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HARDY INEQUALITIES IN HALF SPACES OF THE HEISENBERG GROUP

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ABSTRACT. In this paper we prove some Hardy inequalities in the half space of the Heisenberg group and indicate the sharp constant.

1. Introduction

The well-known Hardy inequality in the Euclidean space plays an important role in the study of linear and nonlinear partial differential equations. The inequality and its generalizations have been established by using many different methods. There exists a large literature dealing with it.

The Hardy inequalities related to vector fields have also been widely studied and applied in recent years. The following Hardy inequality on the Heisenberg group H^n was first established by Garofalo and Lanconelli [2]:

$$\left(\frac{Q-2}{2}\right)^2 \int_{H^n} u^2 \frac{|x|^2+|y|^2}{\left(|x|^2+|y|^2\right)^2+t^2} \le \int_{H^n} |\nabla_{H^n} u|^2, \tag{1}$$

where $u \in C_0^\infty\left(H^n \backslash \{(0,0,0)\}\right)$, $Q = 2n+2$. The L^p version of the inequality (1) has been produced by Niu, Zhang, and Wang [5]. A different proof of (1) with the sharp constant $\left(\frac{Q-2}{2}\right)^2$ appeared in Goldstein and Zhang [3]. In [1], D'Ambrosio set up weighted Hardy inequalities on H^n. Han and Niu [4] examined some Hardy inequalities in the domains of H^n.

Recently, Tidblom [6] deliberated some Hardy inequalities in the half-space of the Euclidean space $\mathbb{R}^n$. Aroused by [6] we will make some Hardy inequalities in the half-space of the Heisenberg group H^n in this paper. The results here do not include in [1].

Now we recall some notations and basic facts about the Heisenberg group H^n. More detailed information can be found in [2] and references therein.

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Let $\xi = (x_1, \dots, x_n, y_1, \dots, y_n, t) = (x, y, t) \in \mathbb{R}^{2n+1}$, where $n \in \mathbb{N}$. The Heisenberg group H^n is the nilpotent Lie group of step two owing underlying manifold $\mathbb{R}^{2n+1}$ and the group law

$$\hat{\xi} \circ \tilde{\xi} = \left(\hat{x} + \tilde{x}, \hat{y} + \tilde{y}, \hat{t} + \tilde{t} + 2 \sum_{i=1}^{n} (\tilde{x}_i \hat{y}_i - \hat{x}_i \tilde{y}_i) \right), \quad \hat{\xi}, \tilde{\xi} \in H^n. \tag{2}$$

A left invariant basis for the Lie algebra of H^n has the form

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad i = 1, \dots, n.$$

We denote the associated Heisenberg gradient by

$$\nabla_{H^n} = (X_1, \dots, X_n, Y_1, \dots, Y_n)$$

and the Heisenberg divergence by $\operatorname{div}_{H^n} F = \nabla_{H^n} \cdot F$, $F \in \mathbb{R}^{2n}$, respectively. The sub-Laplacian Δ_{H^n} on H^n is $\Delta_{H^n} = \sum_{i=1}^{n} \left(X_i^2 + Y_i^2 \right)$. The homogeneous dimension of H^n is $Q = 2n + 2$.

We endow H^n with the norm

$$|\xi|_{H^n} = \left(\left(|x|^2 + |y|^2 \right)^2 + t^2 \right)^{\frac{1}{4}}, \tag{3}$$

and using the notation ξ^{-1} for the inverse of ξ with respect to the operation $\circ$ (note that $\xi^{-1} = -\xi$), introduce the distance $d(\xi, \eta) = \left| \eta^{-1} \circ \xi \right|_{H^n}$. For convenience we write $(z_1, \dots, z_{2n}) = (x_1, \dots, x_n, y_1, \dots, y_n)$ in the sequel.

The main results in this paper are the following.

Theorem 1.1. *Let $\Omega = \{(x, y, t) \in H^n | x_1 > 0\}$, $p > 1$ and $u \in C_0^\infty(\Omega)$. Then for $k = 2, \dots, 2n$,*

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|u|^p}{x_1^p} + c(p) \left(\frac{p-1}{p} \right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1} \left(x_1^2 + z_k^2 \right)^{\frac{1}{2}}}, \tag{4}$$

where

$$c(p) = \begin{cases} \frac{2}{3p+2}, & \text{if } 1 < p < 2, \\ \frac{1}{4(p-1)}, & \text{if } p \geq 2. \end{cases}$$

Moreover, the constant $c(p)$ in (4) is sharp.

We generalize the above inequality by drawing a parameter τ.

Theorem 1.2. *Let $0 < \tau \leq \frac{2}{2-p}$ when $1 < p < 2$ and $\tau > 0$ when $p \geq 2$, $\Omega = \{(x, y, t) \in H^n | x_1 > 0\}$ and $u \in C_0^\infty(\Omega)$. Then for $k = 2, \dots, 2n$,*

(5)

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p} \right)^p \int_\Omega \frac{|u|^p}{x_1^p} + c(p, \tau) \left(\frac{p-1}{p} \right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-\tau} \left(x_1^2 + z_k^2 \right)^{\frac{\tau}{2}}},$$

where

$$c(p,\tau)=\begin{cases}\frac{1+p\tau^2-\sqrt{1+2p\tau^2}}{p^2\tau^2}, & \textit{if } 1<p<2,\\ \frac{\left(\sqrt{1+4\tau^2}-1\right)^2}{8(p-1)\tau^2}, & \textit{if } p\geq 2,\end{cases}$$

and the constant $c(p,\tau)$ *in* (5) *is sharp.*

Remark 1.3. Our method dealt with the constants is different from that in [6] and ensures sharpness of the constants in Theorem 1.1 and Theorem 1.2. In fact this approach can be used to obtain the sharp constants in the inequalities of [6].

2. Hardy inequalities

Let Ω be a domain in H^n with boundary $\partial\Omega$ and $\delta=\delta(g)=\text{dist}(g,\partial\Omega), g\in \Omega\subset H^n$. The following lemma is our starting point.

Lemma 2.1. *Let* $u\in C_0^\infty(\Omega)$, $d\in(-\infty,mp-1)$ *with* $m\in\mathbb{N}$ *and* $1<p<Q$, $F=(F_1,\dots,F_{2n})$ *be a vector field in* $\mathbb{R}^{2n}$ *with components in* $C^1(\Omega)$ *and* $w(x,y,t)\in C^1(\Omega)$ *be a nonnegative weight function. Then*

$$\begin{aligned}&\int_\Omega \frac{|\nabla_{H^n}u|^p\,w}{\delta^{(m-1)p-d}}\\ &\geq \left(\frac{mp-d-1}{p}\right)^p\int_\Omega\left(\frac{|\nabla_{H^n}\delta|^2}{\delta^{mp-d}}-\frac{p}{mp-d-1}\frac{\Delta_{H^n}\delta}{\delta^{mp-d-1}}\right)|u|^p\,w\\ &\quad+\left(\frac{mp-d-1}{p}\right)^p\times\int_\Omega\left[\frac{p}{mp-d-1}\,\text{div}_{H^n}F\right.\\ &\qquad\left.+\frac{p-1}{\delta^{mp-d}}\left(|\nabla_{H^n}\delta|^2-\left|\nabla_{H^n}\delta-\delta^{mp-d-1}F\right|^{\frac{p}{p-1}}\right)\right]|u|^p\,w\\ &\quad+\left(\frac{mp-d-1}{p}\right)^{p-1}\int_\Omega\nabla_{H^n}w\cdot\left(F-\frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}}\right)|u|^p.\end{aligned}\tag{6}$$

Proof. Hölder's inequality and integration by parts give

$$\begin{aligned}&p^p\left(\int_\Omega\frac{|\nabla_{H^n}u|^p\,w}{\delta^{(m-1)p-d}}\right)\cdot\left(\int_\Omega\left|\frac{\nabla_{H^n}\delta}{\delta^{m(p-1)+\frac{d}{p}-d}}-\delta^{m-1-\frac{d}{p}}F\right|^{\frac{p}{p-1}}|u|^p\,w\right)^{p-1}\\ &\geq p^p\left|\int_\Omega\left(\frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}}-F\right)w\cdot(\text{sign }u)\,|u|^{p-1}\,\nabla_{H^n}u\right|^p\\ &=\left|\int_\Omega\left(\frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}}-F\right)w\cdot\nabla_{H^n}|u|^p\right|^p\\ &=\left|-\int_\Omega|u|^p\,\text{div}_{H^n}\left[\left(\frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}}-F\right)w\right]\right|^p\\ &=(mp-d-1)^p\end{aligned}$$

$$\times \left| \int_\Omega \left[\left(\frac{|\nabla_{H^n}\delta|^2}{\delta^{mp-d}} - \frac{\Delta_{H^n}\delta}{(mp-d-1)\delta^{mp-d-1}} + \frac{\operatorname{div}_{H^n} F}{mp-d-1} \right) w + \frac{\nabla_{H^n} w}{mp-d-1} \cdot \left(F - \frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}} \right) \right] |u|^p \right|^p.$$

Then

$$\begin{aligned}
\int_\Omega \frac{|\nabla_{H^n} u|^p w}{\delta^{(m-1)p-d}} \geq & \left(\frac{mp-d-1}{p} \right)^p \\
& \times \left| \int_\Omega \left[\left(\frac{|\nabla_{H^n}\delta|^2}{\delta^{mp-d}} - \frac{\Delta_{H^n}\delta}{(mp-d-1)\delta^{mp-d-1}} + \frac{\operatorname{div}_{H^n} F}{mp-d-1} \right) w \right.\right. \\
& \left.\left. + \frac{\nabla_{H^n} w}{mp-d-1} \cdot \left(F - \frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}} \right) \right] |u|^p \right|^p \\
& \div \left(\int_\Omega \left| \frac{\nabla_{H^n}\delta}{\delta^{m(p-1)+\frac{d}{p}-d}} - \delta^{m-1-\frac{d}{p}} F \right|^{\frac{p}{p-1}} |u|^p w \right)^{p-1} \\
\geq & \left(\frac{mp-d-1}{p} \right)^p \\
& \times \left\{ \int_\Omega \left[\left(\frac{p|\nabla_{H^n}\delta|^2}{\delta^{mp-d}} - \frac{p}{mp-d-1} \frac{\Delta_{H^n}\delta}{\delta^{mp-d-1}} + \frac{p}{mp-d-1} \operatorname{div}_{H^n} F \right) w \right.\right. \\
& \left. + \frac{p}{mp-d-1} \nabla_{H^n} w \cdot \left(F - \frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}} \right) \right] |u|^p \\
& \left. - (p-1) \int_\Omega \left| \frac{\nabla_{H^n}\delta}{\delta^{m(p-1)+\frac{d}{p}-d}} - \delta^{m-1-\frac{d}{p}} F \right|^{\frac{p}{p-1}} |u|^p w \right\} \\
= & \left(\frac{mp-d-1}{p} \right)^p \int_\Omega \left(\frac{|\nabla_{H^n}\delta|^2}{\delta^{mp-d}} - \frac{p}{mp-d-1} \frac{\Delta_{H^n}\delta}{\delta^{mp-d-1}} \right) |u|^p w \\
& + \left(\frac{mp-d-1}{p} \right)^p \times \int_\Omega \left[\frac{p}{mp-d-1} \operatorname{div}_{H^n} F \right. \\
& \left. + \frac{p-1}{\delta^{mp-d}} \left(|\nabla_{H^n}\delta|^2 - \left| \nabla_{H^n}\delta - \delta^{mp-d-1} F \right|^{\frac{p}{p-1}} \right) \right] |u|^p w \\
& + \left(\frac{mp-d-1}{p} \right)^{p-1} \int_\Omega \nabla_{H^n} w \cdot \left(F - \frac{\nabla_{H^n}\delta}{\delta^{mp-d-1}} \right) |u|^p,
\end{aligned}$$

where we use the inequality $\frac{|A|^p}{B^{p-1}} \geq pA - (p-1)B \quad (B > 0)$. Then we have proved the claim. □

The special case of Lemma 2.1 when $m = 1$, $d = 0$ and $w \equiv 1$ gives

$$(7) \quad \int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla_{H^n}\delta|^2}{\delta^p} |u|^p + \left(\frac{p-1}{p}\right)^p \int_\Omega \left[\frac{p}{p-1} \mathrm{div}_{H^n} F - \frac{p}{p-1} \frac{\Delta_{H^n}\delta}{\delta^{p-1}} + \frac{p-1}{\delta^p} \left(|\nabla_{H^n}\delta|^2 - \left|\nabla_{H^n}\delta - \delta^{p-1} F\right|^{\frac{p}{p-1}} \right) \right] |u|^p .$$

Now we describe a general Hardy type inequality that is valid for any domain in H^n.

Proposition 2.2. *Let $u \in C_0^\infty(\Omega)$ and $1 < p < Q$. Then*

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|\nabla_{H^n}\delta|^2}{\delta^p} |u|^p + \left(\frac{p-1}{p}\right)^p \int_\Omega \left[-\frac{p}{p-1} \frac{\Delta_{H^n}\delta}{\delta^{p-1}} + \frac{p-1}{\delta^p} \left(|\nabla_{H^n}\delta|^2 - |\nabla_{H^n}\delta|^{\frac{p}{p-1}} \right) \right] |u|^p .$$

Proof. Simply put $F \equiv 0$ in (7). □

Let $\Omega = \{(x, y, t) \in H^n | x_i > 0\}$ $(i = 1, \ldots, n)$, $g = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) \in \Omega$, $g_0 = (x_{01}, \ldots, x_{0,i-1}, 0, x_{0,i+1}, \ldots, x_{0n}, y_{01}, \ldots, y_{0n}, t_0) \in \partial\Omega$. Denote the Heisenberg distance from g to $\partial\Omega$ by

$$\begin{aligned} \delta(g) &= \mathrm{dist}(g, \partial\Omega) = \inf_{g_0 \in \partial\Omega} \left|g_0^{-1} \circ g\right|_{H^n} \\ &= \inf_{g_0 \in \partial\Omega} \left\{ \left[\sum_{\substack{j=1 \\ j \neq i}}^n (x_j - x_{0j})^2 + x_i^2 + \sum_{j=1}^n (y_j - y_{0j})^2 \right]^2 \right. \\ &\qquad \left. + \left(t - t_0 + 2 \sum_{\substack{j=1 \\ j \neq i}}^n x_{0j} y_j - 2 \sum_{j=1}^n x_j y_{0j} \right)^2 \right\}^{\frac{1}{4}} . \end{aligned}$$

It is easily check that $\delta(g) = x_i = \mathrm{dist}(g, g_{\inf})$, where

$$g_{\inf} = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n, y_1, \ldots, y_n, t - 2x_i y_i) \in \Omega.$$

Analogously, we can get that $\delta(g) = y_i$ when $\Omega = \{(x, y, t) \in H^n | y_i > 0\}$ $(i = 1, \ldots, n)$.

Substituting $\delta(g) = x_i$, $|\nabla_{H^n}\delta| = 1$ and $\Delta_{H^n}\delta = 0$ into (7) yields for $i = 1, \dots, n$,

$$(8)\int_\Omega |\nabla_{H^n}u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_i^p} + \left(\frac{p-1}{p}\right)^p \int_\Omega \left[\frac{p}{p-1}\mathrm{div}_{H^n}F + \frac{p-1}{x_i^p}\right.$$
$$\left.\left(1 - \left|(0,\dots,0,1,0,\dots,0,0,\dots,0) - x_i^{p-1}F\right|^{\frac{p}{p-1}}\right)\right] |u|^p.$$

We will apply this inequality to two different vector fields F and then together the corresponding inequalities to get the desired result.

Proof of Theorem 1.1. We will split the proof into two parts. This leads to two different inequalities. Each part, will in turn, be divided into two cases depending on whether $1 < p < 2$ or $p \geq 2$. The inequalities in the two parts will then be combined to prove the theorem.

Part 1

Apply (8) to the vector field

$$F = \left(\frac{a_p}{x_1^{p-2}(x_1^2+z_k^2)^{\frac{1}{2}}}, 0, \dots, 0\right), \quad 0 \leq a_p \leq 1, \quad k = 2, \dots, 2n$$

and note that

$$(9) \qquad \begin{aligned} &\frac{p}{p-1}\mathrm{div}_{H^n}F \\ = &\frac{p(2-p)a_p}{(p-1)x_1^{p-1}(x_1^2+z_k^2)^{\frac{1}{2}}} - \frac{pa_p}{(p-1)x_1^{p-3}(x_1^2+z_k^2)^{\frac{3}{2}}} \\ \geq &\frac{p(2-p)a_p}{(p-1)x_1^{p-1}(x_1^2+z_k^2)^{\frac{1}{2}}} - \frac{pa_p}{(p-1)x_1^{p-2}(x_1^2+z_k^2)}. \end{aligned}$$

Now we estimate the expression

$$(10) \qquad \begin{aligned} &\frac{p-1}{x_1^p}\left(1 - \left|(1,0,\dots,0,0,\dots,0) - x_1^{p-1}F\right|^{\frac{p}{p-1}}\right) \\ = &\frac{p-1}{x_1^p}\left[1 - \left(1 - \frac{a_p}{x_1^{-1}(x_1^2+z_k^2)^{\frac{1}{2}}}\right)^{\frac{p}{p-1}}\right]. \end{aligned}$$

Case 1. $1 < p < 2$

Lemma 2.2 in [6] contributes us to estimate (10) from below by

$$\frac{pa_p}{x_1^{p-1}(x_1^2+z_k^2)^{\frac{1}{2}}} - \frac{p}{2(p-1)}\frac{a_p^2}{x_1^{p-2}(x_1^2+z_k^2)}.$$

Using (9) and the above estimates lead to the inequality

$$(11)\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + a_p \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}\left(x_1^2+z_k^2\right)^{\frac{1}{2}}} - \left(a_p + \frac{a_p^2}{2}\right)\left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-2}\left(x_1^2+z_k^2\right)}.$$

Case 2. $p \geq 2$

By Lemma 2.3 in [6] we know that (10) is bounded from below by

$$\frac{pa_p}{x_1^{p-1}\left(x_1^2+z_k^2\right)^{\frac{1}{2}}} - \frac{p}{2}\frac{a_p^2}{x_1^{p-2}\left(x_1^2+z_k^2\right)}.$$

It follows from the estimate and (9) that

$$(12) \qquad \int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + a_p \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}\left(x_1^2+z_k^2\right)^{\frac{1}{2}}} - \left(a_p + \frac{p-1}{2}a_p^2\right)\left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-2}\left(x_1^2+z_k^2\right)}.$$

Part 2

Consider the vector field

$$F = \left(\frac{c_p x_1^{3-p}}{x_1^2+z_k^2}, 0, \ldots, 0, \frac{c_p x_1^{2-p} z_k}{x_1^2+z_k^2}, 0, \ldots, 0\right), \quad 0 \leq c_p \leq 1, \quad k = 2, \ldots, 2n.$$

Clearly, we have

$$\frac{p}{p-1}\operatorname{div}_{H^n} F = \frac{p(2-p)c_p}{p-1}\frac{1}{x_1^{p-2}\left(x_1^2+z_k^2\right)}$$

and

$$(13) \qquad \frac{p-1}{x_1^p}\left(1 - \left|(1,0,\ldots,0,0,\ldots,0) - x_1^{p-1}F\right|^{\frac{p}{p-1}}\right) = \frac{p-1}{x_1^p}\left[1 - \left(1 + \frac{\left(c_p^2 - 2c_p\right)x_1^2}{x_1^2+z_k^2}\right)^{\frac{p}{2(p-1)}}\right].$$

Again, the estimates of the expressions above will be differently treated.

Case 1. $1 < p < 2$

Since

$$\left(1 + \frac{\left(c_p^2 - 2c_p\right)x_1^2}{x_1^2+z_k^2}\right)^{\frac{p}{2(p-1)}} \leq 1 + \frac{\left(c_p^2 - 2c_p\right)x_1^2}{x_1^2+z_k^2},$$

then (13) is not less than $\frac{(p-1)\left(2c_p-c_p^2\right)}{x_1^{p-2}\left(x_1^2+z_k^2\right)}$.

Applying the above estimates to (8) we conclude that

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{(p^2-2p+2)c_p-(p-1)^2c_p^2}{p-1}\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^{p-2}\left(x_1^2+z_k^2\right)}.\end{aligned}$$

Noticing $\max\limits_{c_p}\left\{(p^2-2p+2)c_p-(p-1)^2c_p^2\right\}=1$, shows

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{1}{p-1}\left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^{p-2}\left(x_1^2+z_k^2\right)}.\end{aligned} \tag{14}$$

By multiplying (14) by $C=p\left(a_p+\frac{a_p^2}{2}\right)$ and adding the result to (11) we get

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{a_p}{C+1}\left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}\left(x_1^2+z_k^2\right)^{\frac{1}{2}}}.\end{aligned}$$

Since $\max\limits_{a_p}\frac{a_p}{C+1}=\max\limits_{a_p}\frac{a_p}{p\left(a_p+\frac{a_p^2}{2}\right)+1}=\frac{2}{3p+2}$, it follows

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{2}{3p+2}\left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}\left(x_1^2+z_k^2\right)^{\frac{1}{2}}}.\end{aligned} \tag{15}$$

At the moment we deduce the desired statement for $1<p<2$.

Case 2. $p\geq 2$

If $a\geq -1$ and $0\leq\lambda\leq 1$, then $(1+a)^\lambda\leq 1+\lambda a$ by a variant of Bernoulli's inequality. This infers

$$\left(1+\frac{\left(c_p^2-2c_p\right)x_1^2}{x_1^2+z_k^2}\right)^{\frac{p}{2(p-1)}}\leq 1+\frac{p}{2(p-1)}\frac{\left(c_p^2-2c_p\right)x_1^2}{x_1^2+z_k^2},$$

and therefore

$$\frac{p-1}{x_1^p}\left(1-\left|(1,0,\ldots,0,0,\ldots,0)-x_1^{p-1}F\right|^{\frac{p}{p-1}}\right)\geq\frac{p\left(2c_p-c_p^2\right)}{2x_1^{p-2}\left(x_1^2+z_k^2\right)}. \tag{16}$$

In this case, (16) applied to (8) follows

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + \frac{(1-p)c_p^2 + 2c_p}{2} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-2}(x_1^2+z_k^2)}.$$

Because of $\max\limits_{c_p} \left\{(1-p)c_p^2 + 2c_p\right\} = \frac{1}{p-1}$, we obtain

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + \frac{1}{2(p-1)} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-2}(x_1^2+z_k^2)}. \tag{17}$$

This inequality may be multiplied by $D = 2(p-1)a_p + (p-1)^2 a_p^2$ which, when added to (12), promotes

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + \frac{a_p}{D+1} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}(x_1^2+z_k^2)^{\frac{1}{2}}}.$$

Using $\max\limits_{a_p} \frac{a_p}{D+1} = \max\limits_{a_p} \frac{a_p}{[(p-1)a_p+1]^2} = \frac{1}{4(p-1)}$, we have

$$\int_\Omega |\nabla_{H^n} u|^p \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} + \frac{1}{4(p-1)} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-1}(x_1^2+z_k^2)^{\frac{1}{2}}}. \tag{18}$$

The statement for $p \geq 2$ is proved. □

The proof of Theorem 1.2 will be a slightly modified variant of the proof above.

Proof of Theorem 1.2. Remember (8) and put

$$F = \left(\frac{c x_1^{\tau-p+1}}{(x_1^2+z_k^2)^{\frac{\tau}{2}}}, 0, \ldots, 0\right), \quad c = c(p,\tau), \quad 0 \leq c \leq 1, \quad k = 2, \ldots 2n.$$

This implies

$$
\begin{aligned}
(19)\qquad & \frac{p}{p-1}\mathrm{div}_{H^n}F \\
= \; & \frac{p(\tau-p+1)c}{p-1}\frac{1}{x_1^{p-\tau}(x_1^2+z_k^2)^{\frac{\tau}{2}}} - \frac{p\tau c}{p-1}\frac{x_1^\tau}{x_1^{p-2}(x_1^2+z_k^2)^{\frac{\tau}{2}+1}} \\
\geq \; & \frac{p(\tau-p+1)c}{p-1}\frac{1}{x_1^{p-\tau}(x_1^2+z_k^2)^{\frac{\tau}{2}}} - \frac{p\tau c}{p-1}\frac{1}{x_1^{p-2}(x_1^2+z_k^2)}
\end{aligned}
$$

and

$$
\begin{aligned}
(20)\qquad & \frac{p-1}{x_1^p}\left(1-\left|(1,0,\ldots,0,0,\ldots,0)-x_1^{p-1}F\right|^{\frac{p}{p-1}}\right) \\
= \; & \frac{p-1}{x_1^p}\left[1-\left(1-\frac{cx_1^\tau}{(x_1^2+z_k^2)^{\frac{\tau}{2}}}\right)^{\frac{p}{p-1}}\right].
\end{aligned}
$$

As before the estimate of (20) will depend on whether $1<p<2$ or $p\geq 2$.

Case 1. $1<p<2$

According to Lemma 2.2 in [6] we have

$$
\begin{aligned}
& \left(1-\frac{cx_1^\tau}{(x_1^2+z_k^2)^{\frac{\tau}{2}}}\right)^{\frac{p}{p-1}} \\
\leq \; & 1-\frac{pc}{p-1}\frac{x_1^\tau}{(x_1^2+z_k^2)^{\frac{\tau}{2}}}+\frac{pc^2}{2(p-1)^2}\frac{x_1^{2\tau}}{(x_1^2+z_k^2)^{\tau}},
\end{aligned}
$$

and therefore, (20) is not less than

$$
\begin{aligned}
& \frac{p-1}{x_1^p}\left[\frac{pc}{p-1}\frac{x_1^\tau}{(x_1^2+z_k^2)^{\frac{\tau}{2}}}-\frac{pc^2}{2(p-1)^2}\frac{x_1^{2\tau}}{(x_1^2+z_k^2)^{\tau}}\right] \\
\geq \; & \left(pc-\frac{pc^2}{2(p-1)}\right)\frac{1}{x_1^{p-\tau}(x_1^2+z_k^2)^{\frac{\tau}{2}}}.
\end{aligned}
$$

Substituting these estimates into (8) implies

$$
\begin{aligned}
\int_\Omega |\nabla_{H^n}u|^p \;\geq\; & \left(\frac{p-1}{p}\right)^p\int_\Omega\frac{|u|^p}{x_1^p} \\
& +\frac{c(2\tau-c)}{2}\left(\frac{p-1}{p}\right)^p\int_\Omega\frac{|u|^p}{x_1^{p-\tau}(x_1^2+z_k^2)^{\frac{\tau}{2}}} \\
& -\tau c\left(\frac{p-1}{p}\right)^{p-1}\int_\Omega\frac{|u|^p}{x_1^{p-2}(x_1^2+z_k^2)}.
\end{aligned}
$$

The inequality is multiplied by $E = \frac{1}{p\tau c}$ which, when added to (14), gives

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{E}{E+1} \cdot \frac{c(2\tau - c)}{2} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-\tau} (x_1^2 + z_k^2)^{\frac{\tau}{2}}}.\end{aligned}$$

Furthermore,

(21)
$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{1 + p\tau^2 - \sqrt{1 + 2p\tau^2}}{p^2\tau^2} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-\tau} (x_1^2 + z_k^2)^{\frac{\tau}{2}}}\end{aligned}$$

by using $\max_c \left\{\frac{E}{E+1} \cdot \frac{c(2\tau - c)}{2}\right\} = \max_c \frac{c(2\tau - c)}{2(1 + p\tau c)} = \frac{1 + p\tau^2 - \sqrt{1 + 2p\tau^2}}{p^2\tau^2}$. It reaches the desired statement for $1 < p < 2$.

Case 2. $p \geq 2$

By Lemma 2.3 in [6] we have

$$\begin{aligned}& \left(1 - \frac{c x_1^\tau}{(x_1^2 + z_k^2)^{\frac{\tau}{2}}}\right)^{\frac{p}{p-1}} \\ \leq & \ 1 - \frac{pc}{p-1} \frac{x_1^\tau}{(x_1^2 + z_k^2)^{\frac{\tau}{2}}} + \frac{pc^2}{2(p-1)} \frac{x_1^{2\tau}}{(x_1^2 + z_k^2)^\tau},\end{aligned}$$

and therefore, (19) is greater or equal to

$$\begin{aligned}& \frac{p-1}{x_1^p} \left[\frac{pc}{p-1} \frac{x_1^\tau}{(x_1^2 + z_k^2)^{\frac{\tau}{2}}} - \frac{pc^2}{2(p-1)} \frac{x_1^{2\tau}}{(x_1^2 + z_k^2)^\tau}\right] \\ \geq & \left(pc - \frac{pc^2}{2}\right) \frac{1}{x_1^{p-\tau} (x_1^2 + z_k^2)^{\frac{\tau}{2}}}.\end{aligned}$$

Instituting this and (19) into (8) leads to

(22)
$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{c(c - pc + 2\tau)}{2} \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-\tau} (x_1^2 + z_k^2)^{\frac{\tau}{2}}} \\ & - \tau c \left(\frac{p-1}{p}\right)^{p-1} \int_\Omega \frac{|u|^p}{x_1^{p-2} (x_1^2 + z_k^2)}.\end{aligned}$$

The inequality is multiplied by the constant $H = \frac{1}{2(p-1)\tau c}$ which, when added to (17), concludes

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \ \geq \ & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{H}{H+1}\frac{c(c-pc+2\tau)}{2}\left(\frac{p-1}{p}\right)^{p-1}\int_\Omega \frac{|u|^p}{x_1^{p-\tau}\left(x_1^2+z_k^2\right)^{\frac{\tau}{2}}}.\end{aligned}$$

Noting $\max_c \left\{\frac{H}{H+1}\frac{c(c-pc+2\tau)}{2}\right\} = \max_c \frac{c(c-pc+2\tau)}{2[1+2(p-1)\tau c]} = \frac{\left(\sqrt{1+4\tau^2}-1\right)^2}{8(p-1)\tau^2}$, we have

$$\begin{aligned}\int_\Omega |\nabla_{H^n} u|^p \geq & \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{x_1^p} \\ & + \frac{\left(\sqrt{1+4\tau^2}-1\right)^2}{8(p-1)\tau^2}\left(\frac{p-1}{p}\right)^{p-1}\int_\Omega \frac{|u|^p}{x_1^{p-\tau}\left(x_1^2+z_k^2\right)^{\frac{\tau}{2}}}.\end{aligned} \tag{23}$$

It proves the desired statement for $p \geq 2$. □

Remark 2.3. When z_k^2 is replaced by $z_{j_1}^2 + \cdots + z_{j_r}^2$ $(2 \leq j_1 < \cdots < j_r \leq 2n,\ 1 \leq r \leq 2n-1)$, results in Theorem 1.1 and Theorem 1.2 still hold. We can also consider corresponding Hardy inequalities in the half space $\Omega = \{(x,y,t) \in H^n | t > 0\}$. To do so, we use the distance

$$\rho(g_0, g) = \left[\left((x_0-x)^2 + (y_0-y)^2\right)^2 + (t_0-t)^2\right]^{\frac{1}{4}}.$$

With a simple calculation, it has $\delta(g) = \operatorname{dist}(g, \partial\Omega) = t^{\frac{1}{2}} = \operatorname{dist}(g, g_{\inf})$, where $g = (x,y,t) \in \Omega$, $g_{\inf} = (x,y,0)$. Then we get $\nabla_{H^n}\delta = \left(yt^{-\frac{1}{2}}, -xt^{-\frac{1}{2}}\right)$, $|\nabla_{H^n}\delta|^2 = \left(|x|^2+|y|^2\right)t^{-1}$, $\Delta_{H^n}\delta = -\left(|x|^2+|y|^2\right)t^{-\frac{3}{2}}$. By these and Proposition 2.2 we obtain

$$\begin{aligned}&\int_\Omega |\nabla_{H^n} u|^p \\ \geq \ & p\left(\frac{p-1}{p}\right)^{p-1}\int_\Omega \left(|x|^2+|y|^2\right) t^{-\frac{p+2}{2}} |u|^p \\ & -(p-1)\left(\frac{p-1}{p}\right)^p \int_\Omega \left(|x|^2+|y|^2\right)^{\frac{p}{2(p-1)}} t^{-\frac{p^2}{2(p-1)}} |u|^p.\end{aligned}$$

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