NEW WEIGHTED OSTROWSKI-GRÜSS-ČEBYŠEV TYPE INEQUALITIES

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Abstract. In this paper, by introducing parameter $r > 1$, new weighted Ostrowski-Grüss-Čebyšev type inequalities for $1/p + 1/q = 1 - 1/r$ are established.

1. Introduction

In 1938, Ostrowski proved the following interesting integral inequality [7]:

**Theorem 1.1.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable in $(a, b)$ and its derivative $f' : (a, b) \to \mathbb{R}$ is bounded in $(a, b)$, that is, $\|f'\|_{\infty} := \sup_{t \in (a, b)} |f'(x)| < \infty$. Then for any $x \in [a, b]$, we have the inequality:

\[
\left| f(x) - \frac{1}{b - a} \int_{a}^{b} f(t) dt \right| \leq \left[ \frac{1}{4} + \frac{(x - \frac{a + b}{2})^2}{(b - a)^2} \right] (b - a) \|f'\|_{\infty}.
\]

The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one.

For two absolutely continuous functions $f, g : [a, b] \to \mathbb{R}$, consider the functional

\[
T(f, g) = \frac{1}{b - a} \int_{a}^{b} f(x) g(x) dx - \left( \frac{1}{b - a} \int_{a}^{b} f(x) dx \right) \left( \frac{1}{b - a} \int_{a}^{b} g(x) dx \right)
\]

provided the involved integrals exist. In 1882, Čebyšev [6] proved that, if $f'$, $g' \in L^{\infty}[a, b]$, then

\[
|T(f, g)| \leq \frac{1}{12} (b - a)^2 \|f'\|_{\infty} \|g'\|_{\infty}.
\]
In 1934, Grüss [6] showed that

\[
|T(f, g)| \leq \frac{1}{4}(M - m)(N - n)
\]

provided \(m, M, n, N\) are real numbers satisfying the condition \(-\infty < m \leq f(x) \leq M < \infty, -\infty < n \leq g(x) \leq N < \infty\) for all \(x \in [a, b]\).

During the past few years many researchers have given considerable attention to the above inequalities and various generalizations, extensions and variants of these inequalities have appeared in the literature, see [1, 2, 3, 4, 5, 9] and the references cited therein. In [10], Rafiq et al. gave a weighted Ostrowski type inequality for differentiable mappings whose first derivatives belong to \(L^p[a, b], p > 1\).

**Theorem 1.2.** Let \(f : [a, b] \to \mathbb{R}\) be a differentiable mapping on \([a, b]\), whose first derivative, i.e., \(f' : [a, b] \to \mathbb{R}\), belongs to \(L^p[a, b], p > 1\) for any \(x \in [a, b]\). Then, we have the inequality:

\[
\left| f(x) - \frac{1}{m(a, b)} \int_a^b w(t)f(t)dt \right| \leq \frac{(x - a)^{1+1/q} + (b - x)^{1+1/q}}{m(a, b)(q + 1)^{1/q}} \|f'\|_{w,p},
\]

where \(w(t)\) and \(m(a, b)\) be given in Section 2, and the weighted norm of differentiable function whose derivatives belong to \(L^p[a, b]\) is defined as

\[
\|\phi\|_{w,p} = \left( \int_a^b |w(t)\phi(t)|^p dt \right)^{1/p}.
\]

Recently, Pachpatte [8] established a new Grüss type inequality involving two functions and their derivatives.

**Theorem 1.3.** Let \(f, g : [a, b] \to \mathbb{R}\) be continuous on \([a, b]\) and differentiable in \((a, b)\), whose derivatives \(f', g' : (a, b) \to \mathbb{R}\) are bounded in \((a, b)\), i.e., \(\|f'\|_{\infty} < \infty, \|g'\|_{\infty} < \infty\). Then

\[
|T(f, g)| \leq \frac{1}{2(b - a)^2} \int_a^b \left\{|g(x)|\|f'\|_{\infty} + |f(x)|\|g'\|_{\infty}\right\} \left( \int_a^b |x - y|dy \right) dx.
\]

Motivated by the results of Pachpatte and Rafiq, in the present paper we establish some new weighted Ostrowski-Gruess-Cebyshev type inequalities for \(1/p + 1/q = 1 - 1/r\) by introducing parameter \(r > 1\). The analysis used in the proofs is elementary and based on the use of integral identities proved in [8].

2. Main results

Let the weight \(w : [a, b] \to [0, \infty)\), be non-negative and integrable, i.e., \(\int_a^b w(t)dt < \infty\). The domain of \(w\) is finite. We denote the zero moment as

\[
m(a, b) = \int_a^b w(t)dt.
\]
For suitable functions \( f, g : [a, b] \to \mathbb{R} \) we set
\[
T_w(f, g) = \frac{1}{m(a, b)} \int_a^b w(x)f(x)g(x)dx
- \left( \frac{1}{m(a, b)} \int_a^b w(x)f(x)dx \right) \left( \frac{1}{m(a, b)} \int_a^b w(x)g(x)dx \right).
\] (8)

Then the following theorem holds:

**Theorem 2.1.** Let \( f, g : [a, b] \to \mathbb{R} \) be absolutely continuous and have bounded first derivative, \( w \in L^p[a, b] \), \( p > 1, q > 0, r > 1 \) and \( 1/p + 1/q = 1 - 1/r \). Then we have the inequalities
\[
|T_w(f, g)| \leq \frac{1}{2m^2(a, b)} \int_a^b w(x)M_w(x, r)\left\{ |g(x)||f'|_\infty + |f(x)||g'|_\infty \right\}dx
\] (9)
and
\[
|T_w(f, g)| \leq \frac{\|f\|_\infty \|g\|_\infty}{2m^2(a, b)} \int_a^b w(x)N_w(x, r)dx,
\] (10)
where
\[
M_w(x, r) = \|w\|_p \left( \frac{q + r}{q + r + qr} \right) \frac{q+r}{qr} \left[ (x-a)^{\frac{q+r+2ar}{qr}} + (b-x)^{\frac{q+r+2ar}{qr}} \right]
\]
and
\[
N_w(x, r) = \|w\|_p \left( \frac{q + r}{q + r + 2qr} \right) \frac{q+r}{qr} \left[ (x-a)^{\frac{q+r+2qr}{q+r}} + (b-x)^{\frac{q+r+2qr}{q+r}} \right] \frac{q+r}{qr}.
\]

**Proof.** For any \( x, y \in [a, b] \) we have the following identities (see [8]):
\[
f(x) - f(y) = \int_y^x f'(t)dt,
\] (11)
\[
g(x) - g(y) = \int_y^x g'(t)dt.
\] (12)

Multiplying both sides of (11) and (12) by \( g(x) \) and \( f(x) \) respectively and adding the resulting identities, we have
\[
2f(x)g(x) - [g(x)f(y) + f(x)g(y)] = g(x) \int_y^x f'(t)dt + f(x) \int_y^x g'(t)dt
\] (13)
for \( x \in [a, b] \). Multiplying both sides of (13) by \( w(y) \) and integrating with respect to \( y \) over \([a, b]\), we have

\[
2f(x)g(x) \int_{a}^{b} w(y)dy
- \left[ g(x) \int_{a}^{b} w(y)f(y)dy + f(x) \int_{a}^{b} w(y)g(y)dy \right]
= \int_{a}^{b} w(y) \left[ g(x) \int_{y}^{x} f'(t)dt + f(x) \int_{y}^{x} g'(t)dt \right] dy.
\]

(14)

Multiplying both sides of (14) by \( \frac{w(x)}{2m^2(a, b)} \), we obtain

\[
\frac{w(x)f(x)g(x)}{m(a, b)}
- \frac{w(x)g(x)}{2m^2(a, b)} \int_{a}^{b} w(y)f(y)dy + \frac{w(x)f(x)}{2m^2(a, b)} \int_{a}^{b} w(y)g(y)dy
= \frac{w(x)}{2m^2(a, b)} \int_{a}^{b} w(y) \left[ g(x) \int_{y}^{x} f'(t)dt + f(x) \int_{y}^{x} g'(t)dt \right] dy.
\]

(15)

Integrating both sides of (15) with respect to \( x \) over \([a, b]\), we have

\[
T_w(f, g)
= \frac{1}{2m^2(a, b)} \int_{a}^{b} w(x) \left\{ \int_{a}^{b} w(y) \left[ g(x) \int_{y}^{x} f'(t)dt + f(x) \int_{y}^{x} g'(t)dt \right] dy \right\} dx.
\]

Using the properties of modulus, we get

(16)

\[
|T_w(f, g)|
\leq \frac{1}{2m^2(a, b)} \int_{a}^{b} w(x) \left( \int_{a}^{b} |x - y| |w(y)dy| \right) \{ |g(x)||f'||\infty + |f(x)||g'||\infty \} dx.
\]

By Hölder inequality, we have

\[
\int_{a}^{b} |x - y| w(y)dy
= \int_{a}^{x} (x - y) w(y)dy + \int_{x}^{b} (y - x) w(y)dy
\leq \|w\|_p \left[ \left( \int_{a}^{x} (x - y) \frac{q}{r} dy \right)^{\frac{q+r}{q}} + \left( \int_{x}^{b} (y - x) \frac{q}{r} dy \right)^{\frac{q+r}{q}} \right]
= \|w\|_p \left( \frac{q + r}{q + r + qr} \right)^{\frac{q+r}{qr}} \left[ (x - a)^{\frac{q+r}{qr}} + (b - x)^{\frac{q+r}{qr}} \right]
:= M_w(x, r).
\]

(17)

From (16) and (17), we obtain (9).
Multiplying the left and right side of (11) and (12), we get

\[
\begin{aligned}
f(x)g(x) - [g(x)f(y) + f(x)g(y)] + f(y)g(y) &= \left( \int_x^y f'(t) dt \right) \left( \int_y^x g'(t) dt \right) \\
&\quad + \int_a^b w(y) f(y) g(y) dy
\end{aligned}
\]
for \( x \in [a, b] \). Multiplying both sides of (18) by \( w(y) \) and integrating with respect to \( y \) over \([a, b]\), we have

\[
\begin{aligned}
f(x)g(x) \int_a^b w(y) dy - \left[ g(x) \int_a^b w(y) f(y) dy + f(x) \int_a^b w(y) g(y) dy \right] \\
+ \int_a^b w(y) f(y) g(y) dy
&= \int_a^b w(y) \left( \int_y^x f'(t) dt \right) \left( \int_y^x g'(t) dt \right) dy.
\end{aligned}
\]

Multiplying both sides of (19) by \( \frac{w(x)}{m^2(a, b)} \), we obtain

\[
\begin{aligned}
\frac{w(x)f(x)g(x)}{m(a, b)} - \left[ \frac{w(x)g(x)}{m^2(a, b)} \int_a^b w(y) f(y) dy + \frac{w(x)f(x)}{m^2(a, b)} \int_a^b w(y) g(y) dy \right] \\
+ \frac{w(x)}{m^2(a, b)} \int_a^b w(y) f(y) g(y) dy
&= \frac{w(x)}{m^2(a, b)} \int_a^b w(y) \left( \int_y^x f'(t) dt \right) \left( \int_y^x g'(t) dt \right) dy.
\end{aligned}
\]

Integrating both sides of (20) with respect to \( x \) over \([a, b]\), we have

\[
T_w(f, g) = \frac{1}{2m^2(a, b)} \int_a^b w(x) \left\{ \int_a^b w(y) \left( \int_y^x f'(t) dt \right) \left( \int_y^x g'(t) dt \right) dy \right\} dx.
\]

Using the properties of modulus, we get

\[
|T_w(f, g)| \leq \frac{\|f\|_\infty \|g\|_\infty}{2m^2(a, b)} \int_a^b w(x) \left( \int_a^b (x - y)^2 w(y) dy \right) dx.
\]

By Hölder inequality, we have

\[
\begin{aligned}
\int_a^b (x - y)^2 w(y) dy
&\leq \|w\|_p \left[ \int_a^x (x - y)^{\frac{2q}{q+r}} dy + \int_x^b (y - x)^{\frac{2q}{q+r}} dy \right]^{\frac{q+r}{q}}
\leq \|w\|_p \left( \frac{q + r}{q + r + 2qr} \right)^{\frac{q+r}{rq}} \left[ (x - a)^{\frac{q + 2q + r}{q + r}} + (b - x)^{\frac{q + 2q + r}{q + r}} \right]^{\frac{q+r}{rq}} \\
&:= N_w(x, r).
\end{aligned}
\]
From (21) and (22), we obtain (10).

Remark 2.2. We note that in the special cases, if we set \( w(x) = 1, r \to \infty \) and \( q \to 1 \) in (9), we obtain

\[
|T(f, g)| \leq \frac{1}{2(b-a)^2} \int_a^b \left| g(x) \right| \left| f'(x) \right| \|g'\| \left[ \frac{2}{2} (x-a)^2 + (b-x)^2 \right] dx,
\]

which recaptures the inequality (6) since \( \int_a^b |x-y| dy = \frac{(x-a)^2 + (b-x)^2}{2} \). Taking \( w(x) = 1, r \to \infty \) and \( q \to 1 \) in (10), we have the inequality

\[
|T(f, g)| \leq \frac{\|f\|_\infty \|g\|_\infty}{2(b-a)^2} \int_a^b \frac{(x-a)^3 + (b-x)^3}{3} dx,
\]

which recaptures the inequality (3) since \( \int_a^b \frac{(x-a)^3 + (b-x)^3}{3} dx = \frac{1}{6}(b-a)^4 \).

Corollary 2.3. Under the assumptions of Theorem 2.1 with \( r \to \infty \), we have the inequalities

\[
|T_w(f, g)| \leq \frac{1}{2m^2(a, b)} \int_a^b w(x)M_w(x) \left( |g(x)| \left| f'(x) \right| \|f\|_\infty + |f(x)| \|g'\|_\infty \right) dx,
\]

and

\[
|T_w(f, g)| \leq \frac{\|f\|_\infty \|g\|_\infty}{2m^2(a, b)} \int_a^b w(x)N_w(x) dx,
\]

where

\[
M_w(x) = \|w\|_p \left[ \frac{(x-a)^{1+1/q} + (b-x)^{1+1/q}}{(q+1)^{1/q}} \right]
\]

and

\[
N_w(x) = \|w\|_p \left[ \frac{(x-a)^{2q+1} + (b-x)^{2q+1}}{(2q+1)^{1/q}} \right].
\]

Corollary 2.4. Under the assumptions of Theorem 2.1 with \( w(x) = 1 \), we have the inequalities

\[
|T(f, g)| \leq \frac{1}{2(b-a)^2} \int_a^b M(x, r) \left| g(x) \right| \left| f'(x) \right| \|g'\|_\infty dx,
\]

and

\[
|T_w(f, g)| \leq \frac{\|f\|_\infty \|g\|_\infty}{2(b-a)^2} \int_a^b N(x, r) dx,
\]

where

\[
M(x, r) = \left( \frac{q+r}{q+r+qr} \right)^{\frac{q+r}{qr}} \left[ (x-a)^{\frac{q+r}{qr}} + (b-x)^{\frac{q+r}{qr}} \right]
\]

and

\[
N(x, r) = \left( \frac{q+r+qr}{q+r} \right)^{\frac{q+r+qr}{qr}} \left[ (x-a)^{\frac{q+r+qr}{qr}} + (b-x)^{\frac{q+r+qr}{qr}} \right].
\]
and
\[ N(x, r) = \left( \frac{q + r}{q + r + 2qr} \right)^{\frac{q+r}{2q}} \left[ (x - a)^{\frac{q+r+2qr}{q+r}} + (b - x)^{\frac{q+r+2qr}{q+r}} \right]^{\frac{q+r}{2q}}. \]

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