f(i_1 f_{11}^{-1}, i_2)][i_1 f_{11}, i_2].
\]
Since
\[
f(i_1 f_{11}^{-1}, i_2) = f(i_1 f_{11}^{-1}) = \phi_1 f_1 f_{11}^{-1} = i_1 p_1 f_1 f_{11}^{-1} = i_1 f_{11} f_{11}^{-1} = i_1,
\]
we have \( f(i_1 f_{11}^{-1}, i_2) \in \text{Aut}^X_*(X_1 \vee X_2) \subseteq \Phi(X_1, X_2) \). Since \( \text{Aut}^X_*(X_1 \vee X_2) \) is a group, we have
\[
[f(i_1 f_{11}^{-1}, i_2)]^{-1} = (i_1 f_{11}, i_2)f^{-1} \in \text{Aut}^X_*(X_1 \vee X_2) \subseteq \Phi(X_1, X_2).
\]
It is easy to verify that \( (i_1 f_{11}^{-1}, i_2) \in \Phi(X_1, X_2) \), so we have
\[
f^{-1} = (i_1 f_{11}^{-1}, i_2)(i_1 f_{11}, i_2)f^{-1} \in \Phi(X_1, X_2).
\]
This means that \( \Phi(X_1, X_2) \) is closed under formation of inverses.

Suppose that \( \Phi(X_1', \ldots, X_n') \) is closed under formation of inverses for any simply connected CW-complexes \( X_1', \ldots, X_{n-1}' \) with \( n > 2 \). Then given any \( f \in \Phi(X_1, \ldots, X_n) \subseteq \Phi(X_1 \vee X_2, X_3, \ldots, X_n) \) (by Proposition 3.9), we get \( f^{-1} \in \Phi(X_1 \vee X_2, X_3, \ldots, X_n) \) according to the inductive step. It remains to prove that \( f^{-1} i_1 = \phi_1 f^{-1} i_1 \), or equivalently that \( p_{X_1 \vee X_2} f^{-1} i_{X_1 \vee X_2} \in \Phi(X_1, X_2) \).

Since \( p_{X_1 \vee X_2} f i_{X_1 \vee X_2} \in \Phi(X_1, X_2) \) which is a group, it follows that
\[
p_{X_1 \vee X_2} f^{-1} i_{X_1 \vee X_2} = (p_{X_1 \vee X_2} f i_{X_1 \vee X_2})^{-1} \in \Phi(X_1, X_2).
\]
According to the above results, we see that \( \Phi(X_1, \ldots, X_n) \) is a group.

Similarly, we can prove that \( \Psi(X_1, \ldots, X_n) \) is also a group. \( \square \)

Now we can prove our main result as follows.

**Theorem 3.11.** \( \text{Aut}_*(X_1 \vee \cdots \vee X_n) = \Psi(X_1, \ldots, X_n) \Phi(X_1, \ldots, X_n) \).

**Proof.** Since \( \Psi(X_1, \ldots, X_n) \cap \Phi(X_1, \ldots, X_n) = id \), it is sufficient to prove inductively the existence of the factorization of the self-equivalence in \( \text{Aut}_*(X_1 \vee \cdots \vee X_n) \).

When \( n = 2 \), any \( f \in \text{Aut}_*(X_1 \vee X_2) \) can be decomposed as
\[
f = [(f_1, i_2)(i_1 f_{11}^{-1}, i_2)][(i_1 f_{11}, i_2)(f_1, i_2)^{-1} f].
\]
For \( (f_1, i_2)(i_1 f_{11}^{-1}, i_2) \in \text{Aut}^{X_2}_*(X_1 \vee X_2) \), since
\[
p_1(f_1, i_2)(i_1 f_{11}^{-1}, i_2)i_1 = p_1(f_1, i_2)i_1 f_{11}^{-1} = f_{11} f_{11}^{-1} = id_{X_1},
\]
than \( (f_1, i_2)(i_1 f_{11}^{-1}, i_2) \in \Psi(X_1, X_2) \).
From \((f_1, i_2)^{-1}(f_1, i_2) = ((f_1, i_2)^{-1}f_1, i_2) = \text{id}_{X_1 \vee X_2} = (i_1, i_2)\), we get \((f_1, i_2)^{-1}f_1 = i_1\). It follows that
\[(f_1, i_2)^{-1}f = (f_1, i_2)^{-1}(f_1, f_2) = ((f_1, i_2)^{-1}f_1, (f_1, i_2)^{-1}f_2) = (i_1, (f_1, i_2)^{-1}f_2)\]
Then we have \((f_1, i_2)^{-1}f \in \text{Aut}_{\text{d}}^X(X_1 \vee X_2) \subseteq \Phi(X_1, X_2)\). Since obviously \((i_1f_{11}, i_2) \in \Phi(X_1, X_2)\), we have \((i_1f_{11}, i_2)(f_1, i_2)^{-1}f \in \Phi(X_1, X_2)\).

For any \(f \in \text{Aut}_{\text{d}}(X_1 \vee \cdots \vee X_n) = \text{Aut}_{\text{d}}((X_1 \vee X_2) \vee X_3 \vee \cdots \vee X_n)\), we can assume inductively that \(f = f' f''\), where \(f' \in \Phi(X_1 \vee X_2, X_3, \ldots, X_n)\) and \(f'' \in \Phi(X_1 \vee X_2, X_3, \ldots, X_n)\).

Let \(p_{i_2} = p_{X_1 \vee X_2}, i_{12} = i_{X_1 \vee X_2}\) and define \(\bar{f} := p_{i_2} f'' i_{12}\). Since \(f''\) is reducible, we have \(f \in \text{Aut}_{\text{d}}(X_1 \vee X_2)\) and then \(\bar{f} = \psi \phi\), where \(\psi \in \Phi(X_1, X_2)\) and \(\phi \in \Phi(X_1, X_2)\). Define \(\bar{\psi} := (i_{12} \psi, i_3, \ldots, i_n)\), then \(\bar{\psi} \in \Psi(X_1, \ldots, X_n) \cap \Phi(X_1 \vee X_2, X_3, \ldots, X_n)\). Since \(f'' \in \Phi(X_1 \vee X_2, X_3, \ldots, X_n)\), we have \(\bar{\psi}^{-1} f'' i_{12} = \phi_{X_1 \vee X_2} f'' i_{12} = i_{12} p_{i_2} f'' i_{12}\). Then for \(\bar{\psi}^{-1} f'' \in \Phi(X_1 \vee X_2, X_3, \ldots, X_n)\), we have
\[p_{i_2} \bar{\psi}^{-1} f'' i_{12} = (p_{i_2} \bar{\psi}^{-1} i_{12})(p_{i_2} f'' i_{12}) = \psi^{-1} \bar{f} = \phi \in \Phi(X_1, X_2)\).

This implies that \(\bar{\psi}^{-1} f'' \in \Phi(X_1, \ldots, X_n)\). It follows that \(f = (f' \bar{\psi}) (\bar{\psi}^{-1} f'')\),
where \(f' \bar{\psi} \in \Psi(X_1, \ldots, X_n)\) and \(\bar{\psi}^{-1} f'' \in \Phi(X_1, \ldots, X_n)\). This finishes the proof of the theorem.

\[\square\]

4. \(\text{Aut}_{\Sigma}(X)\)

As a subgroup of \(\text{Aut}_{\text{d}}(X)\), \(\text{Aut}_{\Sigma}(X)\) is dual to \(\text{Aut}_{\Omega}(X)\). However, we can not find \(\text{Aut}_{\Sigma}(X)\) appears in any other reference. So in the follows, we will simply describe the general property of \(\text{Aut}_{\Sigma}(X)\) and also list some problems related to it.

First we give a characterization of \(\text{Aut}_{\Sigma}(X)\) as follows.

**Proposition 4.1.** For any pointed space \(X, f \in \text{Aut}_{\Sigma}(X)\) if and only if \(f \in \text{Aut}(X)\) and \(f^* = \text{id} : [X, \Omega Y] \rightarrow [X, \Omega Y]\) for every pointed space \(Y\).

**Proof.** \((\Rightarrow)\) Suppose that \(f \in \text{Aut}_{\Sigma}(X)\), then for any pointed space \(Y\) and \(g \in [X, \Omega Y]\), we have \(\hat{g}(\Sigma f) = \hat{g} f = \text{id}\), where \(\hat{g} : \Sigma X \rightarrow Y\) is the adjoint of \(g\). Take the adjoint of the equation, we have \(g f = g\), i.e., \(f^*(g) = g\). This implies that \(f^* = \text{id}\).

\((\Leftarrow)\) Given any \(f \in \text{Aut}(X)\) such that \(f^* = \text{id} : [X, \Omega Y] \rightarrow [X, \Omega Y]\) for every pointed space, we take \(Y = \Sigma X\) and \(\alpha : X \rightarrow \Omega \Sigma X\) be the adjoint of \(\text{id} : \Sigma X \rightarrow \Sigma X\), then we have \(f^*(\alpha) = \alpha f = \alpha\). By taking adjoint, we get \(\Sigma f = \text{id}\) which implies that \(f \in \text{Aut}_{\Sigma}(X)\).

\(\square\)

In [5], Pavešić proved that if \(X\) is a Co-H-space, then the group \(\text{Aut}_{\Omega}(X)\) is trivial. Dually, we have the following result:

**Corollary 4.2.** If \(X\) is a H-space, then \(\text{Aut}_{\Sigma}(X)\) is trivial.
Proof. Since $X$ is a $H$-space, $X$ is a retract of $\Omega Y$ for some space $Y$ (see p.201 of [7]). Then there exist maps $r : \Omega Y \to X$ and $i : X \to \Omega Y$ such that $ri = id$. Given any $f \in \text{Aut}_\Sigma(X)$, we have $f^*(i) = if = i$ by Proposition 4.1. By applying $r$ to both sides, we get $f = id$ which shows that $\text{Aut}_\Sigma(X)$ is trivial. \hfill \Box

In [11], Pavešić asked that if there is a finite CW-complex $X$ such that $\text{Aut}_\Omega(X) \neq \text{Aut}_{\Sigma} X$). Since $\text{Aut}_\Omega(X)$ is trivial when $X$ is a Co-$H$-space, we may find a finite Co-$H$-space $X$ such that $\text{Aut}_{\Sigma} X \neq \{id\}$. Dually for $\text{Aut}_{\Sigma}(X)$, we have a conjecture as follows.

**Conjecture 4.3.** There is a finite CW complex $X$ such that $\text{Aut}_{\Sigma}(X) \neq \text{Aut}_\Sigma(X)$. By Corollary 4.2, a possible approach to Conjecture 4.3 is to find a finite $H$-space $X$ such that $\text{Aut}_\Sigma(X) \neq \{id\}$.

In [6], Felix and Murillo showed that for pointed CW-complex $X$, $\text{Aut}_\Omega(X)$ is a nilpotent group and its order of nilpotency is bounded by the Ljusternik-Schnirelman category of $X$. Naturally we have the following conjecture:

**Conjecture 4.4.** For any pointed CW-complex $X$, $\text{Aut}_\Sigma(X)$ is a nilpotent group, and its order of nilpotency is bounded by the Ljusternik-Schnirelman cocategory of $X$.

According to the theorem of Maruyama [9], if the above conjecture is correct, then we will ask whether the natural map $\text{Aut}_\Sigma(X) \to \text{Aut}_\Sigma(X_F)$ is a $P$-localization for any set of primes $P$.

Now we turn to the factorization of $\text{Aut}_\Sigma(X_1 \vee \cdots \vee X_n)$ for any pointed simply-connected CW-complexes $X_1, \ldots, X_n$. Since we have already proved that, for pointed simply-connected CW-complexes $X$ and $Y$, all the self-equivalences in $\text{Aut}_\Sigma(X \vee Y)$ are always reducible (see Lemma 3.2), so by a similar proof to that of Lemma 2.1, we have

**Proposition 4.5.** For pointed simply-connected CW-complexes $X$ and $Y$, given any $f \in \text{Aut}_\Sigma(X \vee Y)$, we have $f_{XX} \in \text{Aut}_\Sigma(X)$ and $f_{YY} \in \text{Aut}_\Sigma(Y)$.

This enables us to get the following theorem by a proof similar to that of Theorem 3.7.

**Theorem 4.6.** $\text{Aut}_\Sigma(X_1 \vee \cdots \vee X_n) = \prod_{i=1}^n \text{Aut}_\Sigma^{\psi_i}(X_1 \vee \cdots \vee X_n)$.

Also we can decompose $\text{Aut}_\Sigma(X_1 \vee \cdots \vee X_n)$ as the product of its only two subgroups similarly to Theorem 3.11.

**Theorem 4.7.** $\text{Aut}_\Sigma(X_1 \vee \cdots \vee X_n) = \Psi'(X_1, \ldots, X_n) \Phi'(X_1, \ldots, X_n)$, where $\Psi'(X_1, \ldots, X_n)$ and $\Phi'(X_1, \ldots, X_n)$ are defined similarly to $\Psi(X_1, \ldots, X_n)$ and $\Phi(X_1, \ldots, X_n)$ in Section 3 respectively.

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