REAL HYPERSURFACES IN COMPLEX TWO-PLANE GRASSMANNIANS WITH LIE $\xi$-PARALLEL NORMAL JACOBI OPERATOR

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ABSTRACT. In this paper we give some non-existence theorems for real hypersurfaces in complex two-plane Grassmannians $G_2(C^{n+2})$ with Lie $\xi$-parallel normal Jacobi operator $\hat{R}_N$ and another geometric conditions.

0. Introduction

In the geometry of real hypersurfaces in complex space forms $M_n(c)$ or in quaternionic space forms $Q_n(c)$ Kimura [7] (resp. Pèrez [10]) has classified real hypersurfaces in $M_n(c)$ and (resp. in $Q_n(c)$) with commuting Ricci tensor, that is, $S\phi = \phi S$, (resp. $S\phi_i = \phi_i S$, $i = 1, 2, 3$) where $S$ and $\phi$ (resp. $S$ and $\phi_i$, $i = 1, 2, 3$) denote the Ricci tensor and the structure tensor of a real hypersurface $M$ in $M_n(c)$ (resp. in $Q_n(c)$).

In particular, Kimura and Maeda [8] have considered a real hypersurface $M$ in a complex projective space $P_n(C)$ with Lie $\xi$-parallel Ricci tensor and classified that $M$ is locally congruent to of type $(A)$, a tube over a totally geodesic $P_k(C)$, of type $(B)$, a tube over a complex quadric $Q_{n-1}$, $\cot^2 2r = n - 2$, of type $(C)$, a tube over $P_1(C) \times P_1(C)$, $\cot^2 2r = \frac{1}{n-2}$ and $n$ is odd, of type $(D)$, a tube over a complex two-plane Grassmannian $G_2(C^5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 9$, of type $(E)$, a tube over a Hermitian symmetric space $SO(10)/U(5)$, $\cot^2 2r = \frac{5}{9}$ and $n = 15$. Then it turns out that all of them mentioned above are Hopf hypersurfaces and have commuting Ricci tensors.

If the structure vector $\xi = -JN$ of a real hypersurface $M$ in $P_n(C)$ is invariant by the shape operator, $M$ is said to be a Hopf hypersurface, where $J$ denotes a Kaehler structure of $P_n(C)$, $N$ a unit normal vector of $M$ in $P_n(C)$.

In a quaternionic projective space $Q^{P_m}$ Pèrez and the second author [11] have classified real hypersurfaces in $Q^{P_m}$ with $D^\perp$-parallel curvature tensor $\nabla_{\xi_i} R = 0$, $i = 1, 2, 3$, where $R$ denotes the curvature tensor of $M$ in $Q^{P_m}$.

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and $\mathcal{D}^\perp$ a distribution defined by $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$. In such a case they are congruent to a tube of radius $\frac{\pi}{4}$ over a totally geodesic $\mathbb{Q}P^k$ in $\mathbb{Q}P^m$, $2 \leq k \leq m - 2$.

The almost contact structure vector fields $\{\xi_1, \xi_2, \xi_3\}$ mentioned above are defined by $\xi_i = -J_i N$, $i = 1, 2, 3$, where $\{J_1, J_2, J_3\}$ denote a quaternionic Kähler structure of $\mathbb{Q}P^m$ and $N$ a unit normal field of $M$ in $\mathbb{Q}P^m$.

In quaternionic space forms Berndt [2] has introduced the notion of normal Jacobi operator $\bar{R}_N = \bar{R}(X, N)N \in \text{End} T_x M$, $x \in M$ for real hypersurfaces $M$ in a quaternionic projective space $\mathbb{Q}P^m$ or in a quaternionic hyperbolic space $\mathbb{Q}H^m$, where $\bar{R}$ denotes the curvature tensor of $\mathbb{Q}P^m$ and $\mathbb{Q}H^m$ respectively. He [2] has also shown that the curvature adaptedness, that is, the normal Jacobi operator $\bar{R}_N$ commutes with the shape operator $A$, is equivalent to the fact that the distributions $\mathcal{D}$ and $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator $A$ of $M$, where $T_x M = \mathcal{D} \oplus \mathcal{D}^\perp$, $x \in M$.

Now let us consider a complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ which consists of all complex 2-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Then the situation for real hypersurfaces in $G_2(\mathbb{C}^{m+1})$ with parallel normal Jacobi operator is not so simple and will be quite different from the cases mentioned above.

Now in this paper we consider a real hypersurface $M$ in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$ with Lie $\xi$-parallel normal Jacobi operator, $\mathcal{L}_\xi \bar{R}_N = 0$, where $\bar{R}$ and $\bar{N}$ respectively denotes the curvature tensor of the ambient space $G_2(\mathbb{C}^{m+2})$ and a unit normal vector of $M$ in $G_2(\mathbb{C}^{m+2})$. The curvature tensor $\bar{R}(X, Y)Z$ for any vector fields $X, Y$ and $Z$ on $G_2(\mathbb{C}^{m+2})$ is explicitly defined in section 1. Then the normal Jacobi operator $\bar{R}_N$ for the unit normal vector $N$ can be defined from the curvature tensor $\bar{R}(X, N)N$ by putting $Y = Z = N$.

The ambient space $G_2(\mathbb{C}^{m+2})$ is known to be the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure $J$ and a quaternionic Kähler structure $\bar{J}$ not containing $J$ (See Berndt [3]). So, in $G_2(\mathbb{C}^{m+2})$ we have the two natural geometric conditions for real hypersurfaces that $[\xi] = \text{Span} \{\xi\}$ or $\mathcal{D}^\perp = \text{Span} \{\xi_1, \xi_2, \xi_3\}$ is invariant under the shape operator. By using such kinds of conditions Berndt and the second author [4] have proved the following:

**Theorem A.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathcal{D}^\perp$ are invariant under the shape operator of $M$ if and only if

(A) $M$ is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or

(B) $m$ is even, say $m = 2n$, and $M$ is an open part of a tube around a totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$.

If the structure vector field $\xi$ of a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ is invariant by the shape operator, $M$ is said to be a Hopf hypersurface. In such a case the integral curves of the structure vector field $\xi$ are geodesics (See
Berndt and Suh [5]). Moreover, the flow generated by the integral curves of the structure vector field $\xi$ for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$ is said to be geodesic Reeb flow. Moreover, we say that the Reeb vector field is Killing, that is, $\mathcal{L}_\xi g = 0$ for the Lie derivative along the direction of the structure vector field $\xi$, where $g$ denotes the Riemannian metric induced from $G_2(\mathbb{C}^{m+2})$. Then this is equivalent to the fact that the structure tensor $\phi$ commutes with the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{m+2})$. This condition also has the geometric meaning that the flow of Reeb vector field is isometric. Moreover, Berndt and the second author [5] have proved that real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ with isometric flow is of a tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.

Now by putting a unit normal vector $N$ into the curvature tensor $\bar{R}$ of the ambient space $G_2(\mathbb{C}^{m+2})$, we calculate the normal Jacobi operator $\bar{R}_N$ in such a way that

$$
\bar{R}(X, N)N = X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(X)\xi_{\nu} - \sum_{\nu=1}^{3} \{\eta_{\nu}(\xi)J_{\nu}(\phi X + \eta(X)N) - \eta_{\nu}(\phi X)(\phi_{\nu}\xi + \eta_{\nu}(\xi)N)\}
$$

$$
= X + 3\eta(X)\xi + 3\sum_{\nu=1}^{3} \eta_{\nu}(X)\xi_{\nu} - \sum_{\nu=1}^{3} \{\eta_{\nu}(\xi)(\phi_{\nu}\phi X - \eta(X)\xi_{\nu}) - \eta_{\nu}(\phi X)\phi_{\nu}\xi\}
$$

for any tangent vector field $X$ on $M$ in $G_2(\mathbb{C}^{m+2})$.

On the other hand, we introduce the following theorem due to Pérez and the present authors [6] as follows:

**Theorem B.** Let $M$ be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. If the normal Jacobi and the structure operators both commute with the shape operator, then $M$ is congruent to one of the following:

(A) an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$,

or

(B) an open part of a tube around a totally geodesic and totally real $\mathbb{Q}P^n$, $m = 2n$, in $G_2(\mathbb{C}^{m+2})$.

But related to the normal Jacobi operator $\bar{R}_N$, in this paper we want to give some non-existence theorems for real hypersurfaces $M$ in $G_2(\mathbb{C}^{m+2})$ with Lie $\xi$-parallel normal Jacobi operator, that is, $\mathcal{L}_\xi \bar{R}_N = 0$ as follows:

**Theorem 1.** There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathcal{D}^\perp$.

**Theorem 2.** There do not exist any real hypersurfaces in $G_2(\mathbb{C}^{m+2})$ satisfying $\mathcal{L}_\xi \bar{R}_N = 0$ and $\xi \in \mathcal{D}$.

On the other hand, we say that a real hypersurface $M$ in $G_2(\mathbb{C}^{m+2})$ has commuting shape operator on the distribution $\mathcal{D}^\perp$ if the shape operator $A$ of $M$ commutes with the structure tensor $\phi$ on $\mathcal{D}^\perp$, that is, $A\phi \xi_{\nu} = \phi A\xi_{\nu}$, $\nu = 1, 2, 3$. 

Now in the final section, as an application of Theorems 1 and 2 we consider a real hypersurface \( M \) in \( G_2(\mathbb{C}^{m+2}) \) with Lie \( \xi \)-parallel and commuting shape operator on the distribution \( \mathcal{D}^\perp \). Then by virtue of Theorems 1 and 2 we assert the following:

**Theorem 3.** There do not exist any Hopf real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \) with \( \mathcal{L}_\xi \mathcal{R}_N = 0 \) and commuting shape operator on the distribution \( \mathcal{D}^\perp \).

### 1. Riemannian geometry of \( G_2(\mathbb{C}^{m+2}) \)

In this section we summarize basic material about \( G_2(\mathbb{C}^{m+2}) \), for details we refer to [3], [4], and [5]. By \( G_2(\mathbb{C}^{m+2}) \) we denote the set of all complex two-dimensional linear subspaces in \( \mathbb{C}^{m+2} \). The special unitary group \( G = SU(m+2) \) acts transitively on \( G_2(\mathbb{C}^{m+2}) \) with stabilizer isomorphic to \( K = S(U(2) \times U(m)) \subset G \). Then \( G_2(\mathbb{C}^{m+2}) \) can be identified with the homogeneous space \( G/K \), which we equip with the unique analytic structure for which the natural action of \( G \) on \( G_2(\mathbb{C}^{m+2}) \) becomes analytic. Denote by \( g \) and \( \mathfrak{k} \) the Lie algebra of \( G \) and \( K \), respectively, and by \( m \) the orthogonal complement of \( \mathfrak{k} \) in \( g \) with respect to the Cartan-Killing form \( B \) of \( g \). Then \( g = \mathfrak{k} \oplus m \) is an \( Ad(K) \)-invariant reductive decomposition of \( g \). We put \( o = eK \) and identify \( T_oG_2(\mathbb{C}^{m+2}) \) with \( m \) in the usual manner. Since \( B \) is negative definite on \( m \), its negative restricted to \( m \times m \) yields a positive definite inner product on \( m \). By \( Ad(K) \)-invariance of \( B \) this inner product can be extended to a \( G \)-invariant Riemannian metric \( g \) on \( G_2(\mathbb{C}^{m+2}) \). In this way \( G_2(\mathbb{C}^{m+2}) \) becomes a Riemannian homogeneous space, even a Riemannian symmetric space. For computational reasons we normalize \( g \) such that the maximal sectional curvature of \( (G_2(\mathbb{C}^{m+2}), g) \) is eight. Since \( G_2(\mathbb{C}^3) \) is isometric to the two-dimensional complex projective space \( \mathbb{C}P^2 \) with constant holomorphic sectional curvature eight we will assume \( m \geq 2 \) from now on. Note that the isomorphism \( \text{Spin}(6) \simeq SU(4) \) yields an isometry between \( G_2(\mathbb{C}^4) \) and the real Grassmann manifold \( G_2^+ (\mathbb{R}^6) \) of oriented two-dimensional linear subspaces of \( \mathbb{R}^6 \).

The Lie algebra \( \mathfrak{k} \) has the direct sum decomposition \( \mathfrak{k} = su(m) \oplus su(2) \oplus \mathfrak{r} \), where \( \mathfrak{r} \) is the center of \( \mathfrak{k} \). Viewing \( \mathfrak{k} \) as the holonomy algebra of \( G_2(\mathbb{C}^{m+2}) \), the center \( \mathfrak{r} \) induces a Kähler structure \( J \) and the \( su(2) \)-part a quaternionic Kähler structure \( J \) on \( G_2(\mathbb{C}^{m+2}) \). If \( J_1 \) is any almost Hermitian structure in \( J \), then \( JJ_1 = J_1J \), and \( JJ_1 \) is a symmetric endomorphism with \( (JJ_1)^2 = I \) and \( \text{tr}(JJ_1) = 0 \). This fact will be used frequently throughout this paper.

A canonical local basis \( J_1, J_2, J_3 \) of \( J \) consists of three local almost Hermitian structures \( J_1 \) in \( J \) such that \( J_1 J_1 = J_1^2 = -J_1 \), where the index is taken modulo three. Since \( J \) is parallel with respect to the Riemannian connection \( \nabla \) of \( (G_2(\mathbb{C}^{m+2}), g) \), there exist for any canonical local basis \( J_1, J_2, J_3 \) of \( J \) three local one-forms \( q_1, q_2, q_3 \) such that

\begin{align}
\nabla_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}
\end{align}

for all vector fields \( X \) on \( G_2(\mathbb{C}^{m+2}) \).
Let \( p \in G_2(\mathbb{C}^{m+2}) \) and \( W \) a subspace of \( T_p G_2(\mathbb{C}^{m+2}) \). We say that \( W \) is a quaternionic subspace of \( T_p G_2(\mathbb{C}^{m+2}) \) if \( JW \subset W \) for all \( J \in \mathfrak{J}_p \). And we say that \( W \) is a totally complex subspace of \( T_p G_2(\mathbb{C}^{m+2}) \) if there exists a one-dimensional subspace \( \mathfrak{W} \) of \( \mathfrak{J}_p \) such that \( JW \subset W \) for all \( J \in \mathfrak{W} \) and \( JW \perp W \) for all \( J \in \mathfrak{W}^\perp \subset \mathfrak{J}_p \). Here, the orthogonal complement of \( \mathfrak{W} \) in \( \mathfrak{J}_p \) is taken with respect to the bundle metric and orientation on \( \mathfrak{J} \) for which any local oriented orthonormal frame field of \( \mathfrak{J} \) is a canonical local basis of \( \mathfrak{J} \). A quaternionic (resp. totally complex) submanifold of \( G_2(\mathbb{C}^{m+2}) \) is a submanifold all of whose tangent spaces are quaternionic (resp. totally complex) subspaces of the corresponding tangent spaces of \( G_2(\mathbb{C}^{m+2}) \).

The Riemannian curvature tensor \( \bar{R} \) of \( G_2(\mathbb{C}^{m+2}) \) is locally given by

\[
\bar{R}(X,Y)Z = g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ \\
+ \sum_{\nu=1}^{3} \{ g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z \} \\
+ \sum_{\nu=1}^{3} \{ g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY \},
\]

(1.2)

where \( J_1, J_2, J_3 \) is any canonical local basis of \( \mathfrak{J} \).

2. Some fundamental formulas for real hypersurfaces in \( G_2(\mathbb{C}^{m+2}) \)

Now in this section we want to derive the normal Jacobi operator from the curvature tensor of complex two-plane Grassmannian \( G_2(\mathbb{C}^{m+2}) \) given in (1.2) and the equation of Gauss. Moreover, in this section we derive some basic formulae from the Codazzi equation for a real hypersurface in \( G_2(\mathbb{C}^{m+2}) \) (See [4], [5], [13], [14], and [15]).

Let \( M \) be a real hypersurface of \( G_2(\mathbb{C}^{m+2}) \), that is, a hypersurface of \( G_2(\mathbb{C}^{m+2}) \) with real codimension one. The induced Riemannian metric on \( M \) will also be denoted by \( g \), and \( \nabla \) denotes the Riemannian connection of \( (M, g) \). Let \( N \) be a local unit normal field of \( M \) and \( A \) the shape operator of \( M \) with respect to \( N \). The Kähler structure \( J \) of \( G_2(\mathbb{C}^{m+2}) \) induces on \( M \) an almost contact metric structure \( (\phi, \xi, \eta, g) \). Furthermore, let \( J_1, J_2, J_3 \) be a canonical local basis of \( \mathfrak{J} \). Then each \( J_\nu \) induces an almost contact metric structure \( (\phi_\nu, \xi_\nu, \eta_\nu, g) \) on \( M \). Using the above expression for \( \bar{R} \), the Codazzi equation becomes

\[
(\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi
\]
\[ + \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(X) \phi_{\nu} Y - \eta_{\nu}(Y) \phi_{\nu} X - 2g(\phi_{\nu} X, Y) \xi_{\nu} \right\} \\
+ \sum_{\nu=1}^{3} \left\{ \eta_{\nu}(\phi X) \phi_{\nu} \phi Y - \eta_{\nu}(\phi Y) \phi_{\nu} \phi X \right\} \\
+ \sum_{\nu=1}^{3} \left\{ \eta(X) \eta_{\nu}(\phi Y) - \eta(Y) \eta_{\nu}(\phi X) \right\} \xi_{\nu}. \]

The following identities can be proved in a straightforward method and will be used frequently in subsequent calculations:

\begin{align*}
\phi_{\nu+1} \xi_{\nu} &= -\xi_{\nu+2}, \quad \phi_{\nu} \xi_{\nu+1} = \xi_{\nu+2}, \\
\phi \xi_{\nu} &= \phi_{\nu} \xi, \quad \eta_{\nu}(\phi X) = \eta(\phi_{\nu} X), \\
\phi_{\nu} \phi_{\nu+1} X &= \phi_{\nu+2} X + \eta_{\nu+1}(X) \xi_{\nu}, \\
\phi_{\nu+1} \phi_{\nu} X &= -\phi_{\nu+2} X + \eta_{\nu}(X) \xi_{\nu+1}.
\end{align*}

(2.1)

Now let us put

\begin{align*}
J X &= \phi X + \eta(X)N, \quad J_{\nu} X = \phi_{\nu} X + \eta_{\nu}(X)N
\end{align*}

(2.2)

for any tangent vector $X$ of a real hypersurface $M$ in $G_{2}(\mathbb{C}^{n+2})$, where $N$ denotes a normal vector of $M$ in $G_{2}(\mathbb{C}^{n+2})$. Then from this and the formulas (1.1) and (2.1) we have that

\begin{align*}
(\nabla_{X} \phi) Y &= \eta(Y) AX - g(AX, Y) \xi, \quad \nabla_{X} \xi = \phi AX, \\
(\nabla_{X} \phi_{\nu}) Y &= -q_{\nu+1}(X) \phi_{\nu+2} Y + q_{\nu+2}(X) \phi_{\nu+1} Y + \eta_{\nu}(Y) AX \\
&\quad - g(AX, Y) \xi_{\nu}.
\end{align*}

(2.3)

(2.4)

(2.5)

Summing up these formulas, we find the following

\begin{align*}
\nabla_{X} (\phi \xi_{\nu}) &= \nabla_{X} (\phi_{\nu} \xi_{\nu}) \\
&= (\nabla_{X} \phi) \xi_{\nu} + \phi(\nabla_{X} \xi_{\nu}) \\
&= q_{\nu+2}(X) \phi_{\nu+1} \xi \xi_{\nu} + q_{\nu+1}(X) \phi_{\nu+2} \xi + \phi_{\nu} \phi AX \\
&\quad - g(AX, \xi_{\nu} \xi_{\nu} + \eta(\xi_{\nu}) AX.
\end{align*}

(2.6)

Moreover, from $JJ_{\nu} = J_{\nu} J$, $\nu = 1, 2, 3$, it follows that

\begin{align*}
\phi \phi_{\nu} X &= \phi_{\nu} \phi X + \eta_{\nu}(X) \xi - \eta(X) \xi_{\nu}.
\end{align*}

(2.7)
3. Lie $\xi$-parallel normal Jacobi operator

Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie $\xi$-parallel normal Jacobi operator, that is, $\mathcal{L}_{\xi} \tilde{R}_N = 0$. Then first of all, we write the normal Jacobi operator $\tilde{R}_N$, which is given by

\[(3.1) \quad \tilde{R}_N(X) = \tilde{R}(X, N) N \]
\[= X + 3\eta(X)\xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X)\xi_\nu \]
\[= X + 3\eta(X)\xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X)\xi_\nu \]
\[- \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi_\nu)J_\nu(\phi X + \eta(X)N) - \eta_\nu(\phi X)(\phi_\nu \xi + \eta_\nu(\xi_\nu)N) \right\} \]
\[= X + 3\eta(X)\xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X)\xi_\nu \]
\[- \sum_{\nu=1}^{3} \left\{ \eta_\nu(\xi_\nu)(\phi_\nu \phi X - \eta(X)\xi_\nu) - \eta_\nu(\phi X)\phi_\nu \xi \right\}, \]

where we have used the following

\[g(J_\nu JN, N) = -g(JN, J_\nu N) = -g(\xi, \xi_\nu) = -\eta_\nu(\xi), \]
\[g(J_\nu JX, N) = g(X, JJ_\nu N) = -g(X, J_\nu X) \]
\[= -g(X, \phi_\nu \phi X + \eta(\xi_\nu)N) = -g(X, \phi_\nu \xi_\nu), \]

and

\[J_\nu JN = -J_\nu \xi = -\phi_\nu \xi - \eta_\nu(\xi)N. \]

Of course, by (2.7) we know that the normal Jacobi operator $\tilde{R}_N$ could be symmetric endomorphism of $T_x M$, $x \in M$.

Now let us consider a Lie derivative of the normal Jacobi operator along the direction $\xi$. Then it is given by

\[(3.2) \quad (\mathcal{L}_\xi \tilde{R}_N)X = \mathcal{L}_\xi(\tilde{R}_N X) - \tilde{R}_N(\mathcal{L}_\xi X) \]
\[= [\xi, \tilde{R}_N X] - \tilde{R}_N[\xi, X] \]
\[= (\nabla_\xi \tilde{R}_N)X - \phi A \tilde{R}_N X + \tilde{R}_N \phi A X, \]

where the terms in the right side can be given respectively as follows:

\[(\nabla_\xi \tilde{R}_N)X = 3(\nabla_\xi \eta)(X)\xi + 3\eta(X)\nabla_\xi \xi + 3 \sum_{\nu=1}^{3} (\nabla_\xi \eta_\nu)(X)\xi_\nu \]
\[+ 3 \sum_{\nu=1}^{3} \eta_\nu(X)\nabla_\xi \xi_\nu - \sum_{\nu=1}^{3} \left[ \xi(\eta_\nu(\xi))(\phi_\nu \phi X - \eta(X)\xi_\nu) \right. \]
\[+ \eta_\nu(\xi)(\nabla_\xi \phi_\nu \phi X - (\nabla_\xi \eta)(X)\xi_\nu - \eta(X)\nabla_\xi \xi_\nu \}
\[- (\nabla_\xi \eta_\nu)(\phi X)\phi_\nu \xi - \eta_\nu((\nabla_\xi \phi)(X)\phi_\nu \xi - \eta_\nu(\phi X)\nabla_\xi (\phi_\nu \xi)), \]

\[\phi A \tilde{R}_N X = \phi A X + 3\eta(X)\phi A \xi + 3 \sum_{\nu=1}^{3} \eta_\nu(X)\phi A \xi_\nu \]
\[= \sum_{\nu=1}^{3} \left[ \eta_\nu(\xi)(\phi A_\nu \phi X - \eta(X)\phi A \xi_\nu) - \eta_\nu(\phi X)\phi A_\nu \xi \right]. \]
\[
\bar{R}_N \phi AX = \phi AX + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi AX) \xi_{\nu} - \sum_{\nu=1}^{3} \{\eta_{\nu}(\xi) \phi_{\nu}^2 AX - \eta_{\nu}(\phi^2 AX) \phi_{\nu} \xi\}.
\]

Then by the formulas given in section 2, a real hypersurface \(M\) in \(G_2(\mathbb{C}^{m+2})\) with Lie \(\xi\)-parallel of \(\bar{R}_N\) along the direction of \(\xi\) and satisfies the following (3.3)

\[
(\mathcal{L}_\xi \bar{R}_N)X = (\nabla_\xi \bar{R}_N)X - \phi A \bar{R}_N X + \bar{R}_N \phi AX
\]

\[
= 3g(\phi A \xi, \xi) \xi + 3 \sum_{\nu=1}^{3} g(\phi_{\nu} A \xi, X) \xi_{\nu} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) \phi_{\nu} A \xi \\
- \sum_{\nu=1}^{3} \left[\xi(\eta_{\nu}(\xi))(\phi_{\nu} \phi X - \eta(X) \xi_{\nu}) \right. \\
+ \left. \eta_{\nu}(\xi) \left\{ -q_{\nu+1}(\xi) \phi_{\nu+2} \phi X + q_{\nu+2}(\xi) \phi_{\nu+1} \phi X \right. \right. \\
+ \left. \eta_{\nu}(\phi X) A \xi - g(AX, \phi X) \xi_{\nu} + \eta(X) \phi_{\nu} A \xi \\
- g(AX, \xi) \phi_{\nu} \xi - g(\phi A X, \xi) \xi_{\nu} \right. \\
- \left. \eta(X)(q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2} + \phi A \xi) \right. \\
- \left. g(\phi A X, \phi X) \phi_{\nu} \xi - \eta(X) \eta_{\nu}(AX) \phi_{\nu} \xi + g(AX, \xi) \eta_{\nu}(\phi X) \phi_{\nu} \xi \\
- \left. \eta_{\nu}(\phi X) \left\{ \eta_{\nu}(\xi) A \xi - g(AX, \xi) \xi_{\nu} + \phi_{\nu} A \xi \right\} \right] \\
- 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) A \xi_{\nu} + 3 \sum_{\nu=1}^{3} \{\eta_{\nu}(\xi)(\phi A \phi_{\nu} \phi X - \eta(X) \phi A \xi_{\nu}) \}
\]

\[
- \eta_{\nu}(\phi X) A \phi \xi_{\nu} \} + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi AX) \xi_{\nu} \\
+ \sum_{\nu=1}^{3} \{\eta_{\nu}(\xi) \phi_{\nu} A X - \eta_{\nu}(AX) \phi_{\nu} \xi \} = 0,
\]

where in the second equality we have used the following formulas

\[
3 \sum_{\nu=1}^{3} g(q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2}, X) \xi_{\nu} \\
+ 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) \{q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2}\} = 0
\]

and

\[
\sum_{\nu=1}^{3} \{\eta_{\nu+1}(\phi X) q_{\nu+2}(\xi) \phi_{\nu} \xi - \eta_{\nu+2}(\phi X) q_{\nu+1}(\xi) \phi_{\nu+1} \xi \\
- \eta_{\nu}(\phi X) q_{\nu+1}(\xi) \phi_{\nu+2} \xi + \eta_{\nu}(\phi X) q_{\nu+2}(\xi) \phi_{\nu+1} \xi \} = 0.
\]

From this, by putting \(X = \xi\) and using the formulas in Section 2 we have the following

\[
(\mathcal{L}_\xi \bar{R}_N)\xi = 6 \sum_{\nu=1}^{3} g(\phi_{\nu} A \xi, \xi) \xi_{\nu} + 4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \phi_{\nu} A \xi \\
+ \sum_{\nu=1}^{3} \left[\xi(\eta_{\nu}(\xi)) \xi_{\nu} + \eta_{\nu}(\xi) \left\{q_{\nu+2}(\xi) \xi_{\nu+1} - q_{\nu+1}(\xi) \xi_{\nu+2}\right\} \right] \\
- 4 \sum_{\nu=1}^{3} \eta_{\nu}(\xi) \phi A \xi_{\nu} = 0.
\]
4. Lie $\xi$-parallel normal Jacobi operator for $\xi \in \mathcal{D}^\perp$

In this section we want to give a complete proof of Theorem 1. In order to do this, we consider the case that $\xi \in \mathcal{D}^\perp$. Accordingly, we may put $\xi = \xi_1$. Then (3.1) implies the following for any $X$ on $M$

\begin{equation}
0 = 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_{\nu} A\xi, X)\xi_{\nu} + 3\sum_{\nu=1}^{3} \eta_{\nu}(X)\phi_{\nu} A\xi
+ q_2(\xi)\phi_3 \phi X - q_3(\xi)\phi_2 \phi X + \eta(X)\{q_3(\xi)\xi_2 - q_2(\xi)\xi_3\}
- g(\phi_2 A\xi, \phi X)\xi_3 - \eta(X)\eta_2(A\xi)\xi_3 + g(\phi_3 A\xi, \phi X)\xi_2
+ \eta(X)\eta_3(A\xi)\xi_2 + \alpha\{\eta_2(X)\xi_3 - \eta_3(X)\xi_2\}
+ \eta_3(X)\phi_2 \phi A\xi - \eta_2(X)\phi_3 \phi A\xi
- 3\sum_{\nu=1}^{3} \eta_{\nu}(X)\phi A\xi_{\nu} + \phi A\phi_1 \phi X - \eta(X)\phi A\xi_1
+ \eta_3(X)\phi A\xi_3 + \eta_2(X)\phi A\xi_2 + 3\{\eta_3(A\xi)\xi_2 - \eta_2(A\xi)\xi_3\}
+ \phi_1 A\xi + \eta_2(A\xi)\xi_3 - \eta_3(A\xi)\xi_2,
\end{equation}

where $\alpha$ denotes $g(A\xi, \xi)$.

On the other hand, from $\nabla_X \xi_1 = \nabla_X \xi$ we know that

\begin{equation}
q_2(\xi) = 2g(A\xi, \xi_2), \quad q_3(\xi) = 2g(A\xi, \xi_3).
\end{equation}

By putting $X = \xi_2$ in (4.1), we have

\begin{equation}
0 = (\mathcal{L}_\xi \mathcal{R}_N)\xi_2
= 3g(A\xi, \xi_1)\xi_3 + 3\phi_2 A\xi + q_3(\xi)\xi_1 - \phi_3 \phi A\xi - \phi A\xi_2
+ 2\{\eta_3(A\xi)\xi_2 - \eta_2(A\xi)\xi_3\} + \phi_1 A\xi_2.
\end{equation}

From this, taking an inner product with $\xi_1$, we have

\begin{equation}
0 = 3g(A\xi, \xi_3) + q_3(\xi) + g(A\xi, \xi_3).
\end{equation}

Then from this, together with (4.2), it follows that

\begin{equation}
q_3(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_3) = 0.
\end{equation}

Similarly, by putting $X = \xi_3$ in (4.1) we have

\begin{equation}
0 = (\mathcal{L}_\xi \mathcal{R}_N)\xi_3
= -3g(A\xi, \xi)\xi_2 + 3\phi_3 A\xi - q_2(\xi)\xi_1
+ \phi_2 \phi A\xi - \phi A\xi_3 + \phi_1 A\xi_3
+ 2g(A\xi_3, \xi_3)\xi_2 - 2g(A\xi_3, \xi_2)\xi_3.
\end{equation}

From this, by taking an inner product with $\xi_1$ and using (4.2) we have

\begin{equation}
q_2(\xi) = 0 \quad \text{and} \quad g(A\xi, \xi_2) = 0.
\end{equation}

Then we may summarize such a fact as follows:
Lemma 4.1. Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $A\xi = \alpha \xi + \beta U$, where $U$ is a unit vector field orthogonal to $\xi$ and belongs to $\mathcal{D}$.

From Lemma 4.1 we can prove the following

Lemma 4.2. Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $\beta$ identically vanishes, that is, the structure vector $\xi$ is principal.

Proof. By Lemma 4.1 we may put

$$A\xi = \alpha \xi + \beta U$$

for some unit normal $U$ orthogonal to the structure vector $\xi$. Now let us construct an open set $\mathcal{V}$ in such a way that $\mathcal{V} = \{p \in M | \beta(p) \neq 0\}$. Then on such an open $\mathcal{V}$ we proceed our assertion. Now substituting (4.3) into (4.1), we have the following

$$0 = 3\beta \phi_2 U - \beta \phi_3 \phi U - \phi A\xi_2$$
$$+ 2g(\xi_2, \xi_3)\xi_2 - 2g(\xi_2, \xi_3)\xi_3 + \phi_1 A\xi_2.$$ 

From this, by taking an inner product with $\phi_2 U$ we have

$$0 = -3\beta g(\phi_2 U, \phi_2 U) - \beta g(\phi_3 \phi U, \phi_2 U) - g(\phi A\xi_2, \phi_2 U)$$
$$+ g(\phi_1 A\xi_2, \phi_2 U)$$
$$= 3\beta + \beta g(\phi U, \phi_3 \phi U)$$
$$= 3\beta + \beta g(\phi U, -\phi_1 U + \eta_2(U)\xi_3)$$
$$= 3\beta - \beta g(\phi U, \phi_1 U)$$
$$= 2\beta,$$

where in the second equality we have used $\nabla_{\xi_2} \xi = \nabla_{\xi_2} \xi_1$ and in the final equality we have used the formula $\nabla_{\xi} \xi = \nabla_{\xi_1} \xi$. But this is impossible on the open subset $\mathcal{V}$. Accordingly, such an open $\mathcal{V}$ can not exit on $M$. So we have our assertion. \qed

Lemma 4.3. Let $M$ be a real hypersurface in $G_2(C^{m+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}^\perp$. Then $g(A\mathcal{D}, \mathcal{D}^\perp) = 0$.

Proof. Now we consider (4.1) when the structure vector $\xi$ is principal. Then it follows that

$$(4.4) \quad 0 = -2\eta_2(X)\phi A\xi_2 - 2\eta_3(X)\phi A\xi_3 + \phi A\phi_1 \phi X$$
$$+ 2\eta_3(AX)\xi_2 - 2\eta_3(AX)\xi_3 + \phi_1 AX.$$ 

Now let us take an inner product (4.4) with $\xi_2$. Then it follows that

$$0 = -2\eta_2(X)g(\xi_2, \xi_3) - 2\eta_3(X)g(\xi_3, \xi_3)$$
$$+ g(\phi_1 A\phi X, \xi_2) + 2\eta_3(AX) + g(\phi_1 AX, \xi_2)$$
$$= -2\eta_2(X)g(\xi_2, \xi_3) - 2\eta_3(X)g(\xi_3, \xi_3),$$
where in the first equality we have used the following formula
\[ g(\phi A\phi_1 \phi X, \xi_2) = -g(A\phi_1 \phi X, \phi \xi_2) \]
\[ = g(A\phi_1 \phi X, \xi_3) \]
\[ = g(A\phi_1 X, \xi_3) \]
\[ = -g(\phi_1 X, \nabla_{\xi_3} \xi) \]
\[ = g(\nabla_{\xi_3}(\phi X), \xi) \]
\[ = g(\eta(X)A\xi_3 - g(A\xi_3, X)\xi, \xi) \]
\[ = -g(A\xi_3, X). \]

From this, by putting \( X = \xi_2 \) and \( X = \xi_3 \) we have \( g(A\xi_3, \xi_3) = g(A\xi_2, \xi_3) = 0. \)

On the other hand, by taking an inner product (4.4) with \( \xi_3 \) we have
\[ 2\eta_2(X)g(A\xi_2, \xi_2) + 2\eta_3(X)g(A\xi_3, \xi_2) = 0. \]
Then from this, by putting \( X = \xi_2 \) and \( X = \xi_3 \) we have respectively
\[ g(A\xi_2, \xi_2) = g(A\xi_3, \xi_2) = 0. \]

Summing up these formulas, we conclude that \( g(A\xi_i, \xi_j) = 0 \) for any \( i \) and \( j \) except \( i = j = 1 \). Then we may put \( A\xi_2 = X_2 \) and \( A\xi_3 = X_3 \) for some \( X_2, X_3 \in \mathcal{D}. \)

Now substituting these one into (4.4), we get the following
\[ 0 = g(\phi A\phi_1 X, \xi_2) + 2\eta_3(AX) + g(\phi_1 AX, \xi_2) \]
\[ = -g(A\phi_1 X, \phi \xi_2) + 2g(X_3, X) - g(AX, \xi_3) \]
\[ = g(\phi_1 X, X_3) + g(X_3, X) \]

for any tangent vector field \( X \) on \( M. \) Then from this, by replacing \( X \) by \( \phi_1 X \) we have
\[ 0 = g(\phi_1^2 X, X_3) + g(X_3, \phi_1 X) \]
\[ = -g(X, X_3) + g(X_3, \phi_1 X). \]

Then (4.6) and (4.7) gives \( X_2 \) and \( X_3 \) identically vanishing. That is, \( A\xi_2 = 0 \) and \( A\xi_3 = 0. \) Accordingly, we have our assertion in Lemma 4.2. \( \square \)

Before going to give the proof of Theorem 1 in the introduction let us check that "What kind of model hypersurfaces given in Theorem A satisfy Lie \( \xi \)-parallel normal Jacobi operator." In other words, it will be an interesting problem to know whether there exist any real hypersurfaces in \( G_2(\mathbb{C}^{n+2}) \) satisfying the condition \( L_\xi \bar{R}_N = 0 \) for \( \xi \in \mathcal{D}^\perp. \)

Then by virtue of Lemmas 4.1 and 4.2, we are able to recall a proposition given by Berndt and the second author [4] as follows:

For a tube of type \( A \) in Theorem A we have the following
Proposition A. Let $M$ be a connected real hypersurface of $G_2(C^{m+2})$. Suppose that $A\xi \subset D$, $A\xi = \alpha\xi$, and $\xi$ is tangent to $D_\perp$. Let $J_1 \in J$ be the almost Hermitian structure such that $JN = J_1N$. Then $M$ has three (if $r = \pi/2\sqrt{8}$) or four (otherwise) distinct constant principal curvatures

$$
\alpha = \sqrt{8}\cot(\sqrt{8}r), \quad \beta = \sqrt{2}\cot(\sqrt{2}r), \quad \lambda = -\sqrt{2}\tan(\sqrt{2}r), \quad \mu = 0
$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 2, \quad m(\lambda) = 2m - 2 = m(\mu),$$

and the corresponding eigenspaces we have

$$T_\alpha = \mathbb{R}\xi = \mathbb{R}JN = \mathbb{R}\xi_1,$$
$$T_\beta = \mathbb{C}^\perp\xi = \mathbb{C}^\perp N = \mathbb{R}\xi_2 \oplus \mathbb{R}\xi_3,$$
$$T_\lambda = \{X|X \perp \mathbb{R}\xi, JX = J_1X\},$$
$$T_\mu = \{X|X \perp \mathbb{R}\xi, JX = -J_1X\},$$

where $\mathbb{R}\xi$, $\mathbb{C}\xi$ and $\mathbb{Q}\xi$ respectively denotes real, complex and quaternionic span of the structure vector $\xi$ and $\mathbb{C}^\perp\xi$ denotes the orthogonal complement of $\mathbb{C}\xi$ in $\mathbb{H}\xi$.

Then in the proof of Lemma 4.3 we have asserted that $A\xi_2 = 0$ and $A\xi_3 = 0$. But the principal curvature $\beta = \sqrt{2}\cot(\sqrt{2}r)$ given in Proposition A is never vanishing for any $r \in (0, \pi/4)$. So this makes a contradiction. Accordingly, we completed the proof of our Theorem 1.

5. Lie $\xi$-parallel normal Jacobi operator for $\xi \in D$

In this section, in order to prove our Theorem 2 in the introduction we will give several lemmas. Now we consider for the case that $\xi \in D$. Then using $\xi \in D$ in (3.3) we have the following

$$
(L_\xi \tilde{R}_N)X = (\nabla_\xi \tilde{R}_N)X - \phi A\tilde{R}_N X + \tilde{R}_N \phi AX
$$

$$= 3g(\phi A\xi, X)\xi + 3\sum_{\nu=1}^{3} g(\phi_\nu A\xi, X)\xi_\nu$$
$$+ 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu A\xi$$
$$+ \sum_{\nu=1}^{3} [g(\phi_\nu A\xi, \phi X)\phi_\nu \xi + \eta(X)\eta_\nu(A\xi)\phi_\nu \xi$$
$$+ \eta_\nu(\phi X)\{-g(A\xi, \xi)\xi_\nu + \phi_\nu A\xi\}]$$
$$- 3\sum_{\nu=1}^{3} \eta_\nu(X)\phi AX_\nu - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi AX_\nu$$
$$+ 3\sum_{\nu=1}^{3} \eta_\nu(\phi AX)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu \xi = 0.$$

Then we assert the following
Lemma 5.1. Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{n+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}$. Then the structure vector $\xi$ is principal.

Proof. Now let us put $X = \xi$ in (5.1) and use $\xi \in \mathcal{D}$, we have

$$0 = 3 \sum_{\nu=1}^{3} g(\phi_{\nu} A \xi_{\nu}, \xi_{\nu}) + \sum_{\nu=1}^{3} \eta_{\nu}(A \xi) \phi_{\nu} \xi + 3 \sum_{\nu=1}^{3} \eta_{\nu}(\phi A \xi) \xi_{\nu}$$

$$- \sum_{\nu=1}^{3} \eta_{\nu}(A \xi) \phi_{\nu} \xi$$

$$= 6 \sum_{\nu=1}^{3} g(\phi_{\nu} A \xi, \xi) \xi_{\nu}.$$

From this we assert the following for any $\nu = 1, 2, 3$

(5.2) 

$$g(A \xi, \phi_{\nu} \xi) = 0.$$

On the other hand, let us take an inner product (5.1) with the structure vector $\xi$ and use the fact $\xi \in \mathcal{D}$ and (5.2). Then it follows

$$0 = 3 g(\phi A \xi, X) + 3 \sum_{\nu=1}^{3} \eta_{\nu}(X) g(\phi_{\nu} A \xi, \xi)$$

$$+ \sum_{\nu=1}^{3} \eta_{\nu}(\phi X) g(\phi_{\nu} \phi A \xi, \xi)$$

$$= 3 g(\phi A \xi, X) - \sum_{\nu=1}^{3} \eta_{\nu}(\phi X) \eta_{\nu}(A \xi).$$

Now by putting $X = \phi \xi_{\mu}$ into (5.3) we have

(5.4) 

$$g(A \xi, \xi_{\mu}) = 0$$

for any $\mu = 1, 2, 3$. Then by virtue of (5.2) and (5.4) we may put

(5.5) 

$$A \xi = \alpha \xi + X_0$$

for some $X_0 \in \mathcal{D}$ orthogonal to $\xi, \phi_1 \xi, \phi_2 \xi, \phi_3 \xi$. Then by putting $X = \phi X_0$ in (5.3) we have $g(A \xi, X_0) = 0$. From this, together with (5.5), we have our assertion. \hfill \Box

Then by using Lemma 5.1 we want to verify $g(\mathcal{A} \mathcal{D}, \mathcal{D}^\perp) = 0$. In order to do this, first of all, we should verify the following

Lemma 5.2. Let $M$ be a real hypersurface in $G_2(\mathbb{C}^{n+2})$ satisfying Lie $\xi$-parallel normal Jacobi operator and $\xi \in \mathcal{D}$. Then $g(\mathcal{A} \mathcal{D}, \mathcal{D}^\perp) = 0$. 

Proof. From the results of Lemma 5.1, we have the following

\[
(\mathcal{L}_\xi \tilde{R}_N)X = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, X)\xi_\nu + 3\alpha \sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu \xi \\
- \sum_{\nu=1}^{3} \left[ \xi(\eta_\nu(\xi))\phi_\nu \phi X - \eta(X)\xi_\nu \right] \\
+ \eta_\nu(\xi) \left\{ - q_{\nu+1}(\xi)\phi_{\nu+2} \phi X + q_{\nu+2}(\xi)\phi_{\nu+1} \phi X \\
- \eta(X)(q_{\nu+2}(\xi)\xi_{\nu+1} - q_{\nu+1}(\xi)\xi_{\nu+2} + \alpha \phi_\nu \xi) \right\} \\
- \alpha g(\phi_\nu \xi, \phi X)\phi_\nu \xi \\
- 3 \sum_{\nu=1}^{3} \eta_\nu(X)\phi A \xi_\nu \\
+ \sum_{\nu=1}^{3} \eta_\nu(\xi)(\phi A \phi_\nu \phi X - \eta(X)\phi A \xi_\nu) \\
- \eta_\nu(\phi X)\phi A \phi_\nu \xi + 3 \sum_{\nu=1}^{3} \eta_\nu(\phi A X)\xi_\nu \\
+ \sum_{\nu=1}^{3} \eta_\nu(\xi)\phi A X - \eta_\nu(AX)\phi_\nu \xi \right\} = 0.
\]

(5.6)

Since \( \xi \) is principal and \( \xi \in \mathfrak{D} \), we have

\[
g(AX, \mathfrak{D}^\perp) = 0.
\]

(5.7)

From the formula (5.6) and \( \xi \in \mathfrak{D} \), we have the following

\[
(\mathcal{L}_\xi \tilde{R}_N)X = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, X)\xi_\nu + 4\alpha \sum_{\nu=1}^{3} \eta_\nu(X)\phi_\nu \xi \\
- 3 \sum_{\nu=1}^{3} \eta_\nu(X)\phi A \xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi A \phi_\nu \xi \\
+ 3 \sum_{\nu=1}^{3} \eta_\nu(\phi A X)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu \xi = 0.
\]

(5.8)

Now let us put \( \mathfrak{D}_0(x) = \{X \in \mathfrak{D} | X \perp \xi \} \). From this, for \( X \in \mathfrak{D}_0 \), we have

\[
0 = 4\alpha \sum_{\nu=1}^{3} g(\phi_\nu \xi, X)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(\phi X)\phi A \phi_\nu \xi \\
+ 3 \sum_{\nu=1}^{3} \eta_\nu(\phi A X)\xi_\nu - \sum_{\nu=1}^{3} \eta_\nu(AX)\phi_\nu \xi.
\]

(5.9)

Let us take an inner product the above equation with \( \phi_i \xi \). Then we have

\[
0 = \sum_{\nu=1}^{3} \eta_\nu(\phi X)g(\phi A \phi_\nu \xi, \phi_i \xi) + g(AX, \xi_i).
\]

(5.10)

By the formula (5.10), for \( X \in \mathfrak{D}_1 \), we have

\[
g(AX, \xi_i) = 0, \quad i = 1, 2, 3,
\]

(5.11)
where the distribution $\mathcal{D}_1$ is given by $\mathcal{D}_1 = \{X \in \mathcal{D}_0 | X \perp \phi_i \xi, \ i = 1, 2, 3\}$. On the other hand, by (2.3) and (2.4), we have the following

$$g(A\phi_i \xi, \xi_\mu) = g(A\xi_\mu, \phi_i \xi)$$
$$= g(A\xi_\mu, \phi_i \xi_i)$$
$$= - g(\phi A\xi_\mu, \xi_i)$$
$$= - g(\nabla_{\xi_\mu} \xi, \xi_i)$$
$$= g(\xi, \nabla_{\xi_\mu} \xi_i)$$
$$= g(\xi, \phi_i A\xi_\mu)$$
$$= - g(A\phi_i \xi, \xi_\mu).$$

From the above equation, we have

$$(5.12) \quad g(A\phi_i \xi, \xi_\mu) = 0$$

for any $i, \mu = 1, 2, 3$. Hence, by (5.7), (5.11) and (5.12), we know that

$$g(A\mathcal{D}, \mathcal{D}^\perp) = 0.$$

□

Now by virtue of these Lemmas 5.1 and 5.2 we are able to use Theorem A due to Berndt and the second author [4]. That is, $M$ is locally a tube over a totally geodesic and totally real quaternionic projective space $\mathbb{Q}P^n$, $m = 2n$. So for the geometrical structure for such a tube we recall the following proposition

**Proposition B.** Let $M$ be a connected real hypersurface of $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathcal{D} \subset \mathcal{D}$, $A\xi = \alpha \xi$, and $\xi$ is tangent to $\mathcal{D}$. Then the quaternionic dimension $m$ of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and $M$ has five distinct constant principal curvatures

$$\alpha = -2 \tan(2r), \ \beta = 2 \cot(2r), \ \gamma = 0, \ \lambda = \cot(r), \ \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \ m(\beta) = 3 = m(\gamma), \ m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$T_\alpha = \mathbb{R} \xi, \ T_\beta = \mathbb{J} J \xi, \ T_\gamma = \mathbb{J} \xi, \ T_\lambda, \ T_\mu,$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H} \mathbb{C} \xi)^\perp, \ \mathbb{J} T_\lambda = T_\lambda, \ \mathbb{J} T_\mu = T_\mu, \ J T_\lambda = T_\mu.$$

Now let us construct a subdistribution $\mathcal{D}_0$ in such a way that

$$[\xi] \oplus \mathcal{D}_0 = \mathcal{D},$$

where $[\xi]$ denotes a one-dimensional vector subspace spanned by the structure vector $\xi$. Then $\mathcal{D}_0$ becomes $\mathcal{D}_0 = \{X \in \mathcal{D} | X \perp \xi\}$. Now we substitute any $X \in \mathcal{D}_0$
in (5.17) and use $\xi \in \mathcal{D}$ we have
\[
(L_\xi \tilde{R}_N)X = 4\alpha \sum_{\nu=1}^{3} g(X, \phi_\nu \xi) \xi_\nu - \sum_{\nu=1}^{3} \eta_\nu (\phi X) \phi A \phi_\nu \xi \\
+ 3 \sum_{\nu=1}^{3} \eta_\nu (\phi AX) \xi_\nu - \sum_{\nu=1}^{3} \eta_\nu (AX) \phi_\nu \xi.
\]
From this, putting $X = \phi_\mu \xi$ and using $A \phi_\mu \xi = 0$, $\mu = 1, 2, 3$ in Proposition B, we have
\[
(L_\xi \tilde{R}_N)\phi_\mu \xi = 4\alpha \xi_\mu.
\]
But we have assumed that $L_\xi \tilde{R}_N = 0$. Then this gives $\alpha = 0$. But the constant principal curvature $\alpha = -2 \tan(2r)$ in Proposition B never vanishing for $r \in (0, \frac{\pi}{4})$. This makes a contradiction for this case $\xi \in \mathcal{D}$. So we complete the proof of Theorem 2 in the introduction.

6. Hopf hypersurfaces with $\xi$-parallel normal Jacobi operator

A real hypersurface $M$ in $G_2(\mathbb{C}^{n+2})$ is said to be a Hopf if the structure vector $\xi$ of $M$ is principal. This means that $A\xi = \alpha \xi$, $\alpha = g(A\xi, \xi)$, for the shape operator $A$ of $M$ in $G_2(\mathbb{C}^{n+2})$. Of course, all of hypersurfaces in $G_2(\mathbb{C}^{n+2})$ mentioned in Theorem A are Hopf hypersurfaces. Moreover, by Propositions A and B we have known that the structure vector $\xi$ for real hypersurfaces of type (A) and of type (B) in Theorem A belongs to the distribution $\mathcal{D}^\perp$ and the distribution $\mathcal{D}$ respectively.

In this section we consider a Hopf hypersurface in $G_2(\mathbb{C}^{n+2})$ with Lie $\xi$-parallel normal Jacobi operator $\tilde{R}_N$. Then it will be an interesting fact to check whether Hopf hypersurfaces in $G_2(\mathbb{C}^{n+2})$ with Lie $\xi$-parallel normal Jacobi operator can exist or not.

In order to do this, we prove the following lemma which will be useful in the proof of our Theorem 3 given in the introduction.

**Lemma 6.1.** Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{n+2})$ with Lie parallel normal Jacobi operator along the direction of $\xi$. Then the directional derivative of the principal curvature $\alpha$ is given by
\[
Y\alpha = -4 \sum_{\nu=1}^{3} \eta_\nu (\xi) \eta_\nu (\phi Y)
\]
for any vector field $Y$ on $M$.

**Proof.** Now we assume that $M$ is Hopf. So we may put $A\xi = \alpha \xi$. Then the formula (3.4) implies that
\[
\alpha \sum_{\nu=1}^{3} \eta_\nu (\xi) \phi_\nu \xi = \sum_{\nu=1}^{3} \eta_\nu (\xi) \phi A \xi_\nu.
\]
Now let us consider a vector $U$ defined in such a way that
\[
U = \sum_{\nu=1}^{3} \eta_\nu (\xi) \xi_\nu.
\]
If we put $\xi = X_1 + X_2$ for some vector $X_1$ in the distribution $\mathcal{D}^\perp$ and some vector $X_2$ in $\mathcal{D}$, then we know that $X_1$ becomes the vector $U$. Now hereafter,
unless otherwise stated, let us decompose the structure vector $\xi$ by $\xi = U + X_2$. Then (6.1) can be written as follows

(6.3) \[ \phi AU = \alpha \phi U. \]

Now differentiating (6.2) covariantly and using the formulas given in Section 2, we have

\[ \nabla_X U = \sum_{\nu=1}^{3} \{g(\nabla_X \xi_\nu, \xi) \xi_\nu + g(\xi_\nu, \nabla_X \xi) \xi_\nu + \eta_\nu(\xi) \nabla_X \xi_\nu \}
= 2 \sum_{\nu=1}^{3} g(\xi_\nu, \phi AX) \xi_\nu + \sum_{\nu=1}^{3} \eta_\nu(\xi) \phi_\nu AX. \]

On the other hand, by applying the structure tensor $\phi$ to (6.1) we know the following

(6.4) \[ AU = \alpha U \quad \text{and} \quad AX_2 = \alpha X_2. \]

Now differentiating the first formula of (6.4) and using the above formula, we have the following

\[ (\nabla_X A)U + A \nabla_X U = (X \alpha)U + \alpha \nabla_X U. \]

Then it follows that

\[
g(U, (\nabla_X A)Y)
= g((\nabla_X A)U, Y)
= (X \alpha)g(U, Y) + \alpha g(\nabla_X U, Y) - g(AX_2 U, Y)
\]

(6.5) \[ = (X \alpha) \sum_{\nu=1}^{3} \eta_\nu(\xi_\nu \eta_\nu(Y) + \alpha \{2 \sum_{\nu=1}^{3} g(\xi_\nu, \phi AX) \eta_\nu(Y)
+ \sum_{\nu=1}^{3} \eta_\nu(\xi) g(\phi_\nu AX, Y)\}
- g(2 \sum_{\nu=1}^{3} g(\xi_\nu, \phi AX) AX_\nu + \sum_{\nu=1}^{3} \eta_\nu(\xi) A \phi_\nu AX, Y). \]

From this, let us take a skew-symmetric part of (6.5), then by virtue of the equation of Codazzi the left side becomes

\[
g((\nabla_X A)Y - (\nabla_Y A)X, U)
= \sum_{\nu=1}^{3} \{\eta(X) g(\phi Y, \xi_\nu) \eta_\nu(\xi) - \eta(Y) g(\phi X, \xi_\nu) \eta_\nu(\xi)\}
- 2 \sum_{\nu=1}^{3} g(\phi X, Y) \eta_\nu(\xi)^2 - 2 \sum_{\nu=1}^{3} g(\phi_\nu X, Y) \eta_\nu(\xi)
+ 2 \sum_{\nu=1}^{3} [\eta_\nu(X)\{ - \eta_{\nu+2}(Y) \eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y) \eta_{\nu+2}(\xi)\}
+ \eta_\nu(\phi X)\{ - \eta_{\nu+2}(\phi Y) \eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y) \eta_{\nu+2}(\xi)\}] + \sum_{\nu=1}^{3} \{\eta(X) \eta_\nu(\phi Y) - \eta(Y) \eta_\nu(\phi X)\} \eta_\nu(\xi), \]

(6.6)
where we have used the following
\[ g(\phi, \phi Y, U) = -g(\phi Y, \phi_U U) \]
\[ = -g(\phi Y, \eta_{\nu+1}(\xi)\phi_\nu \xi_{\nu+1} + \eta_{\nu+2}(\xi)\phi_\nu \xi_{\nu+2}) \]
\[ = -\eta_{\nu+2}(\phi Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(\phi Y)\eta_{\nu+2}(\xi) \]
and
\[ g(\phi, Y, U) = -\eta_{\nu+2}(Y)\eta_{\nu+1}(\xi) + \eta_{\nu+1}(Y)\eta_{\nu+2}(\xi). \]
Moreover, the skew-symmetric part in the right side of (6.5) becomes
\[
(X\alpha)\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(Y) - (Y\alpha)\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(X)
+ 2\alpha\sum_{\nu=1}^{3} \{g(\xi_{\nu}, \phi AX)\eta_{\nu}(Y) - g(\xi_{\nu}, \phi AY)\eta_{\nu}(X)\}
+ \alpha\sum_{\nu=1}^{3} \eta_{\nu}(\xi)g((\phi \nu A + A\phi_\nu)X, Y) - 2\sum_{\nu=1}^{3} \{g(\xi_{\nu}, \phi AX)g(A\xi_{\nu}, Y)
- g(\xi_{\nu}, \phi AY)g(A\xi_{\nu}, X)\} - 2\sum_{\nu=1}^{3} \eta_{\nu}(\xi)g(A\phi_\nu AX, Y).
\]
Then by putting \( X = \xi \) into the both sides of the above formulas and using \( A\xi = \alpha\xi \), we have
\[
(6.7)
4\sum_{\nu=1}^{3} \eta_{\nu}(\phi Y)\eta_{\nu}(\xi) = (\xi\alpha)\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(Y) - (Y\alpha)\sum_{\nu=1}^{3} \eta_{\nu}(\xi)^2
+ \alpha^2\sum_{\nu=1}^{3} \eta_{\nu}(\xi)g(\phi_\nu \xi, Y) + \alpha\sum_{\nu=1}^{3} g(\xi_{\nu}, \phi AY)\eta_{\nu}(\xi).
\]
On the other hand, if we differentiate \( A\xi = \alpha\xi \) and take an inner product with \( \xi \), then the Codazzi equation gives the following
\[
-2g(\phi X, Y) + 2\sum_{\nu=1}^{3} \{\eta_{\nu}(X)\eta_{\nu}(\phi Y) - \eta_{\nu}(Y)\eta_{\nu}(\phi X) - g(\phi_\nu X, Y)\eta_{\nu}(\xi)\}
= g((\nabla_X A)Y - (\nabla_Y A)X, \xi)
= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X)
= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y).
\]
From this, if we put \( X = \xi \), then
\[
(6.8)
Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu}(\phi Y).
\]
Then by putting \( Y = X_2 \) in (6.8) we have
\[
(6.9)
X_2\alpha = \|X_2\|^2(\xi\alpha),
\]
where we have used
\[
(6.10) \quad \eta_{\nu}(\phi X_2) = -g(\phi_\nu \xi, X_2) = -g(\phi_\nu U + \phi_\nu X_2, X_2) = 0.
\]
Now we put \( Y = X_2 \) in (6.7), and use (6.8), (6.10) and \( A\xi = \alpha\xi \) in the obtained equation. Then it follows that
\[
(6.11) \quad (X_2\alpha)(1 - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)^2) = (\xi\alpha)\eta(X_2).
\]
Then from (6.9) and (6.11) we have

\[
||X_2||^2(1 - \sum_{\nu=1}^{3} \eta_{\nu}(\xi)^2)(\xi\alpha) = (\xi\alpha)\eta(X_2) = (\xi\alpha)||X_2||^2,
\]

which gives that

\[
(\sum_{\nu=1}^{3} \eta_{\nu}(\xi)^2)||X_2||^2(\xi\alpha) = 0.
\]

(6.12)

From this, together with the decomposition of the structure vector \(\xi\) in the assumption, we have \(\xi\alpha = 0\). Then (6.8) completes the proof of Lemma 6.1. \(\square\)

Now let us show that the structure vector \(\xi\) belongs to either the distribution \(\mathcal{D}\) or the distribution \(\mathcal{D}^\perp\) when a Hopf hypersurface \(M\) in \(G_2(C^{m+2})\) has commuting shape operator, that is \(A\phi = \phi A\) on the distribution \(\mathcal{D}^\perp\). In order to do this we also assumed that the structure vector \(\xi\) is decomposed into two distributions \(\mathcal{D}\) and \(\mathcal{D}^\perp\). That is, \(\xi\) is decomposed into \(\xi = U + X_2\).

Now, by using \(\xi\alpha = 0\) in (6.7) and (6.8), we have

\[
(Y\alpha)(\sum_{\nu=1}^{3} \eta_{\nu}(\xi)^2 - 1) = \alpha^2 g(\phi U, Y) - \alpha g(\phi U, AY).
\]

Moreover, from (6.8) together with \(\xi\alpha = 0\) we have

\[
Y\alpha = 4g(\phi U, Y)
\]

for any tangent vector field \(Y\) on \(M\). So (6.14) gives \(Y\alpha = 0\) for any \(Y\) orthogonal to \(\phi U\). Then from this together with (6.13) we have

\[
\alpha g(A\phi U, Y) = 0
\]

(6.15)

for any \(Y\) orthogonal to \(\phi U\).

For the case where \(\alpha = 0\), by (6.14) we can make a contradiction, because \(\phi U = -\phi X_2\) never vanishing under the decomposition. So we assume that the function \(\alpha \neq 0\). Then (6.15) gives that \(g(A\phi U, Y) = 0\) for any \(Y\) orthogonal to \(\phi U\). So we may put

\[
A\phi U = \beta\phi U.
\]

(6.16)

Now by putting \(Y = \phi U\) in (6.13) and (6.14), and using (6.16), we have

\[
-4||\phi U||^2||X_2||^2 = -(\phi U\alpha)||X_2||^2
= (\alpha^2 - \alpha\beta)||\phi U||^2
= \alpha(\alpha - \beta)||\phi U||^2.
\]

This gives

\[
\alpha(\alpha - \beta) = -4||X_2||^2.
\]

(6.17)

But we have asserted that \(M\) has commuting shape operator on the distribution \(\mathcal{D}^\perp\). This means that \(\phi AU = A\phi U = \alpha\phi U\) for \(U = \sum_{\nu=1}^{3} \eta_{\nu}(\xi)\eta_{\nu} \in \mathcal{D}^\perp\). From this together with (6.17), we can make a contradiction. Then summing up these process and Lemma 6.1 we can assert the following
Lemma 6.2. Let $M$ be a Hopf real hypersurface in $G_2(\mathbb{C}^{m+2})$ with Lie parallel normal Jacobi operator along the direction of $\xi$. If $M$ has commuting shape operator on the distribution $\mathcal{D}^\perp$, then the structure vector $\xi$ belongs to either the distribution $\mathcal{D}$ or the distribution $\mathcal{D}^\perp$.

Accordingly, by Lemma 6.2 and together with Theorem 1 and Theorem 2 for each case $\xi \in \mathcal{D}^\perp$ and $\xi \in \mathcal{D}$ respectively, we give the complete proof of our Theorem 3 mentioned in the introduction.

Remark 6.1. A tube over a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$ in Theorem A has commuting shape operator on the distribution $\mathcal{D}^\perp$. Of course, it is Hopf. But, in section 4 we have asserted that such a hypersurface can not satisfy $\mathcal{L}_{\xi} R_N = 0$.

Remark 6.2. A tube over a totally real totally geodesic $\mathbb{Q}P^n$ in $G_2(\mathbb{C}^{m+2})$ has not commuting shape operator on the distribution $\mathcal{D}^\perp$. In section 5 we have also proved that such a hypersurface is Hopf but can not satisfy $\mathcal{L}_{\xi} R_N = 0$.

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