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COMMON FIXED POINTS UNDER LIPSCHITZ TYPE CONDITION

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Abstract. The aim of the present paper is three fold. Firstly, we obtain common fixed point theorems for a pair of selfmaps satisfying nonexpansive or Lipschitz type condition by using the notion of pointwise R-weak commutativity but without assuming the completeness of the space or continuity of the mappings involved (Theorem 1, Theorem 2 and Theorem 3). Secondly, we generalize the results obtained in first three theorems for four mappings by replacing the condition of noncompatibility of maps with the property (E.A) and using the R-weak commutativity of type (A_g) (Theorem 4). Thirdly, in Theorem 5, we show that if the aspect of noncompatibility is taken in place of the property (E.A), the maps become discontinuous at their common fixed point. We, thus, provide one more answer to the problem posed by Rhoades [11] regarding the existence of contractive definition which is strong enough to generate fixed point but does not forces the maps to become continuous.

1. Introduction

The study of common fixed points of compatible mappings emerged as an area of intense research activity ever since Jungck [2] introduced the notion of compatible mappings in 1986. However, the study of common fixed points of noncompatible mappings is also interesting. Work on these lines was initiated by Pant [4, 5, 6, 7]. In the study of common fixed points of compatible mappings, we often require assumptions on completeness of the space or continuity of the mappings involved besides some contractive condition, but the study of fixed points of noncompatible mappings can be extended to the class of nonexpansive or Lipschitz type mappings pairs [6, 10], even without assuming continuity of the mappings involved or completeness of the space.

Two selfmaps f, g of a metric space (X, d) are called R-weakly commuting (see Pant [4]) if there exists some real number $R > 0$ such that $d(fgx, gfx) \leq$

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$R(d(fx,gx))$ for all x in X. f and g are called pointwise R-weakly commuting if given x in X, there exists $R > 0$ such that $d(fgx, gfx) \leq R(d(fx, gx))$.

It was proved by Pant [4, 5, 8] that pointwise R-weak commutativity is

(i) equivalent to commutativity at coincident points; and
(ii) a necessary, hence minimal, condition for the existence of common fixed points of contractive type mappings.

Two selfmaps f and g of a metric space (X, d) are called compatible (see Jungck [2]) if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. It is clear from the above definition that f and g will be noncompatible if there exists at least one sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either non-zero or non-existent. Compatibility implies pointwise R-weak commutativity since compatible maps commute at their coincidence points. However, as shown in the examples on the following pages, pointwise R-weakly commuting maps need not be compatible.

In 1997, Pathak *et al* [3] gave an analogue of R-weak commutativity by introducing the concept of R-weak commutativity of type (A_g).

Two selfmappings f and g of a metric space (X, d) are called R-weakly commuting *of type* (A_g) (see [3]) if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all x in X. In a recent work, Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the notion of noncompatible maps.

Let f and g be two selfmappings of a metric space (X, d). We say that f and g satisfy the property (E.A) if there exists a sequence $\{x_n\}$ such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X. If two maps are noncompatible, then they satisfy the property $(E.A)$. The converse, however, is not necessarily true. To support our assertion, we quote examples from [1].

Example 1. Let $X = [0, +\infty)$. Define $T, S : X \to X$ by

$$\begin{aligned} Tx &= \frac{x}{4}, \\ Sx &= \frac{3x}{4}, \quad \forall x \in X. \end{aligned}$$

Consider the sequence $\{x_n\} = \frac{1}{n}$. Clearly $\lim_n Tx_n = \lim_n Sx_n = 0$. Then T and S satisfy property (E.A).

Example 2. Let $X = [2, +\infty)$. Define $T, S : X \to X$ by

$$\begin{aligned} Tx &= x + 1, \\ Sx &= 2x + 1, \quad \forall x \in X. \end{aligned}$$

Suppose that property (E.A) holds; then there exists in X a sequence x_n satisfying $\lim_n Tx_n = \lim_n Sx_n = t$ for some $t \in X$. Therefore, $\lim_n x_n = t - 1$ and $\lim_n x_n = \frac{(t-1)}{2}$. Then $t = 1$, which is a contradiction since $1 \notin X$. Hence T and S do not satisfy property (E.A).

In the present paper, we first obtain common fixed point theorems for a pair of mappings, satisfying nonexpansive or Lipschitz type condition, by employing the notion of pointwise R-weak commutativity and simple techniques of contraction maps (Theorem 1, Theorem 2, and Theorem 3). Theorem 4 is a common fixed point theorem for four mappings which are R-weakly commutative of type (A_g) wherein we replace the condition of noncompatibility with the property (E.A). In Theorem 5 we show that if the condition of noncompatibility is used in place of the property (E.A), then the mappings become discontinuous at their common fixed points. Thus, we provide one more answer to the problem regarding the possibility of contractive definition which is strong enough to guarantee the existence of common fixed point but does not forces the maps to become continuous (Rhoades [11]).

2. Main results

Theorem 1. *Let f and g be noncompatible pointwise R-weakly commuting self-mappings of a metric space (X, d) satisfying*

(i) $\overline{fX} \subset gX$, *where* $\overline{fX}$ *denotes the closure of range of* f,
(ii) $d(fx, fy) \leq kd(gx, gy)$, $k \geq 0$, *and*
(iii) $d(fx, f^2x) < \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx),$
$d(f^2x, gfx), d(fx, gfx), d(gx, f^2x)\}$,

whenever $fx \neq f^2x$. Then f and g have a common fixed point.

Proof. Since f and g are noncompatible, there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $gx_n \to t$ for some t in X but $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Then, since $t \in \overline{fX}$ and $\overline{fX} \in gX$ there exists u in X such that $t = gu$. By (ii) we now get

$$d(fx_n, fu) \leq kd(gx_n, gu).$$

On letting $t \to \infty$, we get, $fu = gu$. Pointwise R-weak commutativity of f and g implies that $fgu = gfu$. Also, $ffu = fgu = gfu = ggu$. We claim that $ffu = fu$. If not, by virtue of (iii) we get

$$\begin{aligned} d(fu, ffu) < & \max\{d(gu, gfu), d(ggu, gfu), d(fu, gu), d(ffu, gfu), \\ & d(fu, gfu), d(gu, ffu)\} \\ = & \; d(fu, ffu), \end{aligned}$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. This completes the proof of the theorem. □

We now give an example to illustrate the above theorem.

Example 3. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } > 5 \\ 6 & \text{if } 2 < x \leq 5, \end{cases}$$

$$g(x) = \begin{cases} 2 & \text{if } x = 2 \\ 7 & \text{if } 2 < x \leq 5 \\ \frac{(4x+10)}{15} & \text{if } x > 5. \end{cases}$$

Then f and g satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 2$. In this example $\overline{fX} = 2 \cup 6$ and $gX = [2, 6] \cup 7$. It may be seen that $\overline{fX} \subset gX$. It can be verified also that f and g are pointwise R-weakly commuting maps. f and g are pointwise R-weakly commuting since they commute at their coincidence points. To see that f and g are noncompatible, let us consider a sequence $\{x_n = 5 + \frac{1}{n} : n > 1\}$, then $\lim_n fx_n = 2$, $\lim_n gx_n \to 2$, $\lim_n fgx_n = 6$ and $\lim_n gfx_n = 2$. Hence f and g are noncompatible.

It can be verified that f and g satisfy the Lipschitz type condition $d(fx, fy) \leq kd(gx, gy)$ with $k = 4$ together with the condition

$$d(fx, f^2x) < \max\{d(gx, gfx), d(ggu, gfu), d(fx, gx), d(f^2x, gfx), \\ d(fx, gfx), d(gx, f^2x)\}.$$

In our next result we replace the condition (iii) of the above theorem.

Theorem 2. *Let f and g be noncompatible pointwise R-weakly commuting self-mappings of a metric space (X, d) satisfying condition* (i) *and* (ii) *of Theorem 1 and*

(iii) $$d(fx, f^2x) > \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx), d(f^2x, gfx), \\ d(fx, gfx), d(gx, f^2x)\},$$

whenever $fx \neq f^2x$. Then f and g have a common fixed point.

The theorem can be proved in similar manner as in Theorem 1. To illustrate the theorem we give an example.

Example 4. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } > 5 \\ 8 & \text{if } 2 < x \leq 5, \end{cases}$$

$$g(x) = \begin{cases} 2 & \text{if } x = 2 \\ 7 & \text{if } 2 < x \leq 5 \\ \frac{(4x+10)}{15} & \text{if } x > 5. \end{cases}$$

Then f and g satisfy all the conditions of the above theorem and have a unique common fixed point at $x = 2$.

It may be seen that in the example above f and g satisfy all the conditions of Theorem 2 with $k = 6$.

As a corollary of Theorem 1, we get a common fixed point theorem for nonexpansive type mapping pairs. We formally state it as follows.

Corollary 1. *Let f and g be noncompatible pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying*

(i) $\overline{fX} \subset gX$, *where $\overline{fX}$ denotes the closure of range of f,*

(ii) $d(fx, fy) \leq d(gx, gy)$, *and*

(iii) $$d(fx, f^2x) < \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx), d(f^2x, gfx), d(fx, gfx), d(gx, f^2x)\},$$

whenever $fx \neq f^2x$.

Then f and g have a common fixed point.

Above result can be proved in the similar lines of Theorem 1. To illustrate our argument we now give an example.

Example 5. Let $X = [2, 20]$ and d be the usual metric on X. Define $f, g : X \to X$ as

$$f(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } > 5 \\ 6 & \text{if } 2 < x \leq 5, \end{cases}$$

$$g(x) = \begin{cases} 2 & \text{if } x = 2 \\ 10 & \text{if } 2 < x \leq 5 \\ \frac{(4x+10)}{15} & \text{if } x > 5. \end{cases}$$

Then f and g satisfy all the conditions of the Corollary 1 and have a unique common fixed point at $x = 2$. In this example $\overline{fX} = 2 \cup 6$ and $gX = [2, 6] \cup 10$. It may be seen that $\overline{fX} \subsetneq gX$. It can be verified also that f and g are pointwise R-weakly commuting maps. f and g are pointwise R-weakly commuting since they commute at their coincidence points. To see that f and g are noncompatible, let us consider a sequence $\{x_n = 5 + \frac{1}{n} : n > 1\}$, then $\lim_n fx_n = 2, \lim_n gx_n \to 2, \lim_n fgx_n = 6$ and $\lim_n gfx_n = 2$. Hence f and g are noncompatible. It can be verified that f and g satisfy the condition $d(fx, fy) \leq d(gx, gy)$ together with the condition

$$d(fx, f^2x) > \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx), d(f^2x, gfx), d(fx, gfx), d(gx, f^2x)\}.$$

As a corollary of Theorem 1 and Theorem 2 above, we find the following theorem:

Theorem 3. *Let f and g be noncompatible pointwise R-weakly commuting selfmappings of a metric space (X, d) satisfying*

(i) $\overline{fX} \subset gX$, *where $\overline{fX}$ denotes the closure of range of f,*

(ii) $d(fx,fy) \leq kd(gx,gy), k \geq 0$, *and*

(iii) $d(fx,f^2x) \neq \max\{d(gx,gfx), d(g^2x,gfx), d(fx,gx), d(f^2x,gfx),$
$d(fx,gfx), d(gx,f^2x)\}$,

whenever $fx \neq f^2x$. *Then* f *and* g *have a common fixed point.*

In a recent work, Aamri and Moutawakil [1] introduced the property (E.A) and thus generalized the notion of noncompatible maps. Our next theorem is for R-weakly commutative maps of type (A_g). We use the property (E.A) in place of noncompatibility.

Theorem 4. *Let* f *and* g *be pointwise* R*-weakly commuting selfmappings of type* (A_g) *of a metric space* (X,d) *satisfying*

(i) $\overline{fX} \subset gX$, *where* $\overline{fX}$ *denotes the closure of range of* f,

(ii) $d(fx,fy) \leq kd(gx,gy), k \geq 0$, *and*

(iii) $d(fx,f^2x) < \max\{d(gx,gfx), d(g^2x,gfx), d(fx,gx), d(f^2x,gfx),$
$d(fx,gfx), d(gx,f^2x)\}$,

whenever $fx \neq f^2x$. *Let* f *and* g *satisfy the property* (E.A). *Then* f *and* g *have a common fixed point.*

Proof. Since f and g satisfy the property (E.A), there exists a sequence $\{x_n\}$ such that $fx_n \to t$ and $gx_n \to t$ for some t in X. Then, since $t \in \overline{fX}$ and $\overline{fX} \subset gX$ there exists u in X such that $t = gu$. By (ii) we now get $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $t \to \infty$ we get $fu = gu$. Since f and g are R-weak commutating of type (A_g), we get $d(ffu, gfu) \leq R(d(fu,gu)) = 0$, that is, $ffu = gfu$. If $fu \neq ffu$, using (iii), we get

$$\begin{aligned} d(fu,ffu) <\ & \max\{d(gu,gfu), d(ggu,gfu), d(fu,gu), \\ & d(ffu,gfu), d(f,gfu), d(gu,ffu)\} \\ =\ & d(gu,gfu) = d(fu,ffu), \end{aligned}$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness of the common fixed point follows easily. □

In the next theorem, we show that if we use the notion of noncompatibility in place of the property (E.A), then the mappings become discontinuous at their common fixed point. Thus, we provide one more answer to the problem regarding the existence of contractive definition which is strong enough to guarantee the existence of common fixed point but does not forces the maps to become continuous (Rhoades [11]).

Theorem 5. *Let* f *and* g *be noncompatible selfmappings of a complete metric space* (X,d) *such that*

(i) $\overline{fX} \subset gX$, *where* $\overline{fX}$ *denotes the closure of range of* f,

(ii) $d(fx, fy) \leq kd(gx, gy), k \geq 0$, *and*

(iii) $d(fx, f^2x) < \max\{d(gx, gfx), d(g^2x, gfx), d(fx, gx), d(f^2x, gfx),$
$d(fx, gfx), d(gx, f^2x)\}$,

whenever $fx \neq f^2x$ *and right hand side is positive. If* f *and* g *be* R*-weakly commuting of type of type* (A_g)*, then* f *and* g *have a unique common fixed point and the fixed point is a point of discontinuity.*

Proof. Since f and g are noncompatible maps, there exists a sequence $\{x_n\}$ in X such that

$$\lim_n fx_n = \lim_n gx_n = t \tag{1}$$

for some t in X but either $\lim_n d(fgx_n, gfx_n) = 0$ or the limit does not exist. Since $t \in \overline{fX}$ and $\overline{fX} \subset gX$, there exists some point u in X such that $t = gu$ where $t = \lim_n gx_n$. By (ii) we now get $d(fx_n, fu) \leq kd(gx_n, gu)$. On letting $t \to \infty$ we get $fu = gu$. Since f and g are R-weak commutating of type (A_g), we get $d(ffu, gfu) \leq R(d(fu, gu)) = 0$, that is, $ffu = gfu$. If $fu \neq ffu$, using (iii), we get

$$\begin{aligned} d(fu, ffu) < \; & \max\{d(gu, gfu), d(ggu, gfu), d(fu, gu), \\ & \qquad d(ffu, gfu), d(fu, gfu), d(gu, ffu)\} \\ = \; & d(gu, gfu) = d(fu, ffu), \end{aligned}$$

a contradiction. Hence $fu = ffu = gfu$ and fu is a common fixed point of f and g. Uniqueness of the common fixed point follows easily. We now show that f and g are discontinuous at the common fixed point $t = fu = gu$. If possible, suppose f is continuous. Then considering the sequence $\{x_n\}$ as assumed above, we get $\lim_n ffx_n = ft = t$. R-weak commutativity of type (A_g) implies that $d(ffx_n, gfx_n) \leq Rd(fx_n, gx_n)$. On letting $n \to \infty$ this yields $\lim_n gfx_n = ft = t$. This, in turn, yields $\lim_n d(fgx_n, gfx_n) = d(ft, ft) = 0$. This contradicts the fact that $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent for the sequence $\{x_n\}$ of (1). Hence f is discontinuous at the fixed point. Next, suppose that g is continuous. Then for the sequence x_n of (1), we get $\lim_n gfx_n = gt = t$ and $\lim_n ggx_n = gt = t$. In view of these limits, the inequality

$$\begin{aligned} d(ft, fgx_n) < \max\{& d(gt, ggx_n), d(g^2t, ggx_n), d(ft, gt), \\ & d(fgx_n, gfx_n), d(ft, gfx_n), d(gt, fgx_n)\} \end{aligned}$$

yields a contradiction unless $\lim_n fgx_n = ft = gt$. But $\lim_n fgx_n = gt$ and $\lim_n gfx_n = gt$ contradicts the fact that $\lim_n d(fgx_n, gfx_n)$ is either nonzero or nonexistent. Thus both f and g are discontinuous at their common fixed point. Hence the theorem. □

We now give an example which illustrates the above theorem.

Example 6. Let $X = [2, 20]$ and d be the usual metric on X. Define A, B, S, T: $X \to X$ by

$$A(x) = \begin{cases} 2 & \text{if } x = 2 \\ 3 & \text{if } x > 2, \end{cases}$$

$$S(x) = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2, \end{cases}$$

$$B(x) = \begin{cases} 2 & \text{if } x = 2 \text{ or } 5 \\ 6 & \text{if } 2 < x < 5, \end{cases}$$

$$T(x) = \begin{cases} 2 & \text{if } x = 2 \\ 7 + x & \text{if } 2 < x < 5 \\ \frac{(x+1)}{2} & \text{if } x > 5. \end{cases}$$

Then A, B, S and T satisfy all the conditions of above theorem and have a unique common fixed point $x = 2$. It can be verified in this example that A, B, S and T satisfy contractive condition of the above theorem. It can also be seen that A and S satisfy the property (E.A) and all the mappings A, B, S and T are discontinuous at the common fixed point.

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References

[1] M. Aamri and D. El Moutawakil, *Some new common fixed point theorems under strict contractive conditions*, J. Math. Anal. Appl. **270** (2002), no. 1, 181–188.

[2] G. Jungck, *Compatible mappings and common fixed points*, Internat. J. Math. Math. Sci. **9** (1986), no. 4, 771–779.

[3] H. K. Pathak, Y. J. Cho, and S. M. Kang, *Remarks on R-weakly commuting mappings and common fixed point theorems*, Bull. Korean Math. Soc. **34** (1997), no. 2, 247–257.

[4] R. P. Pant, *Common fixed points of noncommuting mappings*, J. Math. Anal. Appl. **188** (1994), no. 2, 436–440.

[5] ———, *R-weak commutativity and common fixed points of noncompatible maps*, Ganita **49** (1998), no. 1, 19–27.

[6] ———, *Common fixed point theorems for contractive maps*, J. Math. Anal. Appl. **226** (1998), no. 1, 251–258.

[7] ———, *Common fixed points of Lipschitz type mapping pairs*, J. Math. Anal. Appl. **240** (1999), no. 1, 280–283.

[8] ———, *Discontinuity and fixed points*, J. Math. Anal. Appl. **240** (1999), no. 1, 284–289.

[9] ———, *R-weak commutativity and common fixed points*, Soochow J. Math. **25** (1999), no. 1, 37–42.

[10] R. P. Pant and V. Pant, *Lipschitz type condition and common fixed points*, J. Indian Math. Soc. (N.S.) **67** (2000), no. 1-4, 179–182.

[11] B. E. Rhoades, *Contractive definitions and continuity*, Fixed point theory and its applications (Berkeley, CA, 1986), 233–245, Contemp. Math., 72, Amer. Math. Soc., Providence, RI, 1988.

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