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COMPLEX SCALING AND GEOMETRIC ANALYSIS OF SEVERAL VARIABLES

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Abstract. The purpose of this paper is to survey the use of the important method of scaling in analysis, and particularly in complex analysis. Applications are given to the study of automorophism groups, to canonical kernels, to holomorphic invariants, and to analysis in infinite dimensions. Current research directions are described and future paths indicated.

0. Preliminary remarks

It is a classical fact that there is no Riemann mapping theorem in the function theory of several complex variables. Indeed, H. Poincaré proved in 1906 that the unit ball $B = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z| \equiv \sqrt{|z_1|^2 + |z_2|^2} < 1\}$ and the unit bidisc $D^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\}$ are biholomorphically inequivalent. More recently, Burns, Shnider, Wells [12] and Greene, Krantz [35] have shown that two smoothly bounded, strongly pseudoconvex domains (definitions to be discussed below) are generically biholomorphically inequivalent. In particular, if one concentrates attention on smoothly bounded domains that are near the unit ball in some reasonable metric then, with probability one, two randomly selected domains will be biholomorphically inequivalent. Thus one seeks substitutes for the Riemann mapping theorem. In particular, one seeks to classify domains in terms of geometric invariants. Work of Fefferman [28], Bell [6], Bell, Ligocka [7], and others has shown us that a biholomorphic mapping of a reasonable class of smoothly bounded domains will extend to smooth diffeomorphism of the closures.

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Thus it is possible, at least in principle, to carry out Poincaré's original program of determining differential biholomorphic invariants on the boundary. Chern and Moser [18] did the initial work in this direction. (See also [96].) More recent progress has been made by Webster [97], Moser [85] and [86], Moser, Webster [87], Isaev, Kruzhilin [48], and Ejov, Isaev [26]. Another direction, also inspired by Poincaré's work, is to study the automorphism group of a domain. This is a natural biholomorphic invariant, and reflects the Levi and Bergman geometry of the domain in a variety of subtle and useful ways. The purpose of this paper is to develop some techniques connected with the study of automorphism groups of bounded domains in $\mathbb{C}^{n+1}$. In particular, we wish to focus attention on a powerful technique that has become central in the subject. This is the method of *scaling*. A special case of a general technique in differential geometry known as *flattening*, scaling is a method for localizing analysis near a boundary point. This method has been used with considerable effectiveness to study not only automorphism groups ([91], [30], [36]) but also canonical kernels ([88], [67]) and other aspects of classical function theory. The papers [17], [19], and [29] (among many others in the literature) use scaling methods. Certainly the papers [81]–[84] are relevant here as well. Scaling is a far-reaching methodology that has potential applications in many parts of mathematics. We exhibit in this article several different contexts and applications in which the scaling point of view is useful. For our purposes, the theory of automorphism groups is a convenient venue in which to showcase the scaling technique. But it should be of interest to mathematicians with many diverse interests.

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1. Introduction

By a *domain*, we mean a connected open subset in $\mathbb{C}^{n+1}$ for some non-negative integer[1] n. Throughout this paper, we shall use $z = (z_0, z_1, \ldots, z_n)$ for the coordinates of a point in $\mathbb{C}^{n+1}$.

Let Ω be a domain. If its boundary $\partial\Omega$ is a regularly imbedded C^k-hypersurface ($k \geq 2$), then there exists a C^k smooth function $\rho : \mathbb{C}^{n+1} \to \mathbb{R}$ such that

(i) $\Omega = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$
(ii) $\nabla\rho(p) \neq 0$ whenever $p \in \partial\Omega$.

In such a case, Ω is called a domain with C^k smooth boundary. In turn, ρ is called a C^k smooth *defining function* for Ω. Now fix a domain $\Omega \subseteq \mathbb{C}^{n+1}$ with C^2 boundary and defining function ρ. Let $p \in \partial\Omega$. We say that $w \in \mathbb{C}^{n+1}$ is a

[1] It seems notationally convenient in this study to treat domains in $\mathbb{C}^{n+1}$ rather than $\mathbb{C}^n$.

complex tangent vector to $\partial\Omega$ at p if

$$\sum_{j=0}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0 \,.$$

We write $w \in \mathcal{T}_p(\partial\Omega)$. [Observe that w is a *real tangent vector*, which is the classical notion of tangency from differential geometry, if $\mathrm{Re}\, \sum_{j=0}^{n} \frac{\partial \rho}{\partial z_j}(p) w_j = 0$. We write in this case $w \in T_p(\partial\Omega)$.] The *complex normal directions* are the directions in $T_p(\partial\Omega)$ which are complementary to $\mathcal{T}_P(\partial\Omega)$.

We say that $\partial\Omega$ is (weakly) Levi pseudoconvex at p if

$$(*) \qquad \sum_{j,k=0}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k \geq 0$$

for every complex tangent vector w at p. The point p is *strictly* or *strongly pseudoconvex* if the inequality in $(*)$ is strict whenever $0 \neq w \in \mathcal{T}_p(\partial\Omega)$. If each point of $\partial\Omega$ is pseudoconvex, then the domain is said to be pseudoconvex; if each point of $\partial\Omega$ is strongly pseudoconvex, then the domain is said to be strongly pseudoconvex. We note in passing that there is a more general notion of pseudoconvexity due to Hartogs, and which utilizes the theory of plurisubharmonic functions. We shall have no use for that concept here, but see [66]. It is worth noting (see [66]) that if $p \in \partial\Omega$ is a point of strong pseudoconvexity, then there is a choice of defining function $\widetilde{\rho}$ so that

$$\sum_{j,k=0}^{n} \frac{\partial^2 \widetilde{\rho}}{\partial z_j \partial \overline{z}_k}(p) w_j \overline{w}_k > 0$$

for *every* $w \in \mathbb{C}^{n+1} \setminus \{0\}$ (not just the complex tangential w). In fact one may go further. There is a biholomorphic change of coordinates in a neighborhood of p so that the boundary near p is strongly *convex*. This means that, identifying $z_j = t_{2j-1} + i t_{2j}$, and choosing an appropriate defining function $\widetilde{\widetilde{\rho}}$, we have

$$\sum_{j,k=-1}^{2n} \frac{\partial^2 \widetilde{\widetilde{\rho}}}{\partial t_j \partial t_k}(p) a_j a_k > 0$$

for every non-zero real vector $a = (a_{-1}, a_0, \ldots, a_{2n})$.

2. The lore of automorphism groups

Let $\Omega \subseteq \mathbb{C}^{n+1}$ be a domain. The *automorphism group* of Ω is the collection of one-to-one, onto holomorphic mappings $\varphi : \Omega \to \Omega$. It is known that the inverse φ^{-1} is automatically holomorphic. See [66]. Such a mapping is also called a *biholomorphic self-map* of Ω. With the binary operation of composition of mappings, the collection of automorphisms forms a group. We denote this group by $\mathrm{Aut}(\Omega)$. We equip the automorphism group with the topology of uniform convergence on compact sets, equivalently the compact-open topology. If we restrict attention to bounded domains—and in this paper we, for the most part, do just that—then the group Aut (Ω) is in fact a real Lie group (this follows from work of H. Cartan—see [63]). It is never a complex Lie group unless it is discrete. Of special interest are domains having a large or robust group of automorphisms. It is known (see [12], [35]) that strongly pseudoconvex domains which are rigid—i.e., which have no automorphisms except the identity—are generic. On the other hand, every compact Lie group arises as the automorphism group of some bounded, strongly pseudoconvex domain with real analytic boundary (see [3], [93], [38], and [99]). It seems natural, for example, to study a domain with *transitive* automorphism group—this is a domain Ω with the property that if $P, Q \in \Omega$ are arbitrary then there is an automorphism φ of Ω such that $\varphi(P) = Q$. It turns out that the list of such domains is rather restrictive; our knowledge of such domains is essentially complete (see [42], [46]).[2] Perhaps geometrically more natural is to consider domains with *noncompact* automorphism group. A very natural and useful characterization of such domains is contained in the following classical result of Cartan (for which see [89]):

Proposition 2.1. *Let $\Omega \subseteq \mathbb{C}^{n+1}$ be a bounded domain with noncompact automorphism group. Then there are a point $p \in \partial\Omega$, a point $q \in \Omega$, and automorphisms $\varphi_j \in \mathrm{Aut}(\Omega)$ such that $\varphi_j(q) \to p$ as $j \to \infty$. (See* Figure 1.)

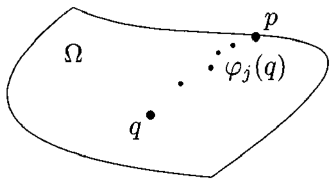

Figure 1

The point p in the proposition is called a *boundary orbit accumulation point.* It is known, in a variety of concrete senses, that the Levi geometry of a boundary orbit accumulation point says a great deal about the domain itself. See [39], [40]. It is a matter of great interest today to classify all possible boundary orbit accumulation points. An important focus of our studies will be (bounded) domains in $\mathbb{C}^{n+1}$ with noncompact automorphism group and their boundary orbit accumulation points. Much is known today about automorphism groups of domains. In classical studies, mathematicians calculated the automorphism groups of very particular domains rather explicitly (see, for instance, the discussion in [66] as well as [44]). Today we have more powerful machinery (the

[2]In fact the only strongly pseudoconvex domain with transitive automorphism group is biholomorphic to the ball. This remarkable fact will be discussed below.

$\overline{\partial}$-Neumann problem, sheaf theory, Levi geometry, Lie group methods, techniques of Riemannian geometry, Kähler theory) that allow us to make qualitative studies of broad classes of domains. The papers [37] and [47] provide a broad overview of the types of results that can be proved with modern techniques. The present paper will introduce the reader to some of the main themes in the subject.

3. The dilatation and scaling sequences

At this stage, we shall only consider the case when the boundary $\partial\Omega$ is $\mathcal{C}^k$ smooth, with $k \geq 2$, in an open neighborhood of $p \in \partial\Omega$. Let $q^\nu = (q_0^\nu, \ldots, q_n^\nu)$ be a sequence of points in the closure $\overline{\Omega}$ of the domain Ω that converges to the boundary point $p = (p_0, \ldots, p_n)$.

3.1. Pinchuk's dilatation sequence

We refer to the source [91] for basic ideas about Pinchuk scaling. Applying a holomorphic coordinate change and the implicit function theorem at p, we may assume that p is the origin and the domain Ω is represented in an open neighborhood of the origin by (i.e., has a defining function given by) the defining inequality

$$\mathrm{Re}\, z_0 \geq \psi(\mathrm{Im}\, z_0, z_1, \ldots, z_n),$$

where:

(i) $\psi \in \mathcal{C}^k$,
(ii) $\psi(0, \ldots, 0) = 0$, and
(iii) $\nabla\psi|_{(0,\ldots,0)} = (0, \ldots, 0)$.

We take $(-1, 0, \ldots, 0)$ to be the unit outward normal vector at p. Now choose the boundary points $p^\nu = (p_0^\nu, \ldots, p_n^\nu)$ satisfying

(i) $p_j^\nu = q_j^\nu$ for every $j = 1, \ldots, n$, and
(ii) $q_0^\nu - p_0^\nu > 0$

for every $\nu = 1, 2, \ldots$. Observe that each p^ν is uniquely determined by these conditions. See Figure 2.

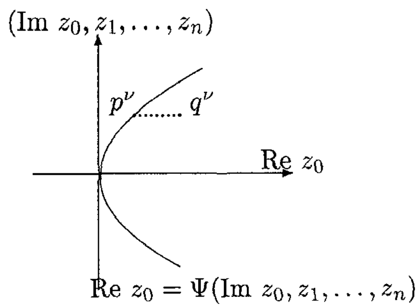

Figure 2: A scaling sequence

Then consider the map $A_\nu : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by $\zeta = A_\nu(z)$ in local coordinates with the explicit expression

$$\begin{aligned} \zeta_0 &= \alpha_0^\nu(z_0 - p_0^\nu) - \sum_{j=1}^{n+1} \alpha_j^\nu(z_j - p_j^\nu) \\ \zeta_1 &= z_1 - p_1^\nu \\ &\vdots \\ \zeta_n &= z_n - p_n^\nu . \end{aligned}$$

See Figure 3.

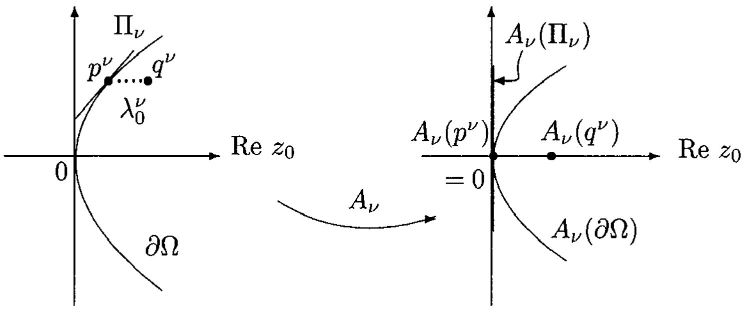

Figure 3: The Centering Process

Here the complex constants $\alpha_0^\nu, \ldots, \alpha_n^\nu$ are chosen so that $\alpha_0^\nu \to 1$ and $\alpha_m^\nu \to 0$ as $\nu \to \infty$ for $m = 1, \ldots, n$, and such that the domain $A_\nu(\Omega)$ is represented in a neighborhood of the origin by a new $\mathcal{C}^k$ defining inequality

$$\operatorname{Re}\zeta_0 > \Psi_\nu(\operatorname{Im}\zeta_0, \zeta_1, \ldots \zeta_2)$$

satisfying

$$\Psi_\nu(0, \ldots, 0) = 0 \text{ and } \nabla\Psi_\nu\big|_0 = (0, \ldots 0).$$

The next step is to consider a sequence of linear maps $L_\nu : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ defined by

$$L_\nu(\zeta_0, \ldots, \zeta_n) = \left(\frac{\zeta_0}{\lambda_0^\nu}, \frac{\zeta_1}{\lambda_1^\nu}, \ldots, \frac{\zeta_n}{\lambda_n^\nu}\right),$$

where

$$\lambda_0^\nu = q_0^\nu - p_0^\nu$$

for each $\nu = 1, 2, \ldots$. The sequence of complex affine mappings $\Lambda_\nu := L_\nu \circ A_\nu$ is the *dilatation sequence* introduced[3] by S. Pinchuk. The choice of $\lambda_1^\nu, \ldots, \lambda_n^\nu$ is an important step in the setup of the dilatation sequence. However, for the sake of smooth exposition, it seems the best to postpone the explication until

[3]Pinchuk originally named it the *stretching coordinates.*

we handle the Wong-Rosay Theorem (our first important application) a bit later in the paper. It should be noted that Pinchuk's dilatation sequence can be defined for a domain with boundary that is not necessarily smooth near the reference boundary point p.

3.2. Pinchuk's scaling sequence with automorphisms

Let Ω be a domain in $\mathbb{C}^{n+1}$ with C^k-smooth boundary $\partial\Omega$, $k \geq 2$. Let p be a boundary point and q^ν be a sequence of points in Ω converging to p as ν tends to infinity. Here we consider the important special case when the sequence q^ν is given by $q^\nu = \varphi_\nu(q)$, where q is a point in Ω and where φ_ν is a holomorphic automorphism of Ω for each $\nu = 1, 2, \ldots$. Let the dilatation sequence $\Lambda_\nu : \Omega \to \mathbb{C}^{n+1}$ be as above, associated with the point sequence q^ν. Then *the scaling sequence* introduced by S. Pinchuk is the following sequence of maps:

$$\sigma_\nu := \Lambda_\nu \circ \varphi_\nu : \Omega \to \mathbb{C}^{n+1}.$$

Once the sequence φ_ν of automorphisms of Ω and a point $q \in \Omega$ is given, the *orbit* $\varphi_\nu(q)$ and the affine adjustments A_ν, which we call the *centering maps*, are defined. The only part of the scaling that needs to be chosen is the sequence of dilating linear maps L_ν. The crux of the matter is to choose L_ν appropriately so that

(i) the σ_ν form a pre-compact normal family, and
(ii) a subsequential limit, say $\widehat{\sigma}$, defines a holomorphic embedding of Ω into $\mathbb{C}^{n+1}$.

We shall see how such a simple idea produces significant results in the subsequent sections. On the other hand, it is not known whether such a choice is always possible so that the scaling sequence converges.

3.3. Frankel's scaling sequence

Before discussing the effect of the scaling method, it should be mentioned that there is another way of constructing a scaling sequence. With the same φ_ν and q as above, S. Frankel in his Ph. D. dissertation introduced the sequence

$$\omega_\nu(z) \equiv [d\varphi_\nu(q)]^{-1}(\varphi_\nu(z) - \varphi_\nu(q)).$$

Notice that each ω_ν embeds Ω into $\mathbb{C}^{n+1}$. From the viewpoint of Pinchuk's scaling, one may see that the differences between the two scaling methods are: **(1)** that the reference points $\varphi_\nu(q)$ are in the interior of Ω, and **(2)** that the sequence $d\varphi_\nu(q)^{-1}$ replaces the role of centering followed by dilation. In the scaling methods, the most delicate and important issues lie in the convergence of the scaling sequence to a biholomorphic embedding of the domain. One would also like to be able to guarantee that the limit mapping is injective. In the ensuing discussion we shall invoke the notion of Kobayashi hyperbolicity. This is an invariant version of the idea of boundedness. A domain Ω or, more generally, a complex manifold is Kobayashi hyperbolic if the Kobayashi distance (see [66]) is positive on Ω. The collection of Kobayashi hyperbolic manifolds is broad. For example, any bounded domain is hyperbolic. Moreover, the upper half-plane $\{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and the Siegel upper half-space $\{z \in \mathbb{C}^{n+1} \mid \operatorname{Im} z_0 > |z_1|^2 + \cdots + |z_n|^2\}$ are Kobayashi hyperbolic.

Theorem 3.1 (Frankel [30]). *Let Ω be a convex, Kobayashi hyperbolic domain in $\mathbb{C}^{n+1}$. For any sequence φ_ν of automorphisms of Ω and a point $q \in \Omega$, the sequence of maps defined by*

$$\omega_\nu(z) = [d\varphi_\nu(q)]^{-1}(\varphi_\nu(z) - \varphi_\nu(q))$$

forms a pre-compact normal family, in the sense that its every subsequence has a subsequence that converges uniformly on compact subsets of Ω. Moreover, every subsequential limit is a holomorphic embedding of Ω into $\mathbb{C}^{n+1}$.

Notice that, from the viewpoint of this article at least, *this theorem is most interesting when $\varphi_\nu(q)$ accumulates at a boundary point.* On the other hand, it turns out, due to work of Kim and Krantz [53], that Pinchuk's scaling sequence can be selected to have the same conclusion in case the domain is convex as in the hypothesis of this theorem. Furthermore, the two scaling methods are indeed equivalent. More precisely the following has been shown in [53]:

Proposition 3.2. *In addition to the hypothesis of Frankel's theorem above, assume that the sequence $\varphi_\nu(q)$ accumulates at a boundary point of Ω as $\nu \to \infty$. Let σ_ν denote Pinchuk's scaling sequence. Then we have the following conclusion:*

(i) *Every subsequence of σ_ν admits a subsequence that converges uniformly on compacta to an injective holomorphic mapping of Ω into $\mathbb{C}^{n+1}$.*
(ii) *Let $\widetilde{\Omega}$ denote the limit domain of the Frankel scaling, and let $\widehat{\Omega}$ the limit domain of the Pinchuk scaling. Then these two domains are biholomorphic to each other by a complex affine linear map.*

It may be noted that the papers [81]–[84] use techniques that relate to Pinchuk and Frankel scaling.

3.4. Normal convergence of sets

We present the concept of Carathéodory kernel convergence of domains which is relevant to the discussion of scaling methods and general normal family of holomorphic mappings. For more detailed discussions on this convergence, see p.76 of [25].

Definition 3.3 (Caratheodory Kernel Convergence). Let Ω_ν be a sequence of domains in $\mathbb{C}^{n+1}$ such that $p \in \bigcap_{\nu=1}^\infty \Omega_\nu$. If p is an interior point of $\bigcap_{\nu=1}^\infty \Omega_\nu$, the *Carathéodory kernel $\widehat{\Omega}$ at p* of the sequence $\{\Omega_\nu\}$ is defined to be the largest domain containing p having the property that each compact subset of $\widehat{\Omega}$ lies in all but a finite number of the domains Ω_ν. If p is not an interior point of $\bigcap_{\nu=1}^\infty \Omega_\nu$, then the Carathéodory kernel $\widehat{\Omega}$ is $\{p\}$. The sequence Ω_ν of domains is said to *converge to its kernel at* p if every subsequence of Ω_ν has the same kernel at p.

We shall also say that a sequence Ω_ν of domains in $\mathbb{C}^{n+1}$ *converges normally* if there exists a point $p \in \bigcap_{\nu=1}^{\infty} \Omega_\nu$ such that Ω_ν converges to its Carathéodory kernel at p.

The motivation for this notion can be seen in the following proposition. We omit the proofs, as they are routine.

Proposition 3.4. *Let Ω_ν form a sequence of domains in $\mathbb{C}^{n+1}$ that converges normally to the domain $\widehat{\Omega}$. Let $W \in \mathbb{C}^m$ be a Kobayashi hyperbolic domain. Then every sequence of holomorphic mappings $f_\nu : \Omega_\nu \to W$ contains a subsequence that converges uniformly on compacta to a holomorphic mapping $\widehat{f} : \widehat{\Omega} \to \overline{W}$. Furthermore, if $\{g_\nu : W \to \Omega_\nu\}$ forms a pre-compact normal family, then every subsequential limit, say $\widehat{g}$, has its image contained in the closure of $\widehat{\Omega}$.*

It may be appropriate to remark that the topological set convergence such as a version of local Hausdorff convergence can replace the normal convergence in case the domains in consideration are convex domains.

4. Domains with noncompact automorphism group

4.1. The theorem of Bun Wong and Jean-Pierre Rosay

Theorem 4.1 (Wong [100], Rosay [92]). *Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with a sequence of automorphisms φ_ν and a point $q \in \Omega$ such that* $\lim_{\nu\to\infty} \varphi_\nu(q)$ *$= p$ for some $p \in \partial\Omega$. If $\partial\Omega$ is C^2 strongly pseudoconvex in a neighborhood U of p, then Ω is biholomorphic to the unit open ball B in $\mathbb{C}^{n+1}$.*

Theorem 4.2 (Wong [100]). *Every smoothly bounded domain in $\mathbb{C}^{n+1}$ with transitive automorphism group is biholomorphic to the unit open Euclidean ball in $\mathbb{C}^{n+1}$.*

Proof of the Wong-Rosay Theorem by the Method of Scaling. This proof is essentially due to S. Pinchuk. It consists of four typical steps for a scaling proof: **(1) preparation, (2) localization, (3) dilatation,** and **(4) synthesis.**

Step 1. Preparation. Without loss of generality, let us assume that p is the origin 0 in $\mathbb{C}^{n+1}$. Since Ω is strongly pseudoconvex at the origin, we may perform a holomorphic coordinate change at the origin so that in an open Euclidean ball $B(0, 10r)$ of radius $10r$ centered at the origin, the set $\Omega \cap B(0, 10r)$ can be defined by an inequality[4]

$$\rho(z) < 0$$

[4]This is a concrete implementation of the statement, discussed earlier, that a strongly pseudoconvex point may be convexified by a biholomorphic mapping.

where

$$\rho(z) = -\mathrm{Re}\, z_0 + |z_0|^2 + \cdots + |z_n|^2 + R(z)$$

and (using Landau's notation)

$$R(z) = o(|z_0|^2 + \cdots + |z_n|^2).$$

In particular, choosing $r \geq 0$ smaller if necessary, we may arrange that

$$|R(z)| \leq \frac{1}{4}(|z_0|^2 + \cdots + |z_n|^2) \ \forall z \in B(0, 2r)$$

and that the boundary $\partial\Omega$ is now strongly convex in $B(0, 2r)$.

It may be appropriate to remark at this juncture that a $\mathcal{C}^\infty$ smooth, strongly pseudoconvex boundary has a 4-th order contact with a sphere. (See Fefferman [28], and also [43].)

Step 2. Localization. As a consequence of the preceding step, we see that there exists a holomorphic function $h : B(0, 2r) \to \mathbb{C}$ such that

$$h(0) = 1 \text{ and } |h(\zeta)| < 1 \text{ for every } \zeta \in \overline{\Omega} \cap B(0, 2r) \setminus \{0\}.$$

Now look at the automorphisms φ_ν of Ω. Since Ω is a bounded domain, every subsequence of φ_ν admits a subsequence that converges to a holomorphic mapping from Ω into the closure $\overline{\Omega}$ of Ω, uniformly on compact subsets. Let Φ be a subsequential limit of a subsequence φ_{ν_k}. Since $\Phi(q) = 0$ and since $\Phi : \Omega \to \overline{\Omega}$ is holomorphic, we may exploit the uniform convergence on compact subsets to see that there exists a relatively compact neighborhood U, say, of q such that $\varphi_{\nu_k}(U) \subset B(0, 2r) \cap \Omega$ for all sufficiently large k. Then consider $h \circ \varphi_{\nu_k}|_U : U \to D$, where D denotes the unit disc. This yields now that $h \circ \Phi(0) = 1$. Hence the Maximum Modulus Principle implies that $\Phi(z) = 0$ for every $z \in U$. Since U contains a non-empty open set, we conclude that Φ vanishes identically in Ω. We may now deduce, replacing $\{\varphi_\nu\}$ by one of its subsequences, that for every compact subset K of Ω there exists a positive integer N such that

$$\varphi_\nu(K) \subset B(0; r) \cap \Omega$$

whenever $\nu \geq N$.

Step 3. Dilatation. Consider now the sequence $\varphi_\nu(q)$ in Ω. Let $q^\nu = \varphi_\nu(q)$ for each ν. Choose the boundary point p^ν as in the construction of Pinchuk's dilatation sequence above. Then choose the centering map A_ν and the dilation map L_ν as above for each ν. We may examine Pinchuk's dilatation sequence $\Lambda_\nu := L_\nu \circ A_\nu$. It is a simple matter to check that the sequence of domains $\Lambda_\nu(\Omega \cap U)$ converges normally to the domain

$$V = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \mathrm{Re}\, z_0 \geq |z_1|^2 + \cdots + |z_n|^2\}.$$

Moreover, one may replace φ_ν by a subsequence again to have that $\Lambda_\nu(\Omega \cap U) \subset \mathcal{E}$ for every ν sufficiently large, where $\mathcal{E} = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : \mathrm{Re}\, z_0 \geq \frac{1}{2}(|z_1|^2 + \cdots + |z_n|^2)\}$.

Step 4. Synthesis via normal families. Take a sequence W_ν of relatively compact subdomains of Ω satisfying

$$\overline{W_\nu} \subset W_{\nu+1} \text{ for every } \nu = 1, 2, \ldots$$

and

$$\bigcup_{\nu=1}^{\infty} W_\nu = \Omega.$$

Consider now the scaling sequence $\sigma_\nu = \Lambda_\nu \circ \varphi_\nu$. Choosing a subsequence, we may assume that

$$\varphi_\nu(W_\nu) \subset \Omega \cap B(0, r)$$

for every $\nu = 1, 2, \ldots$. Then by the preceding step, the scaling sequence $\sigma_\nu := \Lambda_\nu \circ \varphi_\nu|_{W_\mu}$ forms a normal family for every μ as $\nu \to \infty$. Notice also that, for every compact subset K' of V, the sequence σ_ν^{-1} maps K' into Ω for sufficiently large ν. Altogether, one sees that any subsequential limit of the scaling sequence becomes a biholomorphic mapping from Ω onto V. Since V is biholomorphic to the unit open ball, the theorem is now proved. □

4.2. Domains with piecewise Levi-Flat boundary that possess noncompact automorphism group

The main theorem of this section is the following:

Theorem 4.3 (Kim-Krantz-Spiro [55]). *Every generic analytic polyhedron in $\mathbb{C}^2$ with noncompact automorphism group is biholomorphic to the product of the unit open disc in $\mathbb{C}$ and a Kobayashi hyperbolic Riemann surface embedded in $\mathbb{C}^2$.*

A clarification of some terminology is in order. By an *analytic polyhedron* in $\mathbb{C}^{n+1}$, we mean a bounded domain, say Ω in $\mathbb{C}^{n+1}$, admitting an open neighborhood U of the closure $\overline{\Omega}$ of Ω and a finite collection of holomorphic functions $f_j : U \to \mathbb{C}$, $j = 1, \ldots, N$, such that

$$\Omega = \{z \in U : |f_1(z)| < 1, \ldots, |f_N(z)| < 1\}.$$

The collection $\{f_1, \ldots, f_N\}$ is usually called a defining system for Ω. The choice for defining system is not unique. An analytic polyhedron is called *generic* (or, *normal*), if it admits a defining system $\{f_1, \ldots, f_N\}$ satisfying the following additional condition:

$$df_{i_1}|_p \wedge \cdots \wedge df_{i_k}|_p \neq 0$$

whenever the condition $|f_{i_1}(p)| = \cdots = |f_{i_k}(p)| = 1$ holds for any un-repeated indices $i_1, \ldots, i_k \in \{1, \ldots, N\}$. We restrict our attention as usual to the *bounded* analytic polyhedra. By a theorem of H. Cartan mentioned earlier, the automorphism group of our analytic polyhedron is a finite-dimensional Lie group. The non-compactness of the automorphism group is therefore equivalent to the existence of a sequence $\varphi_\nu \in \text{Aut}\,\Omega$ and a point $q \in \Omega$ such that the point sequence $\varphi_\nu(q)$ accumulates at a boundary point.

Notice that Theorem 4.3 improves the following results:

Theorem 4.4 (Kim and Pagano [58]). *Let Ω be a generic analytic polyhedron in $\mathbb{C}^2$ with noncompact automorphism group. Then the holomorphic universal covering space of Ω is biholomorphic to the bidisc.*

It is worth mentioning the following theorem. It concerns only convex analytic polyhedra, but is valid in all dimensions.

Theorem 4.5 (Kim [49]). *Every convex generic analytic polyhedron in $\mathbb{C}^{n+1}$ with noncompact automorphism group is biholomorphic to the product of the unit open disc in $\mathbb{C}$ and a convex domain in $\mathbb{C}^n$. In particular, in case $n = 1$, the product domain is biholomorphic to the bidisc.*

Sketch of proof. We treat all three theorems simultaneously. Let Ω be a generic analytic polyhedron in $\mathbb{C}^{n+1}$ with noncompact automorphism group. Then there exist a boundary point $p \in \partial\Omega$, an interior point $q \in \Omega$ and a sequence $\varphi_j \in \text{Aut}\,\Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$. Since the boundary of Ω is piecewise Levi flat, we divide the proof of each theorem above into the following two cases: **(1)** the case that boundary $\partial\Omega$ is singular at p, and **(2)** the case that boundary $\partial\Omega$ is smooth and Levi flat in a neighborhood of p.

Case 1. The boundary $\partial\Omega$ is singular at p.

Recall that our domain is a generic analytic polyhedron. In complex dimension 2, therefore, it is possible to choose a defining system $f_1, \ldots, f_N$ such that there exist exactly two functions, say f_1, f_2 (shuffling the indices if necessary), such that $|f_1(p)| = |f_1(p)| = 1$ with $df_1|_p \wedge df_2|_p \neq 0$. Hence it is simple to realize that there exists a plurisubharmonic[5] function ψ defined in an open neighborhood of the closure of Ω such that

$$\psi(p) = 0, \text{ and } \psi(z) < 0 \text{ for every } z \in \overline{\Omega} \setminus \{p\}.$$

Such a function is called a *plurisubharmonic* (psh for short) *peak function* for Ω at p. The maximum modulus principle immediately implies in particular that there are no non-trivial analytic varieties in $\partial\Omega$ passing through p. Moreover, it is known that the following localization principle holds (see [8], and also [16] for detailed arguments, for instance):

Let U be an open neighborhood of p. For every compact subset K of Ω, there exists a positive integer j_K such that $\varphi_j(K) \subset U$ for every $j \geq j_K$.

Now, using the mapping $(f_1, f_2) : U \to \mathbb{C}^2$ followed by a linear fractional mapping of $\mathbb{C}^2$, one can construct a biholomorphism-into $\Psi : U \to \mathbb{C}^2$ with $\Psi(p) = (0,0)$ and $\Psi(U \cap \Omega) = \Psi(U) \cap \mathcal{H}^2$ where

$$\mathcal{H}^2 = \{(z_0, z_1) \in \mathbb{C}^2 : \text{Re}\, z_0 > 0, \text{Re}\, z_1 > 0\}.$$

Let us write $\Psi(\varphi_j(q)) = (t_{j,0}, t_{j,1})$. Then consider the dilatation sequence

[5] A real-valued continuous function $\psi : \Omega \to \mathbb{R}$ defined in a domain Ω in $\mathbb{C}^n$ is called *plurisubharmonic* if it is subharmonic when restricted to any complex affine line. See [66].

$$\Lambda_j(z_0, z_1) = \left(\frac{z_0 - \operatorname{Im} t_{j,0}}{\operatorname{Re} t_{j,0}}, \frac{z_1 - \operatorname{Im} t_{j,1}}{\operatorname{Re} t_{j,1}} \right).$$

Finally, consider the scaling sequence

$$\sigma_j := \Lambda_j \circ \Psi \circ \varphi_j$$

for $j = 1, 2, \ldots$. It still requires some checking, but it follows that a subsequential limit of this sequence gives rise to a biholomorphic mapping from Ω onto the domain H^2 which is in turn biholomorphic to the bidisc. Thus Theorems 4.3 and 4.4 are proved in this case. For Theorem 4.5, one takes into consideration that Ω is a convex domain in $\mathbb{C}^{n+1}$. Note that every complex analytic variety contained in a convex Levi-flat hypersurface is an open subdomain of a complex affine hyperplane. Now choose a defining system $f_0, \ldots, f_N$ so that

$$f_0(p) = \cdots = f_k(p) = 1, \quad |f_{k+1}(p)| < 1, \ldots, |f_N(p)| < 1$$

and

$$df_1|_p \wedge \cdots \wedge df_k|_p \neq 0.$$

The maximal variety in the boundary of Ω passing through p is represented by

$$V_p = \{z \in \mathbb{C}^{n+1} : f_0(z) = \cdots = f_k(z) = 1, \quad |f_{k+1}(z)| < 1, \ldots, |f_N(z)| < 1\}.$$

Notice that $\dim_{\mathbb{C}} V_p = n - k$. It is possible that $n = k$, and consequently that V_p is a single point. But it is always the case that $k \geq 0$. Then one may use the mappings $f_0, \ldots, f_k$ to see that there exists an open neighborhood, say W, of V_p such that there exists a holomorphic embedding $\Psi : W \to \mathbb{C}^{n+1}$ such that

(a) $\Psi(V_p) = \{(0, \ldots, 0; z_{k+1}, \ldots, z_n) \in \mathbb{C}^{n+1} : (z_{k+1}, \ldots, z_n) \in \Omega'\}$, where Ω' is a convex domain containing the origin in $\mathbb{C}^{n-k}$, and
(b) $\Psi(W \cap \Omega) = \Psi(W) \cap U$ where

$$U = \{z \in \mathbb{C}^{n+1} : \operatorname{Re} z_0 \geq 0, \ldots, \operatorname{Re} z_k \geq 0, (z_{k+1}, \ldots, z_n) \in \Omega'\}.$$

Write $\Psi \circ \varphi_j(q) = (t_{j,0}, \ldots, t_{j,n})$ and then consider the dilatation map

$$\lambda_j(z) = \left(\frac{z_0 - \operatorname{Im} t_{j,0}}{\operatorname{Re} t_{j,0}}, \ldots, \frac{z_k - \operatorname{Im} t_{j,k}}{\operatorname{Re} t_{j,k}}; z_{k+1}, \ldots, z_n \right).$$

Again it follows that a subsequential limit of the sequence

$$\sigma_j := \Lambda_j \circ \Psi \circ \varphi_j$$

gives rise to a biholomorphic mapping from Ω onto U which in turn is biholomorphic to the product of a $k + 1$ dimensional polydisc and the domain Ω' in $\mathbb{C}^{n-k}$. This proves Theorem 4.5 in the present case.

<u>*Case 2.*</u> The boundary $\partial\Omega$ is smooth and Levi flat at p.

This case is easier to handle when Ω is convex, even when $\dim_{\mathbb{C}^{n+1}} \Omega = n+1$ for an arbitrary non-negative integer n. Consider the maximal variety V_p in the boundary $\partial\Omega$ passing through p. As argued earlier, it follows that V_p is a convex, open subset of a complex affine hyperplane. By a complex affine linear change of coordinates, say by an affine linear biholomorphism $\psi : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$, we may assume that $\psi(p) = 0$, that $\psi(V_p) \subset \{(z_0, \dots, z_n) : z_0 = 0\}$ and that the domain Ω at p is contained in the half-space defined by the inequality $\operatorname{Re} z_0 \geq 0$. Now we apply the scaling method as before. Let $\psi \circ \varphi_j(q) = (t_{j,0}, \dots, t_{j,n})$. Then define the dilatation mapping by

$$\Lambda_j(z_0, \dots, z_n) = \left(\frac{z_0 - \operatorname{Im} t_{j,0}}{\operatorname{Re} t_{j,0}}, z_1, \dots, z_n \right).$$

Then it turns out that the scaling sequence

$$\sigma_j := \Lambda_j \circ \psi \circ \varphi_j$$

yields a subsequential limit which becomes a biholomorphic mapping from Ω onto the product of the upper half plane in $\mathbb{C}$ and the n-dimensional convex domain $\mathbb{C}^{n+1}$. This completes our sketch of the proof to Theorem 4.5. □

In case the analytic polyhedron is merely generic, and not necessarily convex, the situation is much more complicated. Thus it is natural that one focuses on the case of complex dimension two.

Then the maximal variety V_p is a Riemann surface that is Kobayashi hyperbolic. The uniformization theorem of Riemann surface theory yields a holomorphic covering map $\pi : D \to V_p$ from the open unit disc D in $\mathbb{C}$ onto V_p. Then extend it trivially to the map $\widetilde{\pi}(z_1, z_2) = (\pi(z_1), z_2)$. Since the normal bundle for V_p in $\mathbb{C}^2$ is trivial, one may take an open neighborhood U for V_p in $\mathbb{C}^{n+1}$ in such a way that $\Omega \cap U$ is connected and that $\widetilde{\pi}$ gives rise to a local biholomorphism, say $\widehat{\pi}$, from an open neighborhood of $D \times \{0\}$ onto U. One can arrange also that $\widetilde{\pi}(0,0) = p$.

Then consider the domain $\widetilde{\Omega}_{\mathrm{loc}}$ which is a connected component of $\widehat{\pi}^{-1}(\Omega \cap U)$ containing the origin. Then take a lifting of the sequence $\varphi_j(q)$ via $\widehat{\pi}$. There are many liftings. Choose therefore one that converges to the origin $(0,0)$ for instance. Now build a dilatation mapping Λ_j for $\widetilde{\Omega}$, formally the same as in the convex case (with an adjustment; see [55] for details), with respect to the sequence chosen here. Then it turns out that the sequence of mappings

$$\varphi_j^{-1} \circ \widehat{\pi} \circ \Lambda_j$$

yields a subsequential limit, say Ψ, from the product $D \times \mathcal{H}$ of the open unit disc D and the half-plane $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ onto our generic analytic polyhedron Ω. Using normal families arguments, it is not hard to deduce that Ψ is a holomorphic mapping with its Jacobian vanishing nowhere on $D \times \mathcal{H}$. Moreover, it turns out that this map preserves the Kobayashi-Royden infinitesimal metric. Therefore it preserves the Wu metric (see below) as well.

The Wu metric (see [101]) here can be quickly understood as follows. At each point p of a Kobayashi hyperbolic domain G in $\mathbb{C}^{n+1}$, consider the tangent space T_pG and the collection of vectors with Kobayashi length not exceeding 1. This set is sometimes called the Kobayashi indicatrix at p. Endow T_pG with an arbitrarily chosen Hermitian inner product. (This choice of Hermitian inner product is neither unique nor natural, but it will not cause any problems at the end.) Then consider the ellipsoids, say $E = E_H$, in $T_pG(= \mathbb{C}^{n+1})$ given by

$$E_H = \{v \in \mathbb{C}^{n+1} : v^* H v \leq 1\},$$

where H is a positive definite Hermitian $(n+1) \times (n+1)$ matrix and where v^* denotes the conjugate transpose of v. Denote by $\mathcal{Q}$ the set of such E_H containing the Kobayashi indicatrix. Then take $1/(\det H)$ as the volume of E_H. Then it turns out that the element in $\mathcal{Q}$ with the smallest volume is uniquely determined, regardless of the choice of the Hermitian inner product on T_p. See [101]. Now let this minimum volume ellipsoid define a Hermitian inner product, say h_p, on T_pG. The assignment $p \mapsto h_p$ defines the Wu metric on G. It is immediate from the invariance of the Kobayashi metric that the Wu metric is invariant under biholomorphic maps. It is obviously Hermitian, in the sense that it defines a Hermitian inner product on each tangent space. It has been shown that $p \mapsto h_p$ is $\mathcal{C}^0$ (continuous) in general.

Now since the Wu metric $h_{D\times D}$, say, of the bidisc $D \times D$ is real-analytic, so are the Wu metrics $h_{D\times\mathcal{H}}$ of $D\times\mathcal{H}$ and h_Ω of Ω, respectively, since $\Psi_* h_{D\times\mathcal{H}} = h_\Omega$. Since the Kobayashi metric of Ω is complete, so is the Wu metric h_Ω. At this point, one may apply the proof of the Cartan-Hadamard Theorem in Riemannian Geometry to conclude that $\Psi : D \times \mathcal{H} \to \Omega$ is indeed a covering mapping. This yields Theorem 4.4.

Finally, for Theorem 4.3, one has to analyze the covering mapping Ψ as well as its deck transformation group more precisely. Although we are omitting the details here, it should not be difficult for the reader to see that the constructions for $\widetilde{\pi}$ as well as $\widehat{\pi}$ will reflect the nature of the covering map $\pi : D \to V_p$ without any essential changes. Hence it was shown in [55] through a careful analysis that indeed Ω is biholomorphic to the product of the open unit disc and the maximal variety V_p, and that the deck transformation group Γ_Ψ for the covering mapping $\Psi : D \times \mathcal{H} \to \Omega$ is in fact $\Gamma_\pi \times \{\text{id}\}$, where Γ_π denotes the deck-transformation group for the uniformization map $\pi : D \to V_p$. This is how Theorem 4.3 follows.

Notice that this analysis gives a rather complete classification of complex two-dimensional generic analytic polyhedra that possess noncompact automorphism group. Hence it seems appropriate to pose the following question here.

Problem 4.6. Classify all non-generic analytic polyhedra with noncompact automorphism group.

The main difficulty in this problem seems to lie in the question of how to adjust the scaling at a singular boundary point.

4.3. Remarks on the concept of finite type

It is a straightforward calculation (see [66]) to see that a strongly pseudoconvex boundary point p is flat to order 2. That is to say, the maximum possible order of contact of the boundary at p with a one-dimensional complex analytic variety is 2. See Figure 4. It is useful in this subject to be able to generalize this concept. For simplicity in this subsection we restrict our attention to two-dimensional complex space. The entire story in all dimensions is sketched out in [21]. See also [20]

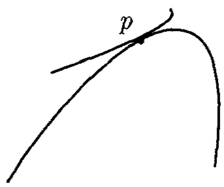

Figure 4: The Order of Contact

Let $\Omega = \{z \in \mathbb{C}^{n+1} : \rho(z) < 0\}$ be a smoothly bounded domain and fix a point $p \in \partial\Omega$. Let V be a nonsingular, one-dimensional complex analytic variety that passes through p. Then the *order of contact* of V with $\partial\Omega$ at p is the greatest positive integer k such that

$$|\rho(z)| \leq C \cdot |z - p|^k$$

for $z \in V$ near p and some constant $C \geq 0$. We say that p is of *finite geometric type* in the sense of orders of contact if there is an upper bound m on the order of contact of analytic varieties with $\partial\Omega$ at p. The least such integer m is called the *type* of the point p. As previously noted, a strongly pseudoconvex point is of type 2. As a very simple illustrative example, for k a positive integer let

$$E_k = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2k} < 1\}.$$

Then one may calculate (again see [66]) that any boundary point of the form $(e^{i\theta}, 0)$ is of finite type $2k$. The notion of "type" is a means of measuring the flatness of the boundary in a complex analytic sense.[6]

One of the most important facts about finite type in complex dimension 2 is that the geometric definition given here is equivalent to an analytic definition involving commutators of vector fields. To wit, we may assume by a normalization of coordinates that $\partial\rho/\partial z_1(p) \neq 0$. Define the vector field

$$L = \frac{\partial\rho}{\partial z_1}(p)\frac{\partial}{\partial z_2} - \frac{\partial\rho}{\partial z_2}(p)\frac{\partial}{\partial z_1}.$$

[6]It is worth noting explicitly that, in complex dimension 2, we can measure the type of a boundary point using the order of contact of *smooth* complex varieties of dimension 1. But, in higher dimensions, work of D'Angelo [20] has taught us that we must examine singular complex varieties as well.

Then L is a tangential holomorphic vector field near p (because $L\rho = 0$). A *first-order commutator* is, for us, an expression of the form $[L, \overline{L}] = L\overline{L} - \overline{L}L$. A *second-order commutator* is the commutator of L or $\overline{L}$ with a first-order commutator. And so forth. We say that p is of *analytic type* m if any commutator of order not exceeding $m - 1$ has no complex normal component but that some commutator of order m *does* have a complex normal component. It is a basic result in dimension two (generalized to higher dimensions by Bloom and Graham [9]) that the boundary point p is of finite geometric type m if and only if it is of finite analytic type m.

One immediate consequence of this characterization is the semicontinuity of type: If $p \in \partial\Omega \subseteq \mathbb{C}^2$ is a point of finite type m, then there is a small boundary neighborhood U of p so that all point of U are of finite type not exceeding m. In complex dimensions greater than 2, this semicontinuity of type (as stated here) fails; but type *is* locally bounded. A substitute result was proved by D'Angelo in [20]. In any event, it follows from these results that if Ω is smoothly bounded and if each point of $\partial\Omega$ is of finite type, then there is an upper bound M so that the type of *every* boundary point does not exceed M.

In deep work [23], Diederich and Fornæss showed that any bounded domain with real analytic boundary, in any complex dimension, is of finite type. Thus domains with real analytic boundary form an important class of examples in this subject. (We recommend readers to read an alternative approach, with a discussion of the relationship to type and subelliptic estimates for the $\overline{\partial}$ problem, in [20].) It is perhaps instructive to contrast such a domain with a boundary that is Levi flat. Such a boundary is foliated by complex analytic varieties, so that one sees immediately that each boundary point is of infinite type. The provenance of the concept of finite type was the study of the $\overline{\partial}$-Neumann problem (see [64]). Since that time, finite type has assumed a rather prominent position in function theory, mapping theory, and related areas. See [20] and [21] for a full account of this central idea. The reference [22] also has many useful ideas.

4.4. A theorem of Bedford-Pinchuk

In the preceding sections, we discussed domains with noncompact automorphism group in the extreme cases when the boundaries are either strongly pseudoconvex or Levi flat. The intermediate concept, called the boundary of finite type in the sense of Catlin/Kohn/D'Angelo, encompasses a large class of weakly pseudoconvex domains with smooth boundary. We now present the following theorem pertaining to this class.

Theorem 4.7 (Bedford-Pinchuk [5]). *A bounded domain $\Omega \subseteq \mathbb{C}^2$ having real analytic boundary and admitting noncompact automorphism group is biholomorphic to*

$$E_m = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2m} < 1\}$$

for some positive integer m.

In fact some unpublished remarks of David Catlin show that the hypothesis "real analytic" may be weakened to "finite type". It should be also mentioned that Bedford and Pinchuk have extended the above theorem to broader classes of domains. Before discussing the proof of this theorem, we remark that several generalizations of this theorem have been carried out, mostly by Bedford and Pinchuk. See [4], [5], [8] for precise results. In particular Bedford and Pinchuk can prove results in higher dimensions with certain restrictions such as convexity. However, we shall focus on the original proof, as it reflects the essential methods for this case.

Sketch of the Proof of Theorem 4.7. The non-compactness of Aut Ω again implies the existence of $\varphi_j \in \text{Aut } \Omega$, $p \in \partial\Omega$ and $q \in \Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$. Then one may choose a holomorphic local coordinate system (z, w) such that p becomes the origin, and such that there exists an open neighborhood U of the origin in which the domain Ω is represented by the inequality

$$\text{Re}\, w \geq H(z) + R(z, w)$$

where:

(i) $H(z)$ is a homogeneous subharmonic polynomial in $z, \bar{z}$ of degree $2m$ without harmonic terms. Here m is a positive integer, and

(ii) $R(z, w) = o(|z|^{2m} + |\text{Im}\, w|)$.

Then it follows by a careful application of the scaling method that Ω is biholomorphic to the domain

$$M(\Omega, p) = \{(z, w) \in \mathbb{C}^2 : \text{Re}\, w \geq H(z)\}.$$

For a detailed argument a useful reference other than the original paper by Bedford-Pinchuk is the theorem on p. 620 of [8] by Berteloot. At this juncture, we simply use the fact that the mapping $(z, w) \mapsto (z, w + it)$ is an automorphism of $M(\Omega, p)$ for every $t \in \mathbb{R}$. This produces a noncompact one-parameter subgroup of automorphisms, say ψ_t, for Ω. But then Bedford and Pinchuk showed that there exists a boundary point $p' \in \partial\Omega$ such that

$$\lim_{t\to\pm\infty} \psi_t(z, w) = p'$$

for every $(z, w) \in \Omega$. Furthermore they show that this turns into a smooth(!) parabolic holomorphic vector field action at p' on the boundary of Ω. For this arguments they exploit the extension theorem by Bell and Ligocka for automorphisms.

The vector field obtained by

$$X := \frac{\partial\psi_t(z, w)}{\partial t}\bigg|_{t=0}$$

is a holomorphic vector field that vanishes at p'. Bedford and Pinchuk studies its expansion, and characterizes what the lowest order terms (in terms of appropriate weights) should be. This analysis allows them to see that the defining function for the domain near p' has to take the form (after a holomorphic change of coordinate system at p')

$$\operatorname{Re} w = |z|^{2m} + \text{terms with higher weights},$$

where p' is now the origin.

Recall that the origin is an accumulation point of the automorphism group orbit. Thus one can scaling the above expression of the local defining function. After a scaling with some care, one can conclude that the original domain is biholomorphic to the domain defined by

$$\operatorname{Re} w = |z|^{2m}.$$

This yields the desired conclusion via a Cayley type transformation. □

As one can see from the proof, the assumption that Ω has global smoothness (indeed, real analyticity) and finite type is essential, as one does not know where p' will be located in $\partial\Omega$. Attempts to obtain the same conclusion from the weaker assumption that $\partial\Omega$ is real analytic of finite type at the initial orbit accumulation point p cannot be successful, as there are famous counterexamples such as the one defined by $\operatorname{Re} z_0 > |z_1|^8 + \frac{15}{7}|z_1|^2 \operatorname{Re} z_1^6$, for instance [7].

5. The Greene-Krantz conjecture

The classification program for domains with noncompact automorphism group is far from being complete, even for the case of very smooth, or piecewise smooth, boundaries. On the other hand, it seems natural at this juncture to mention the following outstanding conjecture by Greene and Krantz:

Conjecture 5.1. *Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with C^∞ boundary. If there exists a sequence φ_ν of automorphisms of Ω and a point $q \in \Omega$ such that the orbit $\varphi_\nu(q)$ accumulates at a boundary point p of Ω, then p is of finite type in the sense of D'Angelo, Catlin, and Kohn.*

The full conjecture is still open. The purpose of this section is to introduce some partial results supporting the conjecture, discovered by means of the scaling method. The first partial result we mention here is the following reformulation of Theorem 4.5:

Proposition 5.2. *Let Ω be a bounded, convex domain in $\mathbb{C}^{n+1}$ with a boundary point $p \in \partial\Omega$ admitting an open neighborhood U such that $\partial\Omega \cap U$ is Levi flat at every point. Then no automorphism orbit of Ω can accumulate at p.*

---

[7]This example indeed is the famous model by Kohn/Nirenberg [65], that cannot be made convex by any holomorphic change of local coordinates at the origin. It should not be too difficult to see that this domain cannot be biholomorphic to $\operatorname{Re} z_0 > |z_1|^8$, when one uses reflection principles for instance.

The basis for this proposition is as follows: If there were an automorphism orbit accumulating at p, then Theorem 4.5 from the hypothesis implies that Ω is biholomorphic to the product of a convex domain and the open unit disc. Therefore, Ω is a (trivial) fiber space over Ω. A theorem of A. Huckleberry ([46]) says that this cannot be biholomorphic to a bounded domain with a strongly pseudoconvex boundary point. Since any bounded domain with entirely smooth boundary must admit[8] a strongly pseudoconvex boundary point, it leads us to a contradiction. Thus the proposition follows immediately. □

It should be observed that an infinite type boundary point need not admit a neighborhood in which every boundary point is of infinite type (consequently Levi-flat). A primary example is given by the origin for the domain defined by

$$\operatorname{Re} w \geq \exp\left(-\frac{1}{|z|^2}\right).$$

Indeed, Greene and Krantz demonstrated the following, when they posed the aforementioned conjecture:

Proposition 5.3. *The automorphism group of the domain in $\mathbb{C}^2$ defined by*

$$|z|^2 + 2\exp\left(-|w|^{-2}\right) < 1$$

is compact. In particular, there is no automorphism orbit accumulating at any boundary point $(e^{i\theta}, 0)$, of infinite type.

The original proof of this proposition exploited the fact that the domain in consideration is Reinhardt, admitting full rotational symmetry. However, it turns out that, for the purpose pertaining to the Greene-Krantz conjecture, the obstruction against the existence of automorphism orbits accumulating at the point of such exponential infinite type boundary point is purely local. Consider the following result.

Theorem 5.4 (Kim and Krantz [53]). *Let Ω be a domain in $\mathbb{C}^2$ with a boundary point p which admits an open neighborhood U and an injective holomorphic mapping $\Psi : U \to \mathbb{C}^2$ such that $\Psi(p) = (0,0)$ and*

$$\Psi(U \cap \Omega) = \{(z,w) \in \Psi(U) : \operatorname{Re} w \geq \psi(|z|)\}$$

where $\psi : \mathbb{R} \to \mathbb{R}$ is a C^∞ smooth function satisfying:

(i) *ψ is C^∞ smooth.*
(ii) *$\psi(t) = 0$, $\forall t \leq 0$, and $\psi''(t) \geq 0$, $\forall t \geq 0$.*
(iii) *$\psi(t) = \exp(-\mu(t)^{-1})$ for some $\mu(t)$ that is a non-negative smooth function vanishing to a finite order at $t = 0$.*

---

[8]Consider the function $f(x) = \|x\|$ that represent the Euclidean distance between the origin and the point $x \in \overline{\Omega}$. Since $\overline{\Omega}$ is compact, the function $f(x)$ assumes the maximum, at a boundary point p, say, of Ω. Then $\partial\Omega$ has a sphere contact at p so that the whole domain Ω is included in the sphere. Then p is in fact a strongly convex (hence, strongly pseudoconvex) boundary point.

Then there is no holomorphic automorphism orbit of Ω *accumulating at* p.

Sketch of the Proof. The detailed argument of the proof given in [53] is long and tedious. On the other hand, the key ideas are as follows. Expecting a contradiction, assume to the contrary that there is an automorphism orbit $\varphi_\nu(q)$ converging to p. Then apply the scaling technique to the domain Ω. Calculations show that the scaled limit domain is biholomorphic to one of the following domains:

(i) the open unit ball $\mathbb{B}$ in $\mathbb{C}^2$,
(ii) the open unit bidisc $D = \{(z,w) \in \mathbb{C}^2 : |z| < 1, |w| < 1\}$
(iii) the domain $T = \{(z,w) \in \mathbb{C}^2 : \operatorname{Re} z \geq \exp(\operatorname{Re} w)\}$.

These three domains occur depending upon the tangency of the orbit $\varphi_\nu(q)$ to the boundary. If the orbit is very tangential to the strongly pseudoconvex part of the boundary, then the first possibility appears. If the orbit is not so tangential to the boundary, than the bidisc shows up as the limit domain of the scaling process. The appropriate intermediate exponential tangency of the orbit to the strongly pseudoconvex part of the boundary produces the 3rd possibility. Now notice that the convergence of scaling (see [53]) implies that the original domain Ω has to be biholomorphic to one of the domains listed above. But none of these possibilities can occur. In the first case, the domain Ω should be homogeneous, as the ball is. Then one may choose a non-tangential sequence of automorphism orbit accumulating at p. Then scaling will show that the domain Ω is biholomorphic to the bidisc. This shows that the domain Ω is biholomorphic to both the ball and the bidisc. This contradicts the theorem of Poincaré which says that there does not exist any biholomorphism between the ball and the bidisc in complex dimension two. For the second case, a mirror-image argument implies the same kind of contradiction. The third case is much the same. Since the domain T contains a real 3-dimensional subgroup without fixed points, one finds an automorphism orbit of Ω that is either non-tangential to the boundary or very tangential to the line of boundary points of infinite type. In either case, one gets the bidisc as the new scaled limit. Then one arrives at a contradiction as before. This, altogether, contradicts the theorem of convergence of Pinchuk's scaling method for the convex case (See [53], stated as Proposition 3.2 in Section 3.3.). Therefore one is led to the conclusion that there are no automorphism orbits in Ω accumulating at p, as claimed. □

Digressing slightly, we note the following results of J. Byun ([13], [14]):

Theorem 5.5 (Byun). *Let* Ω *be a domain in* $\mathbb{C}^2$. *Assume that there exists a point* $p \in \partial\Omega$ *admitting an open neighborhood* U *in* $\mathbb{C}^2$ *satisfying the conditions*

(1) *the boundary* $\partial\Omega$ *is* C^∞, *pseudoconvex, and of finite type in* U, *and*
(2) *the finite type of* $\partial\Omega$ *at* p *is strictly greater than that of other points in* $\partial\Omega \cap U$.

Then there do not exist any automorphism orbits in Ω *accumulating at* p.

Notice that this in particular implies

Corollary 5.6 (Byun). *The Kohn-Nirenberg Domain*

$$\Omega = \{(z,w) \in \mathbb{C}^2 : \operatorname{Re} w + |zw|^2 + |z|^8 + \frac{15}{7}|z|^2 \operatorname{Re} z^6 < 0\}$$

has no automorphism orbit accumulating at the origin $(0,0)$.

In the case that the automorphisms extend to the boundary, one can say more about the Greene-Krantz conjecture. Recent articles [71] by M. Landucci and [15] by J. Byun and H. Gaussier show the following:

> If a domain satisfies Condition R of Bell (that the Bergman projection maps $C^\infty(\overline{\Omega})$ to itself—see [68]), if a straight line segment of positive length lies in its boundary, and if each point of the segment is convexifiable and of maximum type (here infinite type is allowed), then none of the point on the segment can be an orbit accumulation point.

Despite such encouraging and supporting evidences, the most general case of Greene-Krantz conjecture still awaits a solution. At the same time we would like to pose the more restricted problem:

Problem 5.7. Let Ω be a bounded domain in $\mathbb{C}^{n+1}$, for some $n \geq 1$, with C^∞ smooth boundary. Then can one show that an isolated infinite type boundary point cannot be an orbit accumulation point?

6. Asymptotic behavior of holomorphic invariants

6.1. Boundary behavior of the Kobayashi and Carathéodory metrics

Methods of scaling have been used to study the boundary asymptotics of invariant metrics on strongly pseudoconvex domains in $\mathbb{C}^{n+1}$. In what follows, if Ω_1, Ω_2 are domains then we let $\Omega_2(\Omega_1)$ denote the collection of holomorphic mappings from Ω_1 to Ω_2. As usual, D denotes the unit disc in $\mathbb{C}$. We begin by defining two important invariant metrics in complex Finsler geometry (for background and details, see [66]). These should be thought of as generalizations of the Poincaré metric from the disc to more general domains.

Definition 6.1. If $\Omega \subseteq \mathbb{C}^{n+1}$ is open, then the *infinitesimal Carathéodory metric* is given by $F_C : \Omega \times \mathbb{C}^{n+1} \to \mathbb{R}$ where

$$F_C(z,\xi) = \sup_{\substack{f \in B(\Omega) \\ f(z)=0}} |f_*(z)\xi| \equiv \sup_{\substack{f \in B(\Omega) \\ f(z)=0}} \left| \sum_{j=0}^{n} \frac{\partial f}{\partial z_j}(z) \cdot \xi_j \right|.$$

Definition 6.2. Let $\Omega \subseteq \mathbb{C}^{n+1}$ be open. Let $e_1 = (1,0,\ldots,0) \in \mathbb{C}^{n+1}$. The infinitesimal form of the *Kobayashi/Royden metric* is given by $F_K : \Omega \times \mathbb{C}^{n+1} \to \mathbb{R}$, where

$$F_K(z,\xi) \equiv \inf\{|\alpha| \ : \ \exists f \in \Omega(B) \text{ with } f(0) = z, (f'(0))(e_1) = \xi/\alpha\}.$$

Some alternative definitions are available. For instance the following invariant form is known:

$$F_K(z,\xi) = \inf\Big\{\frac{|\xi|}{|(f'(0))(e_1)|} : f \in \Omega(B), (f'(0))(e_1) \text{ is a constant multiple of } \xi\Big\}.$$

Very little was known about these metrics—except for rather abstract generalizations—until the seminal work of Ian Graham in 1975 (see [34]). His basic result is this:

Theorem 6.3 (Graham). *Let $\Omega \subset\subset \mathbb{C}^{n+1}$ be a strongly pseudoconvex domain with C^2 boundary. Fix $P \in \partial\Omega$. Let $\xi \in \mathbb{C}^{n+1}$, and write $\xi = \xi_T + \xi_N$, the decomposition of ξ into complex tangential and normal components relative to the geometry at the point P. Let ρ be a defining function for Ω normalized so that $|\nabla\rho(P)| = 1$. Let $\Gamma_\alpha(P)$ be a non-tangential approach region at P. If F represents either the Carathéodory or Kobayashi/Royden metric on Ω, then*

$$\lim_{\Gamma_\alpha(P)\ni z\to P} d_\Omega(z)\cdot F(z,\xi) = \frac{1}{2}|\xi_N|.$$

Here $|\ |$ denotes Euclidean length and $d_\Omega(x)$ is the distance of x to the boundary of Ω. If $\xi = \xi_T$ is complex tangential, then we have

$$\lim_{\Gamma_\alpha(P)\ni z\to P} \sqrt{d_\Omega(z)}\cdot F(z,\xi) = \frac{1}{2}\mathcal{L}(\xi,\xi),$$

where

$$\mathcal{L}(\xi,\xi) = \sum_{j,k=0}^{n} \frac{\partial^2\rho}{\partial z_j\partial\bar z_k}\Big|_p \cdot \xi_j\bar\xi_k$$

for $\xi = (\xi_0,\dots,\xi_n)$. Such $\mathcal{L}$ is called the Levi form for the defining function ρ.

Graham's proof is based upon an intricate local analysis with uniform estimates on the $\bar\partial$ operator. Later, it has turned out that the proof using the scaling methods are easier to understand and gives finer analysis. Such subsequent analyses are found in [78], [31], [10], [77], [2], and others. Here we present S. Lee's refinement and proof by the scaling method.

Theorem 6.4 (S. Fu, D. Ma, S. Lee). *Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with a C^2 smooth, strongly pseudoconvex boundary. Let $F_\Omega(p,\xi)$ denote either the Carathéodory or Kobayashi metric of Ω for $\xi \in T_p\Omega = \mathbb{C}^{n+1}$. For each $q \in \Omega$ sufficiently close to the boundary of Ω, choose a boundary point $p \in \partial\Omega$ that is the nearest to q. Then it holds that*

$$\lim_{q\to\partial\Omega}\left(\left(\frac{\|\xi_{N,p}\|}{2d(q,\partial\Omega)}\right)^2 + \frac{L_{\partial\Omega,p}(\xi_{T,p},\xi_{T,p})}{d(q,\partial\Omega)}\right)\cdot F_\Omega(q,\xi)^{-2} = 1,$$

where:

(1) *$d(q,\partial\Omega)$ is the distance between q and $\partial\Omega$,*

(2) $\xi_{N,p}$ *and* $\xi_{T,p}$ *denote the normal and the tangential components to* $\partial\Omega$ *at* p *of the vector* ξ, *understood by a parallel translation as a vector at* $p \in \partial\Omega$, *and*

(3) $L_{\partial\Omega,p}$ *represents the normalized Levi form of* $\partial\Omega$ *at* p.

Notice that this result analyzes the asymptotic boundary behavior of the Carathéodory and Kobayashi metric in all directions, without restricting the trajectory of the point q as it approaches the boundary $\partial\Omega$. Moreover, this shows that the Carathéodory and Kobayashi metrics are asymptotically Hermitian.

Sketch of the proof. Lee's proof proceeds following the scaling method. Let Ω and $p \in \partial\Omega$ be as in the hypothesis of theorem. Following the original work of Graham [34], one first observes that there exists an open neighborhood U of p for which one has

$$F_\Omega(q,\xi) \sim F_{\Omega\cap U}(q,\xi)$$

as q approaches p. Then, shrinking U if necessary, apply the scaling method to $\Omega \cap U$. With the dilatation sequence associated with q above, which is the centering map followed by a stretching map $\Lambda_j = L_j \circ A_j$, one sees that the following hold:

(1) With an appropriate fixed Cayley transform (a linear fractional transformation) Φ, the sequence of domains $\Phi\circ\Lambda_j(\Omega\cap U)$ converges normally to the open unit ball.

(2) $\Phi \circ \Lambda_j(q) = 0$ for every j.

Therefore, by the invariance and interior stability properties of the Kobayashi and Carathéodory metrics, one sees that

$$\begin{aligned} F_{\Omega\cap U}(q,\xi) &= F_{\Phi\circ\Lambda_j(\Omega\cap U)}(0, d[\Phi\circ\Lambda_j]_q(\xi)) \\ &\sim F_{\mathbb{B}}(0, d[\Phi\circ\Lambda_j]_q(\xi)). \end{aligned}$$

Now a careful analysis of the last term yields the desired conclusion. See [77] for the detailed analysis. □

One easily sees in this method that strong pseudoconvexity is not a restricting factor for this type of analysis. Indeed, Lee in the same paper showed how to analyze the boundary behavior of such metrics in a domain with an exponentially flat, infinite type boundary.

6.2. Boundary behavior of the Bergman invariants

The role of the dilatation sequence in the preceding section is two-fold:

(1) It turns the boundary limiting behavior problem into an interior stability problem.

(2) The normal convergence limit of the sequence of sets becomes generally simple.

Therefore study of boundary behavior of several holomorphic invariants can be handled through scaling as long as they are localizable and have interior stability.

Thus it is natural to reprove and refine the important theorem of Klembeck [61] on the boundary behavior of Bergman curvature in the strongly pseudoconvex domain by scaling.

The Bergman kernel, metric and curvatures have to be introduced. For a bounded domain Ω in $\mathbb{C}^{n+1}$, we consider the square integrable holomorphic functions

$$\mathcal{A}^2(\Omega) = \{f : \Omega \to \mathbb{C} : \text{ holomorphic}, \int_\Omega f d\mu < \infty\}$$

where $d\mu$ denotes the standard Lebesgue measure for $\mathbb{C}^{n+1}$. It is obviously a linear subspace of the Lebesgue space $L^2(\Omega)$. It follows by the Cauchy estimate that $\mathcal{A}^2(\Omega)$ is a Hilbert space. Moreover, it is a separable Hilbert space with respect to the standard L^2 inner product.

Let z be a point in Ω. Consider the point evaluation map

$$\Psi_z : \mathcal{A}^2(\Omega) \to \mathbb{C}$$

defined by $\Psi_z(f) = f(z)$. The Cauchy estimates imply that this is a bounded linear functional. Consequently, the Riesz representation theorem implies that there exists a holomorphic function $k_z \in \mathcal{A}^2(\Omega)$ such that

$$\Psi_z(f) = \int_\Omega f(\zeta)\overline{k_z(\zeta)}\, d\mu(\zeta)$$

for every $f \in \mathcal{A}^2(\Omega)$. Let $K_\Omega(z,\zeta) := \overline{k_z(\zeta)}$ for $(z,\zeta) \in \Omega \times \Omega$. It is known that the function satisfies the following properties:

(i) $K_\Omega(z,\zeta)$ is holomorphic in $z = (z_0, \ldots, z_n)$ and conjugate holomorphic in $\zeta = (\zeta_0, \ldots, \zeta_n)$.

(ii) $K_\Omega(z,\zeta) = \overline{K_\Omega(\zeta, z)}$.

(iii) $f(z) = \int_\Omega K(z,\zeta) f(\zeta)\, d\mu(\zeta)$ for every $f \in \mathcal{A}^2(\Omega)$.

The function K_Ω is the *Bergman kernel function* for Ω. Bergman showed that the bilinear form

$$\beta_p \equiv \sum_{j,k=0}^{n} \frac{\partial^2 \log K_\Omega(z,z)}{\partial z_j \partial \bar{z}_k}\Big|_p dz_j \otimes d\bar{z}_k$$

defines a positive definite Hermitian form on the tangent space at p for Ω. This is the Bergman metric. Following the formalism in differential geometry this metric admits various concepts of curvatures including the notion of holomorphic sectional curvature.

Now we are ready to present the main theorem of this subsection.

Theorem 6.5 (Klembeck, Kim/Yu). *Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with a boundary point $p \in \partial\Omega$. Denote by $S(q,\xi)$ the holomorphic sectional* *curvature of the Bergman metric of Ω at $q \in \Omega$ in the direction $\xi \in T_q\Omega$. If there exists an open neighborhood of p in $\mathbb{C}^{n+1}$ such that $\partial\Omega$ is $\mathcal{C}^2$ strongly pseudoconvex at every point in $\partial\Omega \cap U$, then*

$$\lim_{\Omega \ni q \to p} S(q, \xi) = -\frac{4}{n+2}.$$

Sketch of the proof. It seems appropriate to point out that the number $-4/(n+2)$ is actually the holomorphic sectional curvature (from here on, we shall simply say *holomorphic curvature*) of the Bergman metric at the origin of the unit ball B in $\mathbb{C}^{n+1}$. Therefore Klembeck's theorem simply says that the holomorphic curvature of a bounded domain is asymptotically the holomorphic curvature of the unit ball at the origin, as the reference point approaches a strongly pseudoconvex boundary point. Originally, Klembeck proved the above stated theorem with the stronger assumption that the domain has $\mathcal{C}^\infty$ strongly pseudoconvex boundary. He needed such a strong assumption because he used the celebrated asymptotic expansion formula of C. Fefferman for the Bergman kernel function. The improvement by Kim and Yu is in that they avoided using Fefferman's formula by exploiting the convenience of the scaling method for this type of problems, and that as a consequence they could prove the theorem with $\mathcal{C}^2$ smoothness, the optimal regularity assumption for strong pseudoconvexity.

The actual arguments by Kim and Yu proceed much along the line of scaling methods demonstrated above. (In fact, this type of arguments that uses the scaling methods to study asymptotic behavior of holomorphic invariants started with this problem in [51]. See [60] for further developments. Note also that Klembeck's result gives another way to prove Wong-Rosay theorem.)

With the notation in the theorem, we let q represent the general element of a sequence of interior points of Ω approaching p. We shall proceed much in the same way as in the scaling proof of Graham's theorem in Section 6.2.

First one localizes the problem. Namely, for the sequence of non-zero tangent vectors $\xi_q \in T_q\Omega$, one needs to establish that for any open neighborhood V of q, there exists an open set U with $p \in U \subset V$ such that the relation

$$S_\Omega(q; \xi_q) \sim S_{\Omega \cap U}(q; \xi_q)$$

holds as q approaches p. This was done in [60] in detail (see also [56]) using two classic results: the representation of the holomorphic curvature of the Bergman metric by the minimum integrals and the L^2 estimates of $\bar\partial$ operator by Hörmander. It should be mentioned that the argument by Kim-Yu does not use any regularity of the boundary but uses only the existence of holomorphic peak functions at p and the pseudoconvexity of the domain Ω in consideration.

Then the next step is to change the holomorphic coordinates, by a biholomorphism ψ of U onto an open ball $\widetilde{U}$ in $\mathbb{C}^n$ centered at the origin 0 with an appropriate radius, such that $\psi(p) = 0$ and

$$\psi(\Omega \cap U) = \{z \in \widetilde{U} : \operatorname{Re} z_0 > |z_0|^2 + \cdots + |z_n|^2 + o(|z_0|^2 + \cdots + |z_n|^2)\}.$$

Let $\widetilde{q} = \psi(q)$. Note that $\widetilde{q}$ approaches the origin 0 now. Then on $\psi(\Omega \cap U)$ with $\widetilde{q}$, let us build the centering map $A_{\widetilde{q}}$, the stretching map $L_{\widetilde{q}}$ and hence the scaling map $\Lambda_{\widetilde{q}} \equiv L_{\widetilde{q}} \circ A_{\widetilde{q}}$.

Now observe that the sequence of sets $\Lambda_{\widetilde{q}}(\psi(\Omega \cap U))$ converges normally to the Siegel half space $\mathcal{S}$ in $\mathbb{C}^n$ defined by

$$\operatorname{Re} z_0 > |z_1|^2 + \cdots + |z_n|^2$$

and $\Lambda_{\widetilde{q}}(\widetilde{q}) = (1, 0, \ldots, 0)$.

Let Φ be a standard Cayley linear fractional transformation that maps $\mathcal{S}$ bihomorphically onto the unit ball B such that $\Phi(1, 0, \ldots, 0) = (0, \ldots, 0)$.

Since the holomorphic curvature of the Bergman metric is a biholomorphism invariant, one immediately deduces that

$$\begin{aligned} S_\Omega(q, \xi_q) &\sim S_{\Omega \cap U}(q, \xi_q) \\ &= S_{\Phi \circ \Lambda_q(\Omega \cap U)}(\Phi \circ \Lambda_q(q), d(\Phi \circ \Lambda_q)|_q(\xi_q)) \\ &\sim S_B(0, d(\Phi \circ \Lambda_q)|_q(\xi_q)) \\ &= -\frac{4}{n+1}, \end{aligned}$$

which is the conclusion of the theorem. □

It should be obvious at this point that using the same method one can study the boundary behavior of the Bergman metric as well as the kernel itself in several cases including, but not limited to, the strongly pseudoconvex domains. The interested reader may consult the articles such as [10], [56] and the references therein.

6.3. Boundary asymptotics of the Poisson kernel

In the recent paper [67], Krantz uses a scaling method to derive results on the boundary asymptotics of the Poisson kernel on a bounded domain $\Omega \subseteq \mathbb{R}^{n+1}$ with C^2 boundary. A typical result is

Theorem 6.6 (Krantz). *Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a bounded domain with C^2 boundary. Let $P : \Omega \times \partial\Omega \to \mathbb{R}^+$ be the Poisson kernel for Ω. Let $\delta(x) = \delta_\Omega(x)$ denote the distance of x to $\partial\Omega$. Then there are constants $c_1, c_2 \geq 0$ such that*

$$(\star) \qquad c_1 \cdot \frac{\delta(x)}{|x-y|^{n+1}} \leq P(x, y) \leq c_2 \cdot \frac{\delta(x)}{|x-y|^{n+1}}.$$

It is worth mentioning that the scaling sequence here is isotropic, as the Laplace operator should be kept within the same conformal class.

7. Scaling in infinite dimensions

In recent years the complex function theory of infinite dimensions, particularly of Hilbert space, has received considerable attention. This new venue helps to put the classical finite-dimensional situation into new perspective, and offers many new challenges. We offer here one example of a result that can be proved by a scaling method, although it must be emphasized that many aspects of the classical arguments of Wong and Rosay as well as a direct application of scaling methods fail in this context, and many of them require new ideas:

Theorem 7.1 (Kim/Krantz, Byun/Gaussier/Kim, Kim/Ma). *Let Ω be a bounded domain in a separable Hilbert space $\mathcal{H}$. Assume that Ω admits a boundary point $\mathbf{p} \in \partial\Omega$ at which*

(1) *$\partial\Omega$ is C^2 smooth and strongly pseudoconvex in a neighborhood of $\mathbf{p}$, and*
(2) *there exist $\mathbf{q} \in \Omega$ and $f_j \in Aut(\Omega)$ $(j = 1, 2, \ldots)$ such that $f_j(\mathbf{q})$ converges to $\mathbf{p}$ in norm as $j \to \infty$.*

Then Ω is biholomorphic to the unit ball $\mathbb{B} = \{z \in \mathcal{H} : \|\mathbf{z}\| < 1\}$.

We do not include the detailed construction of the scaling sequence in infinite dimensions. The interested reader may consult the article by Kim, Krantz [54] and also Byun, Gaussier, and Kim [16], Kim, Ma [57].

8. Further results

It should be apparent by now that the scaling method is a powerful tool for studying the asymptotic behavior of holomorphic invariants as well as the non-compact orbits of a domain. At least it should be obvious that the scaling method is related to many other problems that have been open for some time and still need to be studied.

8.1. Linearization of holomorphic maps

Linearization is one of the traditional problems in function theory. The original problem is this: given a mapping from a subset (containing the origin) of a Euclidean space into another subset in the same space, can one find a new local coordinate system at the origin for the Euclidean space under consideration so that the original mapping becomes a linear map? Let us consider the case when the subset above is a domain, say Ω, the Euclidean space is the complex space $\mathbb{C}^{n+1}$ and the map $f : \Omega \to \mathbb{C}^{n+1}$ is a holomorphic mapping preserving the origin. The linearization problem now is asking whether there exists a local biholomorphic mapping near the origin preserving the origin, say h, such that $h \circ f \circ h^{-1}$ is the restriction of a complex linear mapping from $\mathbb{C}^{n+1}$ into itself.

As studied earlier by Lattès, Poincaré, Dulac and others (see [95] and the references therein), one may ask whether there is a linear mapping $L : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ such that the sequence $\psi_j \equiv L^{-j} \circ f^j$ converges uniformly on compact subsets of U to an injective holomorphic mapping, as j tends to infinity. Here, $f^{j+1} = f^j \circ f$ for each $j = 1, 2, \ldots$. Let f^0 denote the identity map. Let $f^{-k} = (f^{-1})^k$. *If the answer to this last question is affirmative*, let $\psi =$

$\lim_{j\to\infty} L^{-j} \circ f^j$. Then it follows that

$$\begin{aligned}\psi &= \lim_{j\to\infty} L^{-j} \circ f^j \\ &= L^{-1} \circ (\lim_{j\to\infty} L^{-j+1} \circ f^{j-1}) \circ f \\ &= L^{-1} \circ \psi \circ f.\end{aligned}$$

This shows that $\psi \circ f \circ \psi^{-1} = L$, and hence the linearization problem is solved.

Therefore a key question is this:

> Does there exist a linear map L such that the sequence $L^{-j} \circ f^j$ forms a normal family?

In lieu of a complete explication of this problem, we begin by pointing out that one cannot help but notice the strong resemblance of this problem to the scaling method. The conditions for the convergence of the sequence $L^{-j} \circ f^j$ have been studied extensively, from a time much earlier than the time of Pinchuk's initial studies of complex scaling method. For instance, in case the map f is a contraction, in the sense that each eigenvalue of the Jacobian matrix df_0 at the origin has modulus less than 1, then the obstruction to the convergence is known—it comes down to a resonance relation, that is in fact a collection of finitely many algebraic equations between the eigenvalues of df_0. A typical result is that if the eigenvalues of df_0 are free of resonance relations, then the sequence $L^{-j} \circ f^j$ form a normal family in a neighborhood of 0. Consequently, the map f is linearizable.

Focusing still on contractions, we consider the situation related to domains with non-compact automorphism group; this of course is one of the key subjects of the present article. Consider a real hypersurface M in $\mathbb{C}^{n+1}$ passing through the origin 0. Let us assume that there exists a holomorphic mapping f defined in a neighborhood U of 0 mapping U into $\mathbb{C}^{n+1}$ such that

(A) $f(0) = 0$,

(B) f is a contraction,

and

(C) f locally preserves M, i.e., $f(M \cap U) \subset M$.

Now we ask whether f can be linearized. As one readily observes, we are asking whether the condition (C) can replace the resonance-free condition.

In some sense, in case M is a real-analytic hypersurface that is strongly pseudoconvex and not locally biholomorphic to part of the sphere, then (C) does replace the resonance-free condition. (See [69], [59], [27], [95].) But then it is not hard to deduce the following statement:

Proposition 8.1. *Let M be a germ of real-analytic hypersurface in $\mathbb{C}^{n+1}$ passing through the origin. Assume also that M is strongly pseudoconvex. If there exists a local biholomorphism f of $\mathbb{C}^{n+1}$ defined in a neighborhood of 0 preserving the origin and mapping M to M as a contraction (meaning that every* *eigenvalue of df_0 has modulus strictly less than 1), then M is biholomorphic to a germ at 0 of the hypersurface defined by*

$$\operatorname{Re} z_0 = |z_1|^2 + \cdots + |z_n|^2.$$

The proof follows by the linearization. We give a rough sketch only. Expecting a contradiction, let us assume that M is not biholomorphic to a germ of the quadratic surface described above. Then f is linearizable, as mentioned above. Let $\operatorname{Re} z_0 = \rho(z_1, \ldots, z_n, \operatorname{Im} z_1)$ denote the defining relation of M. Let us assume, after a reduction, that f itself is linear. Now replace f by its Jordan canonical form. Roughly speaking, the components $(f_0, \ldots, f_n)$ of the map f will satisfy

$$\operatorname{Re} f_0 = \rho(\operatorname{Im} f_0, f_1, \ldots, f_n).$$

Notice that f is almost a diagonal linear map with the moduli of the eigenvalues all less than one. Give weights to the variables, so that the weight for z_0 is 2, and the weight of z_ℓ is 1 for $\ell = 1, \ldots, n$. Therefore an iteration of this process will imply that all the monomial terms in the Tayler expansion of ρ with degree higher than 2 must vanish. This is strongly analogous to the scaling method described earlier in this paper. But then the conclusion is that M has to be defined by a quadratic equation. Since any strongly pseudoconvex hypersurface defined by a quadratic equation with the prescribed weight has to be linearly biholomorphic to part of sphere, we have arrived at a contradiction. □

This line of investigation gives rise to a number of interesting questions. Notice in particular that the preceding proposition gives a short and simple proof to part of following theorem (see the discussion following for terminology):

Theorem 8.2 (R. Schoen [94]). *Let M be a C^∞ strongly pseudoconvex* CR *manifold of hypersurface type. If a point $p \in M$ admits a* CR *automorphism f of M such that $\lim_{j\to\infty} f^j(q) = p$ for every point q on M except possibly one point, then M is* CR *equivalent to the sphere or the sphere minus one point.*

The concept of abstract CR manifold, of hypersurface type or of more general type, requires a precise introduction. We refer to [11] for details. Nevertheless, it is not so difficult to picture what a CR manifold of hypersurface type should be. First consider a smooth real hypersurface M (of real dimension $2n+1$) in $\mathbb{C}^{n+1}$. For each $p \in M$, the real (extrinsic) tangent space T_pM contains complex n dimensional complex vector subspace in it. This gives rise to a subbundle $\mathcal{D}$ of $\mathcal{T}M$ with complex fibers, with real rank 1 transversal distribution. Implementing this type of bundle with some conditions called integrability in an abstract way formulates the concept of CR manifold. See [11]. Further, it is also possible to define the Levi form from this abstract setting, and hence the concept of strongly pseudoconvex CR manifold of hypersurface type. Due to the famous embedding theorems of strongly pseudoconvex CR manifolds of hypersurface type, except for the case of dimension 3 or 5, the abstract strongly pseudoconvex smooth CR manifolds of hypersurfaces type are locally equivalent to smooth strongly pseudoconvex CR hypersurfaces in a complex Euclidean space. (See [70], [1], [98], and [90].)

Continuing discussions from the above-stated theorem of Schoen, we feel that it is natural at this juncture to pose the following:

Problem 8.3. Let M be a $\mathcal{C}^\infty$ smooth, pseudoconvex CR hypersurface of finite type in $\mathbb{C}^{n+1}$ passing through the origin, not locally biholomorphic to a sphere. If there is a contracting local biholomorphic mapping f of $\mathbb{C}^{n+1}$ preserving the origin and the surface M, then show that f is linearizable.

8.2. CR hypersurfaces with special CR automorphisms

We shall continue with the discussion above. Focusing more on CR hypersurfaces, it may be appropriate to point out that non-degeneracy of the Levi form (which is often called Levi non-degeneracy) is a more natural concept than strong pseudoconvexity, at least for the CR hypersurfaces. Then the representative model for such Levi non-degenerate hypersurfaces should be hyperquadrics. On the other hand, it turns out that the existence of a contracting holomorphic map that preserves the CR hypersurface is a condition not in general restrictive enough to conclude that the CR hypersurface with a contracting holomorphic automorphism must be biholomorphic to a hyperquadric. A correct condition has been found by Kim and Schmalz in [59].

Call a local biholomorphic map f at 0 preserving a CR hypersurface germ M at 0 in $\mathbb{C}^{n+1}$ a CR *hyperbolic automorphism* of M if df_0 is expanding along the normal direction to M while df_0 is contracting along a complex tangential direction. Then one has the following result.

Theorem 8.4 (Kim and Schmalz). *Let $(M, 0)$ denote a germ of a real-analytic Levi non-degenerate hypersurface at 0 in $\mathbb{C}^{n+1}$ $(n \geq 2)$. If M admits a* CR *hyperbolic automorphism, then M is biholomorphic to a germ of hyperquadric in $\mathbb{C}^{n+1}$.*

Again, it is expected that the $\mathcal{C}^\infty$ version (or even $\mathcal{C}^2$) of this theorem should be true, but the more general result remains open at this time.

8.3. Existence of a one-parameter family of automorphisms

Another interesting problem (communicated to us by Wu-yi Hsiang) that is closely related to the subject of this article is as follows:

Question 8.5. Let Ω be a bounded domain in $\mathbb{C}^{n+1}$ with $\mathcal{C}^\infty$ boundary. If there exists a sequence φ_ν of automorphisms of Ω and a point $q \in \Omega$ such that the orbit $\varphi_\nu(q)$ accumulates at a boundary point p of Ω, then does the holomorphic automorphism group $\operatorname{Aut} \Omega$ contain a noncompact one-parameter subgroup?

Recall that the scaling method (when it applies) usually finds that the given domain is biholomorphic to a domain represented by an inequality of type

$$\operatorname{Re} z_0 > \psi(z_1, \ldots, z_n).$$

Hence there is a 1-parameter translation automorphism subgroup along the $\operatorname{Im} z_0$ direction. So the answer to the above question is positive in several cases listed below in which the scaling sequence always converges to a nice domain having such a translation and/or dilation:

- Ω is convex and Kobayashi hyperbolic (but the boundary need not be smooth (see [30], [50]));
- Ω is a subdomain of $\mathbb{C}^2$ and bounded, or if $\Omega \subset \mathbb{C}^2$ is defined by a pluri-subharmonic defining function, say ρ, with $\int_\Omega dd' \log \rho = +\infty$ (see [8]).

Since the convergence of scaling in the general case is quite difficult, it seems reasonable to appeal to a different viewpoint. In particular, we present a recent result that has exploited the circle of ideas surrounding the linearization problem described above.

Theorem 8.6 (K. T. Kim and S. Y. Kim). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^{n+1}$ with a real-analytic boundary, and with no non-trivial complex analytic variety. If there exists $f \in \operatorname{Aut}(\overline{\Omega})$ such that $f(p) = p$ for some $p \in \partial\Omega$ and such that f is a contraction at p, then $\dim_{\mathbb{R}} \operatorname{Aut} \Omega \geq 2$.*

Before sketching the proof, we point out that the recent result by K. Diederich and S. Pinchuk on the reflection principle ([24]) implies that every holomorphic automorphism of such Ω extends holomorphically across its boundary (see also [45]). Hence the assumption that f should belong to $\operatorname{Aut}(\overline{\Omega})$ is not so restrictive in this case.

Now, we sketch the proof. The argument is computational, and is rather intricate and involved. However, the nub of the proof is that f can be linearized to a diagonal matrix unless the boundary $\partial\Omega$ near p is biholomorphic to a CR hypersurface defined by a weighted homogeneous polynomial function. Then it is shown in [52] that the linearizability again implies that the hypersurface has to be defined by a weighted homogeneous polynomial. Altogether, one can conclude from this, using normal families argument, that the domain Ω is biholomorphic to a domain in $\mathbb{C}^{n+1}$ defined by a weighted homogeneous polynomial. But this latter domain admits two non-compact 1-parameter families of automorphisms: dilation and translation. Thus the conclusion of the theorem is obtained. □

8.4. New family of homogeneous spaces with non integrable almost complex structures

Since the seminal paper by M. Gromov [41], the study of almost complex structures has not only been revived, but it has emerged as one of the central objects of study, in particular in symplectic geometry and the related researches. It is generally believed (and said openly) that one should study the pseudo-holomorphic curves (rather than pseudo-holomorphic functions) i.e., the J-holomorphic mappings from the unit open disc in $\mathbb{C}$ with the standard almost complex structure, say J_{st}, into a manifold with an almost complex structure J, because the pseudo-holomorphic mappings (i.e., (J, J')-holomorphic mappings from an almost complex manifold (M, J) into another such (M', J') *rarely* exist. On the other hand, it has turned out that in real dimension 6 or higher, K. H. Lee in his Ph. D. thesis ([73]–[76]) discovered that there are infinitely many mutually inequivalent almost complex manifolds that have the following two distinctive features:

(1) their pseudo-holomorphic automorphisms act transitively on the manifold,

(2) and yet, their almost complex structures are not integrable.

The purpose of this subsection is to introduce the recent results obtained by K. H. Lee briefly, and then suggest several problems that should be studied, in lieu of other results discovered recently.

We start by introducing K. H. Lee's theorem. For this we introduce the following examples:

Let $t \in \mathbb{R}$. For $\mathbb{C}^3$, use the coordinate presentation $z_j = x_j + iy_j$ for $j = 1, 2, 3$ and for $i = \sqrt{-1}$. Define the real 6×6 matrix

$$J_t = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & tx^2 \\ 1 & 0 & 0 & 0 & tx_2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let

$$\Omega_t = \{z \in \mathbb{C}^3 \mid x_1 + t(|z_2|^2 + |z_3|^2) < 0\}.$$

Then one has

Theorem 8.7 (K. H. Lee [75]). *Let $0 \leq t < 8$. For the one parameter family (Ω_t, J_t) of almost complex manifolds, the following properties hold:*

(i) *Each almost complex manifold (Ω_t, J_t) is homogeneous, in the sense that the group $\mathrm{Aut}_J(\Omega_t)$ of J-holomorphic diffeomorphisms of Ω_t onto itself acts transitively on Ω_t.*

(ii) *For every $t \neq 0$, J_t is non-integrable.*

(iii) *(Ω_t, J_t) and (Ω_s, J_s) are biholomorphic (i.e., equivalent via a (J_t, J_s)-holomorphic diffeomorphism) for $s, t \in [0, 8)$ if, and only if, $s = t$.*

(iv) *Each (Ω_t, J_t) is Kobayashi hyperbolic.*

(v) *Any Kobayashi hyperbolic domain in an almost complex manifold of real dimension 6 with an automorphism group orbit accumulating at a strongly pseudoconvex boundary point is equivalent to (Ω_t, J_t) for some $t \in [0, 8)$.*

Furthermore, Lee shows in his papers [73] (See also [75] and [76]) that $\mathrm{Aut}_J(\Omega_t)$ is a real Lie group of real dimension 9 for every $t \in (0,8)$, whereas the automorphism group of (Ω_0, J_0) is of dimension 15 (because it is biholomorphic to the standard ball). We must point out also that Lee has described all the J_t-automorphisms in his papers.

For the interested reader, we remark that Lee expanded his analysis and classified such models in all dimensions.

It is appropriate to provide some explication on how Lee constructs such a family of examples. Lee's starting point is a surprising discovery by H. Gaussier and A. Sukhov [32], [33]. Let $n \geq 2$ and let Ω be a domain in $\mathbb{R}^{2n}$ with a C^2 smooth boundary with C^∞ smooth almost complex structure J. Assume that (Ω, J) admits a sequence of J-automorphisms, an interior point $p \in \Omega$ and a boundary point $q \in \partial\Omega$ at which $\partial\Omega$ is strongly pseudoconvex admitting a sequence of J-holomorphic automorphisms φ_j of Ω such that $\{\varphi_j(q)\}$ accumulates at p (despite that this assumption may possibly be a priori empty).

Then one applies Pinchuk's scaling method here. But, one has to be careful because the Pinchuk scaling sequence, say $\Lambda_\nu : (\Omega, J) \to (\mathbb{C}^3, J_{\mathrm{st}})$, is not (J, J_{st})-holomorphic in general. Therefore, one must equip the image $\Lambda_\nu(\Omega)$ with the almost complex structure $J_\nu \equiv (\Lambda_\nu)_* J$. Then Gaussier and Sukhov (also Lee) show that the sequence of domains $\Lambda_\nu(\Omega)$ converges as ν tends to infinity, as well as the sequence J_ν of almost complex structures converge to a limit almost complex structure, say $\widehat{J}$.

Then, in real dimension 4 (i.e., the case $n = 2$), Gaussier and Sukhov showed, through an intricate analysis, that the limit domain equipped with the limit almost complex structure is equivalent to the unit ball in $\mathbb{C}^2$ equipped with the standard structure. But then, to a surprise to many experts, they indicated that, in case of dimension 6 or more, the limit domain may still have an almost complex structure that is not even integrable.

Lee in his Ph. D. dissertation analyzed this situation carefully further, and first came up with the complete list of possible limit domains with limit almost complex structures. It is indeed surprising that those limit domains possess the structures described in the above mentioned theorem, thus introducing a new line of examples that are worthy of further investigations.

We refer the interested readers to Lee's papers [73]–[75] for further information on these new families of almost complex manifolds that are homogeneous, Kobayashi hyperbolic and with non-vanishing Nijenhuis tensors. On the other hand, we would like to pose a few interesting problems for further study in this direction.

Problem 8.8. For a compact almost complex manifold with a symplectic structure (M, J, ω) the Bergman kernel has been constructed by X. Ma and G. Marinescu (See [79].) Can one construct the Bergman kernel for Lee's domains introduced above imitating Ma-Marisnescu method?

Problem 8.9. For homogeneous domains with the standard complex structure, there is a Jordan algebra method of constructing the Bergman kernel. Can one find an analogous method for Bergman kernel constructed via an analogue of the Jordan triple system method? What would such Bergman kernel(s) explain for these domains?

9. Concluding remarks

The study of automorphism groups is an instance of Felix Klein's *Erlangen program.* It provides an algebraic/geometric invariant for distinguishing and comparing domains in a complex space or a complex manifold. It is proving to be a powerful tool in many aspects of geometric analysis and function theory. Certainly the scaling method, which is an outgrowth of the theory of automorphism groups, is finding use in subjects ranging from partial differential equations to differential geometry to complex variables. We hope that this exposition will spur further interest in this circle of ideas.

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