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ON THE VALUE DISTRIBUTION OF DIFFERENTIAL POLYNOMIALS

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Abstract. In this paper we consider the problem of whether certain homogeneous or non-homogeneous differential polynomials in $f(z)$ necessarily have infinitely many zeros. Particularly, this extends a result of Gopalakrishna and Bhoosnurmath [3, Theorem 2] for a general differential polynomial of degree $\overline{d}(P)$ and lower degree $\underline{d}(P)$.

1. Introduction

Let $f(z)$ be a transcendental meromorphic function in the complex plane. It is assumed that the reader is familiar with the usual notations of Nevanlinna theory (See e.g. [4, 9]). We denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to +\infty$, possibly outside a set of finite linear measure E. Throughout this paper we denote by $a_j(z)$ any small meromorphic function satisfying $T(r, a_j) = S(r, f)$, $j = 1, 2, \ldots, n$.

Many mathematicians were interested in the value distribution of different expressions of a meromorphic function $f(z)$ and obtained a lot of fruitful results. In [5], Hayman discussed Picard's values of a meromorphic function $f(z)$ and its derivatives. In particular, he showed that

Theorem A. *Let $f(z)$ be a transcendental entire function. Then*

(a) *for $n \geq 3$ and $a \neq 0$, $\Psi(z) = f'(z) - a[f(z)]^n$ assumes all finite values infinitely often;*

(b) *for $n \geq 2$, $\Psi(z) = f'(z)[f(z)]^n$ assumes all finite values except possibly zero infinitely often.*

Later in 1964, Hayman showed further in his monograph [4] that

Theorem B. *If $f(z)$ is meromorphic and transcendental in the plane and has only a finite number of poles and zeros, then every meromorphic function $\Psi(z)$ of the form $\Psi(z) = \sum_{i=1}^{n} a_i f^{(i)}(z)$ assumes every finite complex value except possibly zero infinitely often, or else $\Psi(z)$ is constant.*

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In 1967, Clunie [2] proved Theorem A(b) for $n \geq 1$ and later on Sons [6] generalized Theorem A(b) and in fact, he proved the following result on a monomial in $f(z)$

Theorem C. *If $f(z)$ is a transcendental entire function and*

$$\Psi(z) = [f(z)]^{n_0}[f'(z)]^{n_1} \cdots [f^{(k)}(z)]^{n_k},$$

where $n_0 \geq 2, n_k \geq 1$ and $n_i \geq 0$ for $i \neq 0, k$, then $\delta(a, \Psi) < 1$ for $a \neq 0, \infty$. Moreover if $N_{(1)}(r, \frac{1}{f}) = S(r, f)$, then for $n_0 \geq 1$ the same conclusion holds good, where in $N_{(1)}(r, \frac{1}{f})$ we count only simple zeros of $f(z)$.

Regarding the deficiencies of a monomial in $f(z)$, Yang [7, 8] further generalized Theorem C to meromorphic functions as follows

Theorem D. *Let f be transcendental meromorphic with*

$$N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f) \tag{1}$$

and $\Psi(z) = \sum a(z)[f(z)]^{p_0}[f'(z)]^{p_1} \cdots [f^{(k)}(z)]^{p_k}$ with no constant term. If the degree n of the homogeneous differential polynomial $\Psi(z)$ is greater than one and $p_0 < n, 0 \leq p_i \leq n$ for $i \neq 0$, then $\delta(a, \Psi) < 1$ for all $a \neq 0, \infty$.

Theorem E. *Let $f(z)$ and $\Psi(z)$ be as in Theorem* D *and all the terms of $\Psi(z)$ have different degrees at least two, i.e., $\Psi(z)$ is non-homogeneous. Then we have $\delta(a, \Psi) \leq 1 - \frac{1}{2n}$ for $a \neq \infty$.*

Independently, by generalizing Theorem B as Gopalakrishna and Bhoosnurmath's goal, they actually obtained a result which was a generalization of Theorem D above and the argument they used is much simpler and elegant than that of Yang applied. In fact, they proved the following

Theorem F ([3, Theorem 2]). *Let $f(z)$ be a transcendental meromorphic function satisfying* (1) *and let $P[f]$ be a homogeneous differential polynomial in $f(z)$. If $P[f]$ does not reduce to a constant, then $\delta(a, P[f]) = 0$ for $a \neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.*

In this paper, two results are proved. In Theorem 1, we try to obtain bounds for

$$\overline{\lim_{r\to+\infty}} \frac{T(r, P[f])}{T(r, f)} \quad \text{and} \quad \underline{\lim_{r\to+\infty}} \frac{T(r, P[f])}{T(r, f)},$$

where $P[f]$ is a differential polynomial in $f(z)$. Then as a consequence we can obtain the result of Theorem F as a special case of Theorem 2.

2. Definitions and lemmas

For a positive integer j, by a monomial in $f(z)$ we mean an expression of the type

$$M_j[f] = a_j(z)[f(z)]^{n_{0j}}[f'(z)]^{n_{1j}} \cdots [f^{(k)}(z)]^{n_{kj}},$$

where $n_{0j}, n_{1j}, \ldots, n_{kj}$ are non-negative integers. We define $d(M_j) = \sum_{i=0}^{k} n_{ij}$ as the degree of $M_j[f]$ and $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$ as the weight of $M_j[f]$.

Next, a differential polynomial in $f(z)$ is a finite sum of such monomials, i.e.,

$$P[f] = \sum_{j=1}^{n} a_j\,(z)\,M_j[f].$$

We define

$$\overline{d}(P) = \max_{1\leq j\leq n}\{d(M_j)\}, \quad \underline{d}(P) = \min_{1\leq j\leq n}\{d(M_j)\} \quad \text{and} \quad \Gamma_P = \max_{1\leq j\leq n}\{\Gamma_{M_j}\}$$

as the *degree*, the *lower degree* and the *weight* of $P[f]$ respectively. If, in particular, $\overline{d}\,(P) = \underline{d}\,(P)$, then $P[f]$ is called *homogeneous* and *non-homogeneous* otherwise.

Lemma 1 ([1]). *Let $f(z)$ be a meromorphic function and $P[f]$ be a differential polynomial with coefficient $a_j\,(z)$ and degree $\overline{d}\,(P)$ and lower degree $\underline{d}\,(P)$. Then*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2 ([4, Lemma of the logarithmic derivatives]). *Let $f(z)$ be meromorphic and non-constant in the plane. Then there are positive constants C_1 and C_2 such that*

$$m\left(r, \frac{f'}{f}\right) \leq C_1 \log r + C_2 \log T\,(r,f)$$

as r tends to infinity outside possibly a set E of finite measure.

Consequently, Lemma 2 implies the famous result

$$m\left(r, \frac{f^{(k)}}{f}\right) = S\,(r,f)$$

for any positive integer k (See [4, Theorem 3.1]).

Lemma 3. *Let $f(z)$ be a meromorphic function with a pole of order $p \geq 1$ at z_0. If $P[f]$ is a differential polynomial in $f(z)$ whose coefficient are analytic at z_0, then $P[f]$ has a pole at z_0 of order at most $p\overline{d}(P) + \Gamma_P - \overline{d}(P)$.*

Proof. Now $P[f]$ is a sum of terms of the form $a_j f^{n_{0j}}(f')^{n_{1j}} \cdots (f^{(k)})^{n_{kj}}$ where a_j is analytic at z_0. If this term has a pole at z_0, then its order is at most

$$\begin{aligned} & \max_{1\le j\le n}\left\{\sum_{s=0}^{k}(p+s)n_{sj}\right\} \\ = & \max_{1\le j\le n}\left\{(p-1)\sum_{s=0}^{k} n_{sj} + (n_{0j}+2n_{1j}+\cdots+(k+1)\,n_{kj})\right\} \\ \le & (p-1)\,\overline{d}(P) + \Gamma_P \\ \le & p\overline{d}(P) + \Gamma_P - \overline{d}(P), \end{aligned}$$

completing the proof of the lemma. □

3. Our main results

Theorem 1. *Let $f(z)$ be a transcendental meromorphic function satisfying condition* (1) *and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\overline{d}(P)$ and lower degree $\underline{d}(P)$. Then*

$$\underline{d}(P) \le \varliminf_{r\to+\infty} \frac{T(r,P[f])}{T(r,f)} \le \varlimsup_{r\to+\infty} \frac{T(r,P[f])}{T(r,f)} \le 2\overline{d}(P) - \underline{d}(P).$$

Proof. The poles of $P[f]$ can occur only at the poles of f or at the poles of the coefficients a_j of $P[f]$. As $T(r,a_j) = S(r,f)$, we can ignore the poles of the coefficients a_j.

At z_0, a pole of f of order p, it is easily seen from Lemma 3 that $P[f]$ has a pole z_0 of order at most $p\overline{d}(P) + \Gamma_P - \overline{d}(P)$. Hence we have

$$N(r,P[f]) \le \overline{d}(P)\,N(r,f) + [\Gamma_P - \overline{d}(P)]\overline{N}(r,f) + S(r,f) \tag{2}$$

and then this and the assumption (1) give

$$\begin{aligned} N\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) &\le N(r,P[f]) + N\left(r,\frac{1}{f^{\overline{d}(P)}}\right) \\ &\le [\Gamma_P - \overline{d}(P)]\overline{N}(r,f) + \overline{d}(P)\left[N(r,f) + N\left(r,\frac{1}{f}\right)\right] + S(r,f) \\ &= S(r,f). \end{aligned} \tag{3}$$

On the one hand, it follows from (3), Lemma 1 and then the first fundamental theorem that

$$\begin{aligned} T(r,P[f]) &\le T\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) + T\left(r,f^{\overline{d}(P)}\right) \\ &\le m\left(r,\frac{P[f]}{f^{\overline{d}(P)}}\right) + \overline{d}(P)\,T(r,f) \\ &\le \left[\overline{d}(P) - \underline{d}(P)\right] m\left(r,\frac{1}{f}\right) + \overline{d}(P)\,T(r,f) + S(r,f) \end{aligned} \tag{4}$$

$$\begin{aligned}
&\leq [\overline{d}(P) - \underline{d}(P)]T\left(r, \frac{1}{f}\right) + \overline{d}(P)T(r, f) + S(r, f) \\
&= [2\overline{d}(P) - \underline{d}(P)]T(r, f) + S(r, f).
\end{aligned}$$

Thus inequality (4) implies that

$$\overline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)} \leq 2\overline{d}(P) - \underline{d}(P). \tag{5}$$

On the other hand, we also have from the first fundamental theorem, (3) and then Lemma 1 the following

$$\begin{aligned}
\overline{d}(P)T(r, f) &\leq T\left(r, f^{\overline{d}(P)}\right) \\
&\leq T\left(r, \frac{f^{\overline{d}(P)}}{P[f]}\right) + T(r, P[f]) \\
&\leq T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + T(r, P[f]) + O(1) \\
&\leq T(r, P[f]) + [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r, f) \\
&\leq T(r, P[f]) + [\overline{d}(P) - \underline{d}(P)]T(r, f) + S(r, f)
\end{aligned}$$

$$\underline{d}(P)T(r, f) \leq T(r, P[f]) + S(r, f). \tag{6}$$

Thus inequality (6) implies that

$$\underline{d}(P) \leq \underline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)}. \tag{7}$$

Hence by inequalities (5) and (7) we get

$$\underline{d}(P) \leq \underline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)} \leq \overline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)} \leq 2\overline{d}(P) - \underline{d}(P),$$

completing the proof of the theorem. □

Remark 1. In particular, if the given differential polynomial is homogenous, i.e., $\overline{d}(P) = \underline{d}(P) = n$ for some positive integer n, then we obtain

$$n \leq \underline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)} \leq \overline{\lim_{r \to +\infty}} \frac{T(r, P[f])}{T(r, f)} \leq n,$$

so that

$$\lim_{r \to +\infty} \frac{T(r, P[f])}{T(r, f)} = n,$$

outside possibly a set E of finite linear measure. In other words, we have

$$T(r, P[f]) = nT(r, f) + O(1) \tag{8}$$

as $r \to +\infty$ outside possibly a set E of finite linear measure in this case.

Theorem 2. *Let $f(z)$ be a transcendental meromorphic function satisfying the assumption* (1) *and let $P[f]$ be a differential polynomial in $f(z)$ of degree $\overline{d}(P)$ and lower degree $\underline{d}(P)$. Suppose that $P[f]$ does not reduce to a constant.*

(a) *If $P[f]$ is a homogeneous differential polynomial, then we have*

$$\delta(a, P[f]) = 0$$

for any $a \neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

(b) *If $P[f]$ is a non-homogeneous differential polynomial with $2\underline{d}(P) > \overline{d}(P)$, then we have*

$$\delta(a, P[f]) \leq 1 - \frac{2\underline{d}(P) - \overline{d}(P)}{\underline{d}(P)} < 1$$

for any $a \neq 0$, i.e., $P[f]$ assumes all finite complex values except possibly zero infinitely often.

Proof. By Theorem 1, we see that small functions of f are small functions of $P[f]$ and small functions of $P[f]$ are also small functions of f, i.e.,

$$S(r, f) = S(r, P[f]). \tag{9}$$

By (9), it follows from assumption (1) and inequality (2) that

$$N(r, P[f]) = S(r, P[f]).$$

We also have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{P[f]}\right) &\leq \overline{N}\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + \overline{N}\left(r, \frac{f^{\overline{d}(P)}}{P[f]}\right) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{f^{\overline{d}(P)}}{P[f]}\right) \\ &\leq \overline{N}\left(r, \frac{1}{f}\right) + T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + O(1). \end{aligned} \tag{10}$$

Now Lemma 1, inequalities (3) and (9) imply that

$$\begin{aligned} T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) &= m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + N\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \\ &\leq [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r, f) \\ &= [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r, P[f]). \end{aligned} \tag{11}$$

Hence using (11), inequality (10) can be written as

$$\overline{N}\left(r, \frac{1}{P[f]}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r, P[f])$$

and by hypothesis (1) and (9), we get

$$\overline{N}\left(r, \frac{1}{P[f]}\right) \le [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + S(r, P[f]). \tag{12}$$

If $b \neq 0$, then the second fundamental theorem and inequality (12) imply that (13)

$$T(r, P[f]) \le \overline{N}(r, P[f]) + \overline{N}\left(r, \frac{1}{P[f]}\right) + \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f])$$

$$T(r, P[f]) \le [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]).$$

We have the following two cases.

Case (a): If $P[f]$ is a homogeneous differential polynomial, i.e., $\overline{d}(P) = \underline{d}(P)$ then by the above inequality (13) we obtain

$$T(r, P[f]) \le \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]), \tag{14}$$

but it follows from (8) that $P[f]$ is a transcendental meromorphic function and then this relation and inequality (14) imply (a).

Case (b): By Theorem 1, we still have $\underline{d}(P)T(r, f) \le T(r, P[f]) + S(r, f)$ for all sequences of r tending to $+\infty$ outside possibly a set E of finite linear measure. If $P[f]$ is a non-homogeneous differential polynomial with $2\underline{d}(P) > \overline{d}(P)$, then we obtain from inequality (13) that

$$\begin{aligned} T(r, P[f]) &\le [\overline{d}(P) - \underline{d}(P)]m\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]) \\ &\le [\overline{d}(P) - \underline{d}(P)]T(r, f) + \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]) \\ &\le \left[\frac{\overline{d}(P) - \underline{d}(P)}{\underline{d}(P)}\right]T(r, P[f]) + \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]) \end{aligned}$$

$$\left[\frac{2\underline{d}(P) - \overline{d}(P)}{\underline{d}(P)}\right]T(r, P[f]) \le \overline{N}\left(r, \frac{1}{P[f]-b}\right) + S(r, P[f]).$$

Since $2\underline{d}(P) > \overline{d}(P)$, the desired result follows and thus we complete the proof of Theorem 2. □

4. Further remarks

In this section, a few remarks will be given concerning the question we consider in this paper.

Remark 2. Our Theorem 2 is much more general than that of Gopalakrishna and Bhoosnurmath [3, pp. 334–335] because they obtained the inequality (14) for homogeneous $P[f]$ only, but the main inequality we obtain here is (13) which works for *any*, homogeneous or non-homogeneous, differential polynomial $P[f]$.

Remark 3. The following example shows that the condition $2\underline{d}(P) > \overline{d}(P)$ cannot be dropped from Theorem 2(b).

Example 1. Let

$$f(z) = e^z \quad \text{and} \quad P[f] = f^2(z) + af(z) - af'(z) + 1$$

for any complex number a. Then we have $\overline{d}(P) = 2$ and $\underline{d}(P) = 0$. However, $P[f] - 1 = e^z \neq 0$ for any z and hence

$$\delta(1, P[f]) = 1.$$

Remark 4. We note that the condition (1) was used heavily in the proofs of Theorems D to F, and our two theorems here. In the remark made in [7, p. 201], Yang noted that Theorem D is also valid when the condition (1) is replaced by the weaker condition

$$N_1(r, f) + N_1\left(r, \frac{1}{f}\right) = S(r, f), \tag{15}$$

where $N_1(r, f)$ and $N_1\left(r, \frac{1}{f}\right)$ denote the counting functions of simple poles and simple zeros of $f(z)$ in $|z| \leq r$ respectively.

However, Yang [8] and Gopalakrishna and Bhoosnurmath [3] did not say whether Theorems E and F were still valid under the condition (15). Hence it is natural to conjecture that

Conjecture. Theorems 1 and 2 hold good even under the weaker condition (15).

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