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COMMON FIXED POINT THEOREMS FOR CONTRACTIVE TYPE MAPPINGS AND THEIR APPLICATIONS IN DYNAMIC PROGRAMMING

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Abstract. A few sufficient conditions for the existence and uniqueness of fixed point and common fixed point for certain contractive type mappings in complete metric spaces are provided. Several existence and uniqueness results of solution and common solution for some functional equations and system of functional equations in dynamic programming are discussed by using the fixed point and common fixed point theorems presented in this paper.

1. Introduction and Preliminaries

Bellman [2] first studied the existence of solutions for some classes of functional equations arising in dynamic programming. Bellman and Lee [3] pointed out that the basic form of the functional equations in dynamic programming is as follows:

$$f(x) = \operatorname*{opt}_{y\in D} H(x, y, f(T(x, y))), \quad \forall x \in S, \tag{1.1}$$

where opt represents sup or inf, x and y denote the state and decision vectors, respectively, T stands for the transformation of the process, and $f(x)$ represents the optimal return function with the initial state x. Afterwards, Baskaran and Subrahmanyam [1], Bhakta and Choudhury [4], Bhakta and Mitra [5], Chang and Ma [6], Liu [8]-[10], Liu, Agarwal, and Kang [11], Liu and Ume [12], Pathak and Fisher [13], Zhang [15] and others investigated the existence and uniqueness of solution and common solution for some kinds of functional equations and systems of functional equations, which include the functional equation (1.1) as a special case, arising in dynamic programming under several suitable assumptions.

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Ray [14] established two common fixed point theorems for the following self mappings f, g and h in a complete metric space (X, d):

$$d(fx, gy) \le d(hx, hy) - w(d(hx, hy)), \quad \forall x, y \in X. \tag{1.2}$$

Liu [7] introduced and studied a class of contractive type mappings below:

$$\begin{aligned} d(fx, gy) \le{} & \max\{d(hx, hy), d(hx, fx), d(hy, gy)\} \\ & - w(\max\{d(hx, hy), d(hx, fx), d(hy, gy)\}), \quad \forall x, y \in X, \end{aligned} \tag{1.3}$$

and established common fixed point theorems for the class of mappings in a complete metric space (X, d).

The main aim of this paper is to give several sufficient conditions which guarantee the existence and uniqueness of common fixed point for the following contractive type mappings in a complete metric space (X, d):

$$\begin{aligned} & d(fx, gy) \\ \le{} & \max\Big\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)], \\ & \qquad \frac{d(hx, fx)d(hy, gy)}{1 + d(fx, gy)}, \frac{d(hx, fx)d(hy, gy)}{1 + d(hx, hy)}\Big\} \\ & - w\Big(\max\Big\{d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)], \\ & \qquad \frac{d(hx, fx)d(hy, gy)}{1 + d(fx, gy)}, \frac{d(hx, fx)d(hy, gy)}{1 + d(hx, hy)}\Big\}\Big), \quad \forall x, y \in X. \end{aligned} \tag{1.4}$$

As applications we use the fixed point and common fixed point theorems presented in this paper to discuss the existence and uniqueness problems of solution and common solution for the following functional equation (1.5) and system of functional equations (1.6), respectively, arising in dynamic programming:

$$f(x) = \operatorname*{opt}_{y \in D} \{u(x, y) + H(x, y, f(T(x, y)))\}, \quad \forall x \in S \tag{1.5}$$

and

$$f_i(x) = \operatorname*{opt}_{y \in D} \{u(x, y) + H_i(x, y, f_i(T(x, y)))\}, \quad \forall x \in S,\ i \in \{1, 2, 3\}. \tag{1.6}$$

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, ω and $\mathbb{N}$ denote the sets of all nonnegative and positive integers, respectively, and

$$W = \{w \mid w : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is a continuous mapping with } 0 < w(t) < t \text{ for all } t > 0\}.$$

For a self mapping f in a metric space (X, d), define

$$C_f(X) = \{g \mid g : X \to X \text{ is continuous and } gf = fg\}.$$

Let I denote the identity mapping in X.

2. Common fixed point theorems for contractive type mappings

In this section, we prove several fixed point and common fixed point theorems for some classes of contractive type mappings in a complete metric space (X, d). For self mappings f, g and h in (X, d) and $x_0 \in X$, put $d_n = d(hx_n, hx_{n+1})$ for all $n \in \omega$. Our main results are as follows:

Theorem 2.1. *Let (X, d) be a complete metric space and f, g and h be three self mappings in X with $h \in C_f(X) \cap C_g(X)$ and $f(X) \cup g(X) \subseteq h(X)$. If there exists a $w \in W$ satisfying (1.4), then f, g and h have a unique common fixed point in X.*

Proof. Let x_0 be any point in X. According to $f(X) \cup g(X) \subseteq h(X)$, we choose a sequence $\{x_n\}_{n \in \omega} \in X$ such that $fx_{2n} = hx_{2n+1}$ and $gx_{2n+1} = hx_{2n+2}$ for any $n \in \omega$. By (1.4) we deduce that

$$\begin{aligned}
&d(fx_{2n}, gx_{2n+1})\\
&\le \max\Big\{d(hx_{2n}, hx_{2n+1}), d(hx_{2n}, fx_{2n}), d(hx_{2n+1}, gx_{2n+1}),\\
&\qquad \frac{1}{2}[d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})],\\
&\qquad \frac{d(hx_{2n}, fx_{2n}) d(hx_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})},\\
&\qquad \frac{d(hx_{2n}, fx_{2n}) d(hx_{2n+1}, gx_{2n+1})}{1 + d(hx_{2n}, hx_{2n+1})}\Big\}\\
&\quad - w\Big(\max\Big\{d(hx_{2n}, hx_{2n+1}), d(hx_{2n}, fx_{2n}), d(hx_{2n+1}, gx_{2n+1}),\\
&\qquad \frac{1}{2}[d(hx_{2n}, hx_{2n+1}) + d(fx_{2n}, gx_{2n+1})],\\
&\qquad \frac{d(hx_{2n}, fx_{2n}) d(hx_{2n+1}, gx_{2n+1})}{1 + d(fx_{2n}, gx_{2n+1})},\\
&\qquad \frac{d(hx_{2n}, fx_{2n}) d(hx_{2n+1}, gx_{2n+1})}{1 + d(hx_{2n}, hx_{2n+1})}\Big\}\Big), \quad \forall n \in \omega,
\end{aligned}$$

which means that

$$\begin{aligned}
&d_{2n+1}\\
&\le \max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2}[d_{2n} + d_{2n+1}], \frac{d_{2n} d_{2n+1}}{1 + d_{2n+1}}, \frac{d_{2n} d_{2n+1}}{1 + d_{2n}}\Big\}\\
&\quad - w\Big(\max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2}[d_{2n} + d_{2n+1}], \frac{d_{2n} d_{2n+1}}{1 + d_{2n+1}}, \frac{d_{2n} d_{2n+1}}{1 + d_{2n}}\Big\}\Big)\\
&= \max\{d_{2n}, d_{2n+1}\} - w(\max\{d_{2n}, d_{2n+1}\}), \quad \forall n \in \omega.
\end{aligned} \tag{2.1}$$

Suppose that $d_{2n+1} > d_{2n}$ for some $n \in \omega$. In view of (2.1) it is easy to verify that $d_{2n+1} \le d_{2n+1} - w(d_{2n+1}) < d_{2n+1}$, a contradiction. Consequently, we infer that $d_{2n+1} \le d_{2n}$ and so $d_{2n+1} \le d_{2n} - w(d_{2n})$ for any $n \in \omega$ by (2.1). In a similar manner, it can be shown that $d_{2n} \leq d_{2n-1} - w(d_{2n-1})$ for all $n \in \mathbb{N}$. It follows that

$$d_n \leq d_{n-1} - w(d_{n-1}), \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Next, we prove that

$$\lim_{n\to\infty} d_n = 0. \tag{2.3}$$

Note that (2.2) means that

$$\sum_{i=0}^{n} w(d_i) \leq d_0 - d_{n+1} \leq d_0$$

for all $n \in \omega$ and $\{d_n\}_{n\in\omega}$ is a decreasing sequence. Whereas the series $\sum_{n=0}^{\infty} w(d_n)$ and the sequence $\{d_n\}_{n\in\omega}$ are convergent. It is clear that $\lim_{n\to\infty} w(d_n) = 0$ and there exists some point $p \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d_n = p$. In terms of the continuity of w, we derive that $\lim_{n\to\infty} w(d_n) = w(p) = 0$. This means that $p = 0$, that is, (2.3) holds.

In order to show that $\{hx_n\}_{n\in\omega}$ is a Cauchy sequence, we need only to prove that $\{hx_{2n}\}_{n\in\omega}$ is a Cauchy sequence. Suppose that $\{hx_{2n}\}_{n\in\omega}$ is not a Cauchy sequence. Thus there exists some $\epsilon > 0$ such that, for any even integer $2k$, there are even integers $2m(k)$ and $2n(k)$ with $2m(k) > 2n(k) > 2k$ and $d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon$. Further, let $2m(k)$ denote the least even integer exceeding $2n(k)$ which satisfies that $2m(k) > 2n(k) > 2k$,

$$d(hx_{2m(k)-2}, hx_{2n(k)}) \leq \epsilon \quad \text{and} \quad d(hx_{2m(k)}, hx_{2n(k)}) > \epsilon. \tag{2.4}$$

Notice that for any $k \in \mathbb{N}$

$$d(hx_{2m(k)}, hx_{2n(k)}) \leq d_{2m(k)-1} + d_{2m(k)-2} + d(hx_{2m(k)-2}, hx_{2n(k)}),$$

$$\left|d(hx_{2m(k)}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)})\right| \leq d_{2n(k)},$$

$$\left|d(hx_{2m(k)+1}, hx_{2n(k)+1}) - d(hx_{2m(k)}, hx_{2n(k)+1})\right| \leq d_{2m(k)},$$

$$\left|d(hx_{2m(k)+1}, hx_{2n(k)+2}) - d(hx_{2m(k)+1}, hx_{2n(k)+1})\right| \leq d_{2n(k)+1}.$$

Following (2.3), (2.4) and the above inequalities, we infer that

$$\begin{aligned} \epsilon &= \lim_{k\to\infty} d(hx_{2m(k)}, hx_{2n(k)}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+1}) \\ &= \lim_{k\to\infty} d(hx_{2m(k)+1}, hx_{2n(k)+2}). \end{aligned} \tag{2.5}$$

Using (1.4) again, we have, $\forall k \in \mathbb{N}$,

$$\begin{aligned}
&d(fx_{2m(k)}, gx_{2n(k)+1}) \\
&\le \max\Big\{d(hx_{2m(k)}, hx_{2n(k)+1}), d_{2m(k)}, d_{2n(k)+1}, \\
&\qquad \frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1})], \\
&\qquad \frac{d_{2m(k)}d_{2n(k)+1}}{1+d(fx_{2m(k)}, gx_{2n(k)+1})}, \frac{d_{2m(k)}d_{2n(k)+1}}{1+d(hx_{2m(k)}, hx_{2n(k)+1})}\Big\} \\
&\quad - w\Big(\max\Big\{d(hx_{2m(k)}, hx_{2n(k)+1}), d_{2m(k)}, d_{2n(k)+1}, \\
&\qquad \frac{1}{2}[d(hx_{2m(k)}, hx_{2n(k)+1}) + d(fx_{2m(k)}, gx_{2n(k)+1})], \\
&\qquad \frac{d_{2m(k)}d_{2n(k)+1}}{1+d(fx_{2m(k)}, gx_{2n(k)+1})}, \frac{d_{2m(k)}d_{2n(k)+1}}{1+d(hx_{2m(k)}, hx_{2n(k)+1})}\Big\}\Big).
\end{aligned}$$

Letting $k \to \infty$, by (2.5) we deduce that

$$\begin{aligned}
\epsilon &\le \max\{\epsilon, 0, 0, \epsilon, 0, 0\} - w(\max\{\epsilon, 0, 0, \epsilon, 0, 0\}) \\
&= \epsilon - w(\epsilon) < \epsilon,
\end{aligned}$$

which is absurd, and hence $\{hx_n\}_{n\in\omega}$ is a Cauchy sequence. It follows from completeness of (X, d) that $\{hx_n\}_{n\in\omega}$ converges to a point $u \in X$. Since $h \in C_f(X) \cap C_g(X)$, we infer that

$$\begin{aligned}
hu &= \lim_{n\to\infty} fhx_{2n} = \lim_{n\to\infty} hfx_{2n} = \lim_{n\to\infty} hhx_{2n+1} \\
&= \lim_{n\to\infty} ghx_{2n+1} = \lim_{n\to\infty} hgx_{2n+1} = \lim_{n\to\infty} hhx_{2n+2}.
\end{aligned} \tag{2.6}$$

By virtue of (1.4) we get that

$$\begin{aligned}
&d(fu, ghx_{2n+1}) \\
&\le \max\Big\{d(hu, hhx_{2n+1}), d(hu, fu), d(hhx_{2n+1}, ghx_{2n+1}), \\
&\qquad \frac{1}{2}[d(hu, hhx_{2n+1}) + d(fu, ghx_{2n+1})], \\
&\qquad \frac{d(hu, fu)d(hhx_{2n+1}, ghx_{2n+1})}{1+d(fu, ghx_{2n+1})}, \frac{d(hu, fu)d(hhx_{2n+1}, ghx_{2n+1})}{1+d(hu, hhx_{2n+1})}\Big\} \\
&\quad - w\Big(\max\Big\{d(hu, hhx_{2n+1}), d(hu, fu), d(hhx_{2n+1}, ghx_{2n+1}), \\
&\qquad \frac{1}{2}[d(hu, hhx_{2n+1}) + d(fu, ghx_{2n+1})], \\
&\qquad \frac{d(hu, fu)d(hhx_{2n+1}, ghx_{2n+1})}{1+d(fu, ghx_{2n+1})}, \frac{d(hu, fu)d(hhx_{2n+1}, ghx_{2n+1})}{1+d(hu, hhx_{2n+1})}\Big\}\Big).
\end{aligned}$$

As $n \to \infty$ in the above inequality, it follows from (2.6) that

$$\begin{aligned} d(fu, hu) &\le \max\Big\{0, d(hu, fu), 0, \frac{1}{2}d(fu, hu), 0, 0\Big\} \\ &\quad - w\Big(\max\Big\{0, d(hu, fu), 0, \frac{1}{2}d(fu, hu), 0, 0\Big\}\Big) \\ &= d(hu, fu) - w(d(hu, fu)), \end{aligned}$$

this gives that $fu = hu$. Similarly we obtain that $hu = gu$. In light of (1.4) we conclude that

$$\begin{aligned} & d(fhx_{2n}, gx_{2n+1}) \\ \le\ & \max\Big\{d(hhx_{2n}, hx_{2n+1}), d(hhx_{2n}, fhx_{2n}), d(hx_{2n+1}, gx_{2n+1}), \\ & \qquad \frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})], \\ & \qquad \frac{d(hhx_{2n}, fhx_{2n})d(hx_{2n+1}, gx_{2n+1})}{1 + d(fhx_{2n}, gx_{2n+1})}, \\ & \qquad \frac{d(hhx_{2n}, fhx_{2n})d(hx_{2n+1}, gx_{2n+1})}{1 + d(hhx_{2n}, hx_{2n+1})}\Big\} \\ & - w\Big(\max\Big\{d(hhx_{2n}, hx_{2n+1}), d(hhx_{2n}, fhx_{2n}), d(hx_{2n+1}, gx_{2n+1}), \\ & \qquad \frac{1}{2}[d(hhx_{2n}, hx_{2n+1}) + d(fhx_{2n}, gx_{2n+1})], \\ & \qquad \frac{d(hhx_{2n}, fhx_{2n})d(hx_{2n+1}, gx_{2n+1})}{1 + d(fhx_{2n}, gx_{2n+1})}, \\ & \qquad \frac{d(hhx_{2n}, fhx_{2n})d(hx_{2n+1}, gx_{2n+1})}{1 + d(hhx_{2n}, hx_{2n+1})}\Big\}\Big), \quad \forall n \in \omega. \end{aligned}$$

Taking the limit as $n \to \infty$, from (2.6) we conclude that

$$\begin{aligned} d(hu, u) &\le \max\Big\{d(hu, u), d(hu, hu), d(u, u), d(hu, u), \\ &\qquad \frac{d(hu, hu)d(u, u)}{1 + d(hu, u)}, \frac{d(hu, hu)d(u, u)}{1 + d(hu, u)}\Big\} \\ &\quad - w\Big(\max\Big\{d(hu, u), d(hu, hu), d(u, u), d(hu, u), \\ &\qquad \frac{d(hu, hu)d(u, u)}{1 + d(hu, u)}, \frac{d(hu, hu)d(u, u)}{1 + d(hu, u)}\Big\}\Big) \\ &= d(hu, u) - w(d(hu, u)), \end{aligned}$$

which ensures that $hu = u$. Thus, u is a common fixed point of f, g and h.

If $v \in X \setminus \{u\}$ is another common fixed point of f, g and h, from (1.4) we immediately infer that

$$d(u, v) = d(fu, gv) \le d(u, v) - w(d(u, v)),$$

which implies that $u = v$. Hence f, g and h have a unique common fixed point $u \in X$. This completes the proof. □

As in the proof of Theorem 2.1, we have

Theorem 2.2. *Let (X, d) be a complete metric space and f, g and h be three self mappings in X with $h \in C_f(X) \cap C_g(X)$ and $f(X) \cup g(X) \subseteq h(X)$. If there exists a $w \in W$ satisfying*

$$\begin{aligned} d(fx, gy) \le \max\Big\{&d(hx, hy), d(hx, fx), d(hy, gy), \frac{1}{2}[d(hx, hy) + d(fx, gy)]\Big\}, \\ &- w\Big(\max\Big\{d(hx, hy), d(hx, fx), d(hy, gy), \\ &\qquad \frac{1}{2}[d(hx, hy) + d(fx, gy)]\Big\}\Big), \quad \forall x, y \in X, \end{aligned}$$

then f, g and h have a unique common fixed point in X.

Taking $h = I$ in Theorem 2.1, we obtain the following:

Theorem 2.3. *Let f and g be two self mappings from a complete metric space (X, d) into itself. If there exists a $w \in W$ satisfying*

$$\begin{aligned} d(fx, gy) \le \max\Big\{&d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)], \\ &\frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)}, \frac{d(x, fx)d(y, gy)}{1 + d(x, y)}\Big\} \\ &- w\Big(\max\Big\{d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, y) + d(fx, gy)], \\ &\qquad \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)}, \frac{d(x, fx)d(y, gy)}{1 + d(x, y)}\Big\}\Big), \quad \forall x, y \in X, \end{aligned}$$

then f and g have a unique common fixed point in X.

In case $f = g$ in Theorem 2.1, we gain the following:

Theorem 2.4. *Let f and h be two self mappings from a complete metric space (X, d) into itself with $h \in C_f(X)$ and $f(X) \subseteq h(X)$. If there exists a $w \in W$ satisfying*

$$\begin{aligned} &d(fx, fy) \\ &\le \max\Big\{d(hx, hy), d(hx, fx), d(hy, fy), \frac{1}{2}[d(hx, hy) + d(fx, fy)], \\ &\qquad \frac{d(hx, fx)d(hy, fy)}{1 + d(fx, fy)}, \frac{d(hx, fx)d(hy, fy)}{1 + d(hx, hy)}\Big\} \\ &\quad - w\Big(\max\Big\{d(hx, hy), d(hx, fx), d(hy, fy), \frac{1}{2}[d(hx, hy) + d(fx, fy)], \\ &\qquad \frac{d(hx, fx)d(hy, fy)}{1 + d(fx, fy)}, \frac{d(hx, fx)d(hy, fy)}{1 + d(hx, hy)}\Big\}\Big), \quad \forall x, y \in X, \end{aligned}$$

then f and h have a unique common fixed point in X.

Letting $h = I$ in Theorem 2.4, we get the following:

Theorem 2.5. *Let f be a mapping from a complete metric space (X, d) into itself. If there exists a $w \in W$ satisfying*

$$\begin{aligned} d(fx, fy) \le \max\Big\{ & d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,y)+d(fx,fy)], \\ & \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(x,y)} \Big\} \\ - w\Big(\max\Big\{ & d(x,y), d(x,fx), d(y,fy), \frac{1}{2}[d(x,y)+d(fx,fy)], \\ & \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}, \frac{d(x,fx)d(y,fy)}{1+d(x,y)} \Big\}\Big), \quad \forall x, y \in X, \end{aligned}$$

then f has a unique fixed point in X.

3. Existence and uniqueness of common solution for systems of functional equations

Throughout this section, let X and Y be Banach spaces, $S \subseteq X$ be the state space and $D \subseteq Y$ be the decision space. $B(S)$ denotes the set of all real-valued bounded functions on S. Put

$$d(a,b) = \sup_{x \in S} \big|a(x) - b(x)\big|, \quad \forall a, b \in B(S).$$

It is obvious that $(B(S), d)$ is a complete metric space. Define $u : S \times D \to \mathbb{R}$, $T : S \times D \to S$ and $H_i : S \times D \times \mathbb{R} \to \mathbb{R}$ for $i \in \{1, 2, 3\}$.

Now we study those conditions, which guarantee the existence and uniqueness of solution and common solution for the functional equation (1.5) and the system of functional equations (1.6), respectively.

Theorem 3.1. *If the following conditions are satisfied:*

$(C1)$ u and H_i are bounded for $i \in \{1, 2, 3\}$;

$(C2)$ there exists a $w \in W$ satisfying

$$\begin{aligned} & \big|H_1(x,y,a(t)) - H_2(x,y,b(t))\big| \\ \le\ & \max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)], \\ & \qquad \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1+d(f_1a, f_2b)}, \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1+d(f_3a, f_3b)} \Big\} \\ & - w\Big(\max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)], \\ & \qquad \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1+d(f_1a, f_2b)}, \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1+d(f_3a, f_3b)} \Big\}\Big) \end{aligned}$$

for all $(x, y) \in S \times D$, $a, b \in B(S)$ *and* $t \in S$, *where the mappings* f_1, f_2 *and* f_3 *are defined as follows:* $\forall x \in S$, $a_i \in B(S)$, $i \in \{1, 2, 3\}$,

$$f_i a_i(x) = \operatorname*{opt}_{y \in D} \{u(x, y) + H_i(x, y, a_i(T(x, y)))\}; \tag{3.1}$$

$(C3)$ $f_1(B(S)) \cup f_2(B(S)) \subseteq f_3(B(S))$ *and* $f_3 \in C_{f_1}(B(S)) \cap C_{f_2}(B(S))$, *then the system of functional equations* (1.6) *possesses a unique common solution in* $B(S)$.

Proof. It follows from $(C1)$ and $(C2)$ that f_1, f_2 and f_3 are self mapping in $B(S)$. Let $a, b \in B(S)$ and $x \in S$. We now have to consider two possible cases:

Case 1. Suppose that $\operatorname{opt}_{y \in D} = \sup_{y \in D}$. For any $\epsilon > 0$, there exist $y, z \in D$ satisfying

$$\begin{aligned} f_1 a(x) &< u(x, y) + H_1(x, y, a(T(x, y))) + \epsilon, \\ f_2 b(x) &< u(x, z) + H_2(x, z, b(T(x, z))) + \epsilon, \\ f_1 a(x) &\geq u(x, z) + H_1(x, z, a(T(x, z))), \\ f_2 b(x) &\geq u(x, y) + H_2(x, y, b(T(x, y))). \end{aligned} \tag{3.2}$$

Combining (3.2) and $(C2)$, we arrive at

$$\begin{aligned} &\left|f_1 a(x) - f_2 b(x)\right| \\ &< \epsilon + \max\{\left|H_1(x, y, a(T(x, y))) - H_2(x, y, b(T(x, y)))\right|, \\ &\qquad \left|H_1(x, z, a(T(x, z))) - H_2(x, z, b(T(x, z)))\right|\} \\ &\leq \epsilon + \max\Big\{d(f_3 a, f_3 b), d(f_3 a, f_1 a), d(f_3 b, f_2 b), \frac{1}{2}[d(f_3 a, f_3 b) + d(f_1 a, f_2 b)], \\ &\qquad \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_1 a, f_2 b)}, \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_3 a, f_3 b)}\Big\} \\ &\quad - w\Big(\max\Big\{d(f_3 a, f_3 b), d(f_3 a, f_1 a), d(f_3 b, f_2 b), \frac{1}{2}[d(f_3 a, f_3 b) + d(f_1 a, f_2 b)] \\ &\qquad \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_1 a, f_2 b)}, \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_3 a, f_3 b)}\Big\}\Big), \end{aligned}$$

which yields that

$$\begin{aligned} &d(f_1 a, f_2 b) \\ &\leq \epsilon + \max\Big\{d(f_3 a, f_3 b), d(f_3 a, f_1 a), d(f_3 b, f_2 b), \\ &\qquad \frac{1}{2}[d(f_3 a, f_3 b) + d(f_1 a, f_2 b)], \\ &\qquad \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_1 a, f_2 b)}, \frac{d(f_3 a, f_1 a) d(f_3 b, f_2 b)}{1 + d(f_3 a, f_3 b)}\Big\} \end{aligned} \tag{3.3}$$

$$- w\Big(\max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)] \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_1a, f_2b)}, \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_3a, f_3b)} \Big\}\Big).$$

Case 2. Suppose that $\mathrm{opt}_{y\in D} = \inf_{y\in D}$. By using a method similar to the proof of Case 1, we infer that (3.3) holds also.

Letting ϵ tend to zero in (3.3), we gain easily that

$$\begin{aligned} & d(f_1a, f_2b) \\ \le & \max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)], \\ & \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_1a, f_2b)}, \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_3a, f_3b)} \Big\} \\ & - w\Big(\max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)], \\ & \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_1a, f_2b)}, \frac{d(f_3a, f_1a)d(f_3b, f_2b)}{1 + d(f_3a, f_3b)} \Big\}\Big). \end{aligned}$$

Therefore, Theorem 2.1 ensures that f_1, f_2 and f_3 have a unique common fixed point $v \in B(S)$. That is, the system of functional equations (1.6) possesses a unique common solution $v \in B(S)$. This completes the proof. □

It follows from Theorem 2.2 and Theorem 3.1 that

Theorem 3.2. *Assume that* $(C1)$, $(C3)$ *and the following condition are satisfied:*

$(C4)$ *there exists a* $w \in W$ *satisfying*

$$\begin{aligned} & \big| H_1(x, y, a(t)) - H_2(x, y, b(t)) \big| \\ \le & \max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)] \Big\} \\ & - w\Big(\max\Big\{ d(f_3a, f_3b), d(f_3a, f_1a), d(f_3b, f_2b), \frac{1}{2}[d(f_3a, f_3b) + d(f_1a, f_2b)] \Big\}\Big) \end{aligned}$$

for all $(x, y) \in S \times D$, $a, b \in B(S)$ *and* $t \in S$, *where the mappings* f_1, f_2 *and* f_3 *are defined by* (3.1). *Then the system of functional equations* (1.6) *possesses a unique common solution in* $B(S)$.

In case $f_3 = I$ in Theorem 3.1, we conclude that

Theorem 3.3. *Suppose that the following conditions hold:*

$(C5)$ u *and* H_i *are bounded for* $i \in \{1, 2\}$;

$(C6)$ *there exists a* $w \in W$ *satisfying*

$$\begin{aligned}
&\big|H_1(x,y,a(t)) - H_2(x,y,b(t))\big| \\
&\le \max\Big\{d(a,b), d(a,f_1a), d(b,f_2b), \frac{1}{2}[d(a,b)+d(f_1a,f_2b)], \\
&\qquad \frac{d(a,f_1a)d(b,f_2b)}{1+d(f_1a,f_2b)}, \frac{d(a,f_1a)d(b,f_2b)}{1+d(a,b)}\Big\} \\
&\quad - w\Big(\max\Big\{d(a,b), d(a,f_1a), d(b,f_2b), \frac{1}{2}[d(a,b)+d(f_1a,f_2b)], \\
&\qquad \frac{d(a,f_1a)d(b,f_2b)}{1+d(f_1a,f_2b)}, \frac{d(a,f_1a)d(b,f_2b)}{1+d(a,b)}\Big\}\Big)
\end{aligned}$$

for all $(x,y) \in S \times D$, $a,b \in B(S)$ *and* $t \in S$, *where the mappings* f_1 *and* f_2 *are defined as follows:*

$$f_ia_i(x) = \operatorname*{opt}_{y\in D}\big\{u(x,y)+H_i(x,y,a_i(T(x,y)))\big\}, \quad \forall x \in S,\ a_i \in B(S),\ i \in \{1,2\}.$$

Then the system of functional equations

$$f_i(x) = \operatorname*{opt}_{y\in D}\big\{u(x,y)+H_i(x,y,f_i(T(x,y)))\big\}, \quad \forall x \in S,\ i \in \{1,2\}$$

possesses a unique common solution in $B(S)$.

Taking $f_1 = f_2$ in Theorem 3.3, we get the following:

Theorem 3.4. *Suppose that* $(C5)$ *and the following condition hold:*
$(C7)$ *there exists a* $w \in W$ *satisfying*

$$\begin{aligned}
&\big|H_1(x,y,a(t)) - H_2(x,y,b(t))\big| \\
&\le \max\Big\{d(a,b), d(a,fa), d(b,fb), \frac{1}{2}[d(a,b)+d(fa,fb)], \\
&\qquad \frac{d(a,fa)d(b,fb)}{1+d(fa,fb)}, \frac{d(a,fa)d(b,fb)}{1+d(a,b)}\Big\} \\
&\quad - w\Big(\max\Big\{d(a,b), d(a,fa), d(b,fb), \frac{1}{2}[d(a,b)+d(fa,fb)], \\
&\qquad \frac{d(a,fa)d(b,fb)}{1+d(fa,fb)}, \frac{d(a,fa)d(b,fb)}{1+d(a,b)}\Big\}\Big)
\end{aligned}$$

for all $(x,y) \in S \times D$, $a,b \in B(S)$ *and* $t \in S$, *where the mapping* f *is defined as follows:*

$$fa(x) = \operatorname*{opt}_{y\in D}\big\{u(x,y)+H(x,y,a(T(x,y)))\big\}, \quad \forall x \in S,\ a \in B(S).$$

Then the functional equation (1.5) *possesses a unique solution in* $B(S)$.

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