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WEAK FORMS OF SUBTRACTION ALGEBRAS

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Abstract. As a weak form of a subtraction algebra, the notion of weak subtraction algebras is introduced, and its examples are given. A method to make a weak subtraction algebra from a quasi-ordered set is provided.

1. Introduction

B. M. Schein [6] considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [7] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [4] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [3], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Y. B. Jun and K. H. Kim [5] introduced the notion of prime and irreducible ideals of a subtraction algebra, and gave a characterization of a prime ideal. They also provided a condition for an ideal to be a prime/irreducible ideal. In this paper, we introduce the notion of weak subtraction algebras, and give its examples. We investigate relations between a subtraction algebra and a weak subtraction algebra. We give a method to make a weak subtraction algebra from a quasi-ordered set.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;

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(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [4, 5]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1 ([4]). A nonempty subset A of a subtraction algebra X is called an *ideal* of X if it satisfies

- $0 \in A$
- $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Lemma 2.2 ([5]). *An ideal A of a subtraction algebra X has the following property:*

$$(\forall x \in X)(\forall y \in A)(x \leq y \Rightarrow x \in A).$$

3. Weak forms of subtraction algebras

We introduce more weak forms of subtraction algebras.

Definition 3.1. By a *weak subtraction algebra* (WS-algebra), we mean a triplet $(W, -, 0)$, where W is a nonempty set, $-$ is a binary operation on W and $0 \in W$ is a nullary operation, called *zero element*, such that

(b1) $(\forall x \in W)\ (x - 0 = x,\ x - x = 0)$,
(b2) $(\forall x, y, z \in W)\ ((x - y) - z = (x - z) - y)$,
(b3) $(\forall x, y, z \in W)\ ((x - y) - z = (x - z) - (y - z))$.

Example 3.2. Let $W = \{0, a, b, c\}$ be a set with the following Cayley tables.

$-_1$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	0	0	0
c	c	0	0	0

$-_2$	0	a	b	c
0	0	0	0	0
a	a	0	a	a
b	b	b	0	0
c	c	c	0	0

$-_3$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
c	c	c	c	0

$-_4$	0	a	b	c
0	0	0	0	0
a	a	0	0	0
b	b	b	0	b
c	c	c	c	0

$-_5$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

$-_6$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	b	0
c	c	c	c	0	0
d	d	d	c	b	0

$-_7$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	0
c	c	c	c	0	c
d	d	d	d	d	0

$-_8$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	0	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

It is routine to check that $(W, -_1, 0)$, $(W, -_2, 0)$, $(W, -_3, 0)$, $(W, -_4, 0)$, $(W, -_5, 0)$, $(W, -_6, 0)$, $(W, -_7, 0)$ and $(W, -_8, 0)$ are WS-algebras.

Proposition 3.3. *For a WS-algebra* $(W, -, 0)$, *we have*

(i) $(\forall x \in W)\ (0 - x = 0)$,
(ii) $(\forall x, y \in W)\ ((x - y) - x = 0)$,
(iii) $(\forall x, y, z \in W)\ (x - y = 0 \Rightarrow (x - z) - (y - z) = 0)$.

Proof. (i) Putting $x = y = z$ in (b3) and using (b1), we have

$$0 = 0 - 0 = (x - x) - (x - x) = (x - x) - x = 0 - x.$$

(ii) Replacing z by x in (b3) and using (b1) and (i), we get

$$(x - y) - x = (x - x) - (y - x) = 0 - (y - x) = 0.$$

(iii) Let $x, y, z \in W$ be such that $x - y = 0$. Then

$$(x - z) - (y - z) = (x - y) - z = 0 - z = 0.$$

This completes the proof. □

Define a relation $\leq$ on a WS-algebra $(W, -, 0)$ as follows:

$$(\forall x, y \in W)\,(x \leq y \Leftrightarrow x - y = 0).$$

This relation $\leq$ may not be an order relation on a WS-algebra. In fact, in the WS-algebra $(W, -_1, 0)$ in Example 3.2, we can not guarantee the antisymmetry of $\leq$.

Proposition 3.4. *If a WS-algebra* $(W, -, 0)$ *satisfies the identity*

$$(\forall x, y \in W)\,(x - (x - y) = y - (y - x)),$$

then $\leq$ *is an order relation on* W *and* 0 *is the least element.*

Proof. By (b1), $\leq$ is reflexive. Proposition 3.3(i) implies $0 \leq x$ for all $x \in W$. Let $x, y \in W$ be such that $x \leq y$ and $y \leq x$. Then $x - y = 0$ and $y - x = 0$, so

$$x = x - 0 = x - (x - y) = y - (y - x) = y - 0 = y$$

proving the antisymmetry of $\leq$. Now let $x, y, z \in W$ be such that $x \leq y$ and $y \leq z$. Then

$$\begin{aligned} x - z &= (x - 0) - z = (x - (x - y)) - z \\ &= (y - (y - x)) - z = (y - z) - (y - x) \\ &= 0 - (y - x) = 0 \end{aligned}$$

which yields $x \leq z$. Hence $\leq$ is an order relation on W. □

Lemma 3.5. *Every subtraction algebra* X *satisfies the following equality:*

$$(\forall x, y, z \in X)\,((x - y) - z = (x - z) - (y - z)).$$

Proof. For any $x, y, z \in X$, we have

$$\begin{aligned} &((x - z) - (y - z)) - ((x - y) - z) \\ &= (((x - z) - z) - (y - z)) - ((x - y) - z) \quad \text{by (a1)} \\ &\leq ((x - z) - y) - ((x - y) - z) \quad \text{by (a7) and (a9)} \\ &= ((x - y) - z) - ((x - y) - z) \quad \text{by (S3)} \\ &= 0, \end{aligned}$$

and so $((x - z) - (y - z)) - ((x - y) - z) = 0$, that is,

$$(x - z) - (y - z) \leq (x - y) - z.$$

Using (S3), (a3) and (a7), we get

$$\begin{aligned} &((x - y) - z) - ((x - z) - (y - z)) \\ &= ((x - z) - y) - ((x - z) - (y - z)) \\ &\leq (y - z) - y = 0, \end{aligned}$$

and therefore $((x - y) - z) - ((x - z) - (y - z)) = 0$, i.e.,

$$(x - y) - z \leq (x - z) - (y - z).$$

Consequently the desired result is valid. □

Using Lemma 3.5, we have the following theorem.

Theorem 3.6. *Every subtraction algebra is a WS-algebra.*

The converse of Theorem 3.6 may not be true as seen in the following example.

Example 3.7. The WS-algebras in Example 3.2 are not subtraction algebras.

A reflexive and transitive relation $\mathscr{R}$ on a set W is called a *quasi-ordering* of W, and the couple $(W, \mathscr{R})$ is then called a *quasi-ordered set* (see [2, p. 20]).

Proposition 3.8. *Let $\mathscr{R}_W$ be a relation on a WS-algebra W defined by*

$$(\forall x, y \in W)\,((x,y) \in \mathscr{R}_W \Leftrightarrow y - x = 0).$$

Then $\mathscr{R}_W$ is a quasi-ordering of W. Moreover,

(i) $(\forall x \in W)\ ((x,0) \in \mathscr{R}_W)$,
(ii) $(\forall x \in W)\ ((0,x) \in \mathscr{R}_W \Rightarrow x = 0)$.

We then call $\mathscr{R}_W$ the *induced quasi-ordering* of a WS-algebra W.

Proof. Since $x - x = 0$ for all $x \in W$, we have $(x,x) \in \mathscr{R}_W$, that is, $\mathscr{R}_W$ is reflexive. Let $x, y, z \in W$ be such that $(x,y) \in \mathscr{R}_W$ and $(y,z) \in \mathscr{R}_W$. Then $y - x = 0$ and $z - y = 0$. Using (a2) and Lemma 3.5, we have

$$0 = 0 - x = (z-y) - x = (z-x) - (y-x) = (z-x) - 0 = z - x,$$

and hence $(x,z) \in \mathscr{R}_W$, that is, $\mathscr{R}_W$ is transitive. Hence $\mathscr{R}_W$ is a quasi-ordering of W. Moreover, (i) follows directly from Proposition 3.3(i). Now let $x \in W$ be such that $(0,x) \in \mathscr{R}_W$. Then $x = x - 0 = 0$. This completes the proof. □

Proposition 3.9. *Let $\mathscr{R}_W$ be the induced quasi-ordering of a WS-algebra W. Then*

(i) $(\forall x, y, z \in W)\ ((x,y) \in \mathscr{R}_W \Rightarrow (x-z, y-z) \in \mathscr{R}_W)$.
(ii) $(\forall x, y, z \in W)\ ((x,y) \in \mathscr{R}_W \Rightarrow (z-x, z-y) \in \mathscr{R}_W)$.
(iii) $(\forall x, y \in W)\ ((y, x-(x-y)) \in \mathscr{R}_W)$.
(iv) $(\forall x, y, z \in W)\ ((x-y, (x-z)-(y-z)) \in \mathscr{R}_W)$.

Proof. (i) and (ii). Let $x, y, z \in W$ be such that $(x,y) \in \mathscr{R}_W$. Then $y - x = 0$, and so

$$(y-z) - (x-z) = (y-x) - z = 0 - z = 0,$$

and

$$\begin{aligned}(z-x)-(z-y) &= \Big(z-(z-y)\Big) - x = (z-x) - \Big((z-y)-x\Big)\\ &= (z-x) - \Big((z-x)-(y-x)\Big)\\ &= (z-x) - \Big((z-x)-0\Big)\\ &= (z-x)-(z-x) = 0.\end{aligned}$$

Hence $(x-z, y-z) \in \mathscr{R}_W$ and $(z-y, z-x) \in \mathscr{R}_W$ for all $z \in W$.

(iii) is by (b1) and (b2).

(iv) Proposition 3.3(ii) implies that $(x,\ x-z) \in \mathscr{R}_W$ for all $x, z \in W$. It follows from (b2), Lemma 3.5 and Proposition 3.9(i) that

$$\Big(x-y,\ (x-z)-(y-z)\Big) = \Big(x-y,\ (x-y)-z\Big) = \Big(x-y,\ (x-z)-y\Big) \in \mathscr{R}_W$$

for all $x, y, z \in W$. □

For every quasi-ordering $\mathscr{R}$ of W, denote by $\mathscr{E}_{\mathscr{R}}$ the relation on W given by

$$(\forall x, y \in W)\,((x,y) \in \mathscr{E}_{\mathscr{R}} \Leftrightarrow (x,y) \in \mathscr{R},\ (y,x) \in \mathscr{R}).$$

Obviously $\mathscr{E}_{\mathscr{R}}$ is an equivalence relation on W, which is called an *equivalence relation induced by* $\mathscr{R}$. Denote by $[a]_{\mathscr{E}_{\mathscr{R}}}$ the equivalence class containing a and by $W/\mathscr{E}_{\mathscr{R}}$ the set of all equivalence classes of W with respect to $\mathscr{E}_{\mathscr{R}}$, that is,

$$[a]_{\mathscr{E}_{\mathscr{R}}} = \{x \in W \mid (x,a) \in \mathscr{E}_{\mathscr{R}}\} \text{ and } W/\mathscr{E}_{\mathscr{R}} = \{[a]_{\mathscr{E}_{\mathscr{R}}} \mid a \in W\}.$$

Define a relation $\preceq_{\mathscr{R}}$ on $W/\mathscr{E}_{\mathscr{R}}$ by

$$(\forall a, b \in W)\,([a]_{\mathscr{E}_{\mathscr{R}}} \preceq_{\mathscr{R}} [b]_{\mathscr{E}_{\mathscr{R}}} \Leftrightarrow (a,b) \in \mathscr{R}).$$

Then $\preceq_{\mathscr{R}}$ is a partial order on $W/\mathscr{E}_{\mathscr{R}}$, and so $(W/\mathscr{E}_{\mathscr{R}}, \preceq_{\mathscr{R}})$ becomes a poset, which is called a *poset assigned to the quasi-ordered set* $(W, \mathscr{R})$. A relation $\mathscr{R}$ on W is said to be *compatible* if $(x-u, y-v) \in \mathscr{R}$ whenever $(x,y) \in \mathscr{R}$ and $(u,v) \in \mathscr{R}$ for all $x, y, u, v \in W$. A compatible equivalence relation on W is called a *congruence relation* on W. The set

$$[0]_{\mathscr{R}} = \{x \in W \mid (x,0) \in \mathscr{R}\}$$

is called the *kernel* of $\mathscr{R}$.

Theorem 3.10. *Let $\mathscr{R}_W$ be the induced quasi-ordering of a WS-algebra W and let $\Theta = \mathscr{E}_{\mathscr{R}_W}$ be the equivalence relation induced by $\mathscr{R}_W$. Then*

(i) *Θ is a congruence relation on W with kernel $[0]_\Theta = \{0\}$.*

(ii) *the quotient algebra $(W/\Theta, \ominus, [0]_\Theta)$ is a WS-algebra, where the operation $\ominus$ on W/Θ is defined by*

$$[a]_\Theta \ominus [b]_\Theta = [a-b]_\Theta.$$

Proof. (i) Note that Θ is an equivalence relation on W. Let $x, y, u, v \in W$ be such that $(x,y) \in \Theta$ and $(u,v) \in \Theta$. Then $(x,y) \in \mathscr{R}_W$, $(y,x) \in \mathscr{R}_W$, $(u,v) \in \mathscr{R}_W$, and $(v,u) \in \mathscr{R}_W$. Using (i) and (ii) of Proposition 3.9, we obtain $(x-u, x-v) \in \mathscr{R}_W$ and $(x-v, y-v) \in \mathscr{R}_W$. By the transitivity of $\mathscr{R}_W$, we get $(x-u, y-v) \in \mathscr{R}_W$. Similarly, we have $(y-v, x-u) \in \mathscr{R}_W$. Hence $(x-u, y-v) \in \Theta$, that is, Θ is a congruence relation on W. Now if $x \in [0]_\Theta$, then $(x,0) \in \Theta$ and so $(0,x) \in \mathscr{R}_W$. It follows from Proposition 3.8(ii) that $x = 0$. Hence $[0]_\Theta = \{0\}$.

(ii) is straightforward. □

Let W be a WS-algebra and $\emptyset \neq K \subseteq W$. Denote by θ_K the relation on W given by

$$(\forall x, y \in W)\,((x,y) \in \theta_K \Leftrightarrow x-y \in K,\ y-x \in K).$$

Lemma 3.11. *If θ_K is reflexive for every nonempty subset K of a WS-algebra W, then $[0]_{\theta_K} = K$.*

Proof. Suppose that θ_K is reflexive for every nonempty subset K of W. Then $0 = x - x \in K$. If $a \in K$, then $a - 0 = a \in K$ and $0 - a = 0 \in K$. Hence $(a, 0) \in \theta_K$, that is, $a \in [0]_{\theta_K}$. Conversely if $a \in [0]_{\theta_K}$, then $(a, 0) \in \theta_K$ and hence $a = a - 0 \in K$. Therefore $[0]_{\theta_K} = K$. □

Lemma 3.12. *Let K be a nonempty subset of a WS-algebra W. Assume that the relation θ_K is an equivalence relation on W. Then*

$$a \in K,\ a - b \in K \ \text{ and } \ b - a = 0 \ \text{ imply } \ b \in K.$$

Proof. Suppose that $a \in K$, $a - b \in K$ and $b - a = 0$. Then $b - a = 0 \in [0]_{\theta_K} = K$, and so $(a, b) \in \theta_K$. Since θ_K is an equivalence relation on W, a and b belong to the same class of θ_K. Hence $a \in K = [0]_{\theta_K}$ implies $b \in [0]_{\theta_K} = K$. This completes the proof. □

We provide a method to construct a WS-algebra from a quasi-ordered set.

Theorem 3.13. *Let $(W, \mathscr{R})$ be a quasi-ordered set. Suppose $0 \notin W$ and $W_0 = W \cup \{0\}$. Define a binary operation $-$ on W_0 as follows:*

$$x - y = \begin{cases} 0 & \text{if } (x, y) \in \mathscr{R} \\ x & \text{otherwise.} \end{cases}$$

Then $(W_0, -, 0)$ is a WS-algebra.

Proof. Since $\mathscr{R}$ is reflexive, obviously $x - x = 0$ for all $x \in W$. Since $(x, 0) \notin \mathscr{R}$ for every $x \in W$, we have $x - 0 = x$ for all $x \in W$. Note that $0 - x = 0$ for all $x \in W$. Assume that $(x, y) \notin \mathscr{R}$ and $(x, z) \notin \mathscr{R}$. Then

$$(x - y) - z = x - z = x = x - y = (x - z) - y.$$

If $(x, y) \in \mathscr{R}$ and $(x, z) \notin \mathscr{R}$, then

$$(x - y) - z = 0 - z = 0 = x - y = (x - z) - y.$$

Suppose that $(x, y) \notin \mathscr{R}$ and $(x, z) \in \mathscr{R}$. Then

$$(x - y) - z = x - z = 0 = 0 - y = (x - z) - y.$$

If $(x, y) \in \mathscr{R}$ and $(x, z) \in \mathscr{R}$, then

$$(x - y) - z = 0 - z = 0 = 0 - y = (x - z) - y.$$

This proves the condition (b2) holds. To verify the condition (b3), we consider the following cases:

(1) $(x, y) \in \mathscr{R}$ and $(y, z) \in \mathscr{R}$.
(2) $(x, y) \notin \mathscr{R}$ and $(y, z) \in \mathscr{R}$.
(3) $(x, y) \in \mathscr{R}$ and $(y, z) \notin \mathscr{R}$.
(4) $(x, y) \notin \mathscr{R}$ and $(y, z) \notin \mathscr{R}$.

For the case (1), we have $(x, z) \in \mathscr{R}$, and so

$$(x - y) - z = 0 - z = 0 = 0 - 0 = (x - z) - (y - z).$$

Case (2) implies that

$$(x - y) - z = x - z = (x - z) - 0 = (x - z) - (y - z).$$

For the case (3), we get first $(x - y) - z = 0 - z = 0$. If $(x, z) \in \mathscr{R}$, then

$$(x - z) - (y - z) = 0 - (y - z) = 0 = (x - y) - z;$$

if $(x, z) \notin \mathscr{R}$, then

$$(x - z) - (y - z) = x - y = 0 = (x - y) - z.$$

For the case (4), if $(x, z) \in \mathscr{R}$, then

$$(x - y) - z = x - z = 0 = 0 - y = (x - z) - (y - z).$$

If $(x, z) \notin \mathscr{R}$, then

$$(x - y) - z = x - z = x = x - y = (x - z) - (y - z).$$

Hence the condition (b3) is valid. Therefore $(W_0, -, 0)$ is a WS-algebra. □

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