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NULL BERTRAND CURVES IN A LORENTZ MANIFOLD

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ABSTRACT. The purpose of this paper is to study the geometry of null Bertrand curves in a Lorentz manifold.

1. INTRODUCTION

J. Bertrand studied a pair of curves in a 3-dimensional Euclidean space which posses common principal normal direction. Such a curve is now called a Bertrand curve. Bertrand curves are characterized as follows:

Theorem A ([6]). *A curve C in a 3-dimensional Euclidean space, parameterized by the arc length, is a Bertrand curve if and only if C is a plane curve or curves whose curvature κ and torsion τ are in linear relation: $a\kappa + b\tau = 1$ for some constants a and b. The product of torsion of Bertrand pair is constant.*

Extending above result to null curves in 3-dimensional Minkowski space $\mathbf{R}_1^3$, Honda-Inoguchi [10] and Inoguchi-Lee [13] have done some work on a pair of null curves $(C, \bar{C})$, called a null Bertrand pair and their relation with null helices in R_1^3. They have the following result.

Theorem B ([13]). *Let $C(p)$ be a null Cartan curve in $\mathbf{R}_1^3$, where p is a special distinguished parameter. Then C admits a Bertrand mate $\bar{C}$ if and only if C and C have same nonzero constant curvatures. Moreover, $\bar{C}$ is congruent to C.*

Recently, Cöken and Ciftci [3] have followed the 3-dimensional notion of Bertrand curves and proved a theorem for null helices in 4-dimensional Minkowski space $\mathbf{R}_1^4$.

Theorem C ([3]). *A null Cartan curve in $\mathbf{R}_1^4$ is a null Bertrand curve if and only if τ_1 is non-zero and τ_2 is zero.*

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The purpose of this paper is to study null Bertrand curves in a Lorentz manifold. We draw a conclusion which characterizes null Bertrand curves by properties of the second and third curvatures of the general null curves which contain the null Cartan curves as special case: $\kappa_1 = 1$, $\kappa_2 = \tau_1$ and $\kappa_3 = \tau_2$.

2. Frenet and Cartan Equations

Let (M, g) be a real $(m+2)$-dimensional Lorentz manifold and C a smooth null curve in M locally given by

$$x^i = x^i(t), \quad t \in I \subset \mathbf{R}, \quad i \in \{0, 1, \ldots, (m+1)\}$$

for a coordinate neighborhood $\mathcal{U}$ on C. Then the tangent vector field $\xi = C'$ on $\mathcal{U}$ satisfies

$$g(\xi, \xi) = 0.$$

Denote by TC the tangent bundle of C and $TC^\perp$ the TC perpendicular. Clearly, $TC^\perp$ is a vector bundle over C of rank $(m+1)$. Since ξ is null, the tangent bundle TC of C is a vector subbundle of $TC^\perp$, of rank 1. This implies that $TC^\perp$ is not complementary to TC in $TM_{|C}$. Thus we must find complementary vector bundle to TC in TM which will play the role of the normal bundle $TC^\perp$ consistent with the classical non-degenerate theory.

Suppose $S(TC^\perp)$ denotes the complementary vector subbundle to TC in $TC^\perp$, i.e., we have

$$TC^\perp = TC \perp S(TC^\perp)$$

where $\perp$ means the orthogonal direct sum. It follows that $S(TC^\perp)$ is a non-degenerate vector subbundle of TM, of rank m. We call $S(TC^\perp)$ a *screen vector bundle* of C, which being non-degenerate, we have

$$TM|_c = S(TC^\perp) \perp S(TC^\perp)^\perp, \tag{1}$$

where $S(TC^\perp)^\perp$ is a complementary orthogonal vector subbundle to $S(TC^\perp)$ in $TM|_C$ of rank 2.

We denote by $F(C)$ the algebra of smooth functions on C and by $\Gamma(E)$ the $F(C)$ module of smooth sections of a vector bundle E over C. We use the same notation for any other vector bundle.

Theorem 1 ([4], [5]). *Let C be a null curve of a Lorentz manifold (M, g) and $S(TC^\perp)$ be a screen vector bundle of C. Then there exists a unique vector bundle*

$ntr(C)$ over C, of rank 1, such that on each coordinate neighborhood $\mathcal{U} \subset C$ there is a unique section $N \in \Gamma(ntr(C)|_{\mathcal{U}})$ satisfying

$$g(\xi, N) = 1, \quad g(N, N) = g(N, X) = 0, \quad {}^{\forall}X \in \Gamma(S(TC^{\perp})|_{\mathcal{U}}). \tag{2}$$

We call the vector bundle $ntr(C)$ the *null transversal bundle* of C with respect to $S(TC^{\perp})$. Next consider the vector bundle

$$tr(C) = ntr(C) \perp S(TC^{\perp}),$$

which according to (1) and (2) is complementary but not orthogonal to TC in $TM|_C$. More precisely, we have

$$TM|_C = TC \oplus tr(C) = (TC \oplus ntr(C)) \perp S(TC^{\perp}). \tag{3}$$

We call $tr(C)$ the *transversal vector bundle* of C with respect to $S(TC^{\perp})$. The vector field N in Theorem 1 is called the *null transversal vector field* of C with respect to ξ. As $\{\xi, N\}$ is a null basis of $\Gamma((TC \oplus ntr(C))|_{\mathcal{U}})$ satisfying (2), any screen vector bundle $S(TC^{\perp})$ of C is Riemannian.

Let $C = C(p)$ be a smooth null curve, parametrized by the distinguished parameter p ([4]), such that $||C''|| = \kappa_1 \neq 0$. Denote by ∇ the Levi-Civita connection on M. Using (2) and (3) and taking into account that the screen vector bundle $S(TC^{\perp})$ is Riemannian of rank m, we obtain the following Frenet equations ([15])

$$\begin{aligned}
\nabla_\xi \xi &= \kappa_1 W_1, \\
\nabla_\xi N &= \kappa_2 W_1 + \kappa_3 W_2, \\
\nabla_\xi W_1 &= -\kappa_2 \xi - \kappa_1 N, \\
\nabla_\xi W_2 &= -\kappa_3 \xi + \kappa_4 W_3, \\
\nabla_\xi W_3 &= -\kappa_4 W_2 + \kappa_5 W_4, \\
&\cdots\cdots\cdots\cdots \\
\nabla_\xi W_i &= -\kappa_{i+1} W_{i-1} + \kappa_{i+2} W_{i+1}, \quad i \in \{3, \ldots, m-1\}, \\
\nabla_\xi W_m &= -\kappa_{m+1} W_{m-1},
\end{aligned} \tag{4}$$

where $\{\kappa_1, \ldots, \kappa_{m+1}\}$ are smooth functions on $\mathcal{U}$ and $\{W_1, \ldots, W_m\}$ is a certain orthonormal basis of $\Gamma(S(TC^{\perp})|_{\mathcal{U}})$. In general, for any $m > 0$, we call $F = \{\xi, N, W_1, \cdots, W_m\}$ a *natural Frenet frame* on M along C with respect to the screen vector bundle $S(TC^{\perp})$ and the equations (4) are called its *natural Frenet equations* of C. Finally, the functions $\{\kappa_1, \ldots, \kappa_{m+1}\}$ are called *curvature functions* of C with respect to the Frenet frame F.

Note. According to Duggal and Bejancu [4] and Jin [14], a null curve with respect to a distinguished parameter p is called a *null geodesic* if $\kappa_1 = 0$.

Let $C = C(p)$ be a smooth null curve in a Lorentz manifold (M, g), parametrized by a special distinguished parameter p such that $||C''|| = 1$. Also we obtain the following *Cartan equations* due to [5]:

$$
\begin{aligned}
\nabla_\xi \xi &= W_1, \\
\nabla_\xi N &= \tau_1 W_1 + \tau_2 W_2, \\
\nabla_\xi W_1 &= -\tau_1 \xi - N, \\
\nabla_\xi W_2 &= -\tau_2 \xi + \tau_3 W_3, \\
\nabla_\xi W_3 &= -\tau_3 W_2 + \tau_4 W_4, \\
&\cdots\cdots\cdots\cdots \\
\nabla_\xi W_i &= -\tau_i W_{i-1} + \tau_{i+1} W_{i+1}, \quad i \in \{3, \ldots, m-1\}, \\
\nabla_\xi W_m &= -\tau_m W_{m-1}.
\end{aligned}
\tag{5}
$$

We call the frame $F = \{\xi, N, W_1, \cdots, W_m\}$ of the equations (5), its curvature functions and the corresponding curve C the *Cartan frame* on M along C, the *Cartan curvatures* and the *null Cartan curve* respectively ([2], [5]).

3. Null Bertrand Curves

In this section we investigate the properties of the null Bertrand curve C in a Lorentz manifold. Using the usual terminology, the spacelike unit vector field $W_1 = C''/\kappa_1$ will be called *principal normal* vector field of C.

Definition. A pair of null curves $(C, \bar{C})$ in a Lorentz manifold (M, g) is called *null Bertrand pair* if the principal normal directions of C and $\bar{C}$ coincide. We say that $\bar{C}$ is a *null Bertrand mate* for C and vice versa. A null curve C is said to be a *null Bertrand curve* if it admits a null Bertrand mate.

Let $(C, \bar{C})$ be a null Bertrand pair parametrized by their distinguished parameters p and $\bar{p}$ respectively, then $\bar{C}$ is parametrized as

$$\bar{C}(\bar{p}(p)) = C(p) + f(p)\, W_1(p) \tag{6}$$

for some function $f(p) \neq 0$. Without any loss of generality, we assume that

$$\bar{W}_1(\bar{p}(p)) = -W_1(p). \tag{7}$$

Then, using (6), we obtain

$$\frac{d\bar{p}}{dp}\bar{\xi} = (1 - f\kappa_2)\xi - f\kappa_1 N + f' W_1. \tag{8}$$

Taking the scalar product of (8) with W_1 and using (7), we obtain $f' = 0$ and

$$\frac{d\bar{p}}{dp}\bar{\xi} = (1 - f\kappa_2)\xi - f\kappa_1 N. \tag{9}$$

Also, taking the scalar product of both sides in the last equation, we have

$$\kappa_1(1 - f\kappa_2) = 0. \tag{10}$$

Differentiating (9) with respect to p and using (4) and (10), we get

$$\frac{d^2\bar{p}}{dp^2}\bar{\xi} + \bar{\kappa}_1\left(\frac{d\bar{p}}{dp}\right)^2 \bar{W}_1 = -f\kappa_2'\xi - f\kappa_1' N - f\kappa_1\kappa_2 W_1 - f\kappa_1\kappa_3 W_2. \tag{11}$$

Taking the scalar product of (9) and (11) with ξ, we obtain

$$\frac{d\bar{p}}{dp} g(\bar{\xi}, \xi) = -f\kappa_1, \qquad \frac{d^2\bar{p}}{dp^2} g(\bar{\xi}, \xi) = -f\kappa_1', \tag{12}$$

respectively. From the equations (12), we have

$$\frac{d\bar{p}}{dp} = c\,\kappa_1\,; \quad \text{where } c \text{ is a non-zero constant.} \tag{13}$$

Also, by duality, we have

$$\frac{dp}{d\bar{p}} = d\,\bar{\kappa}_1\,; \quad \text{where } d \text{ is a non-zero constant.} \tag{14}$$

Thus we have $\bar{\kappa}_1\,\kappa_1 = \frac{1}{cd}$ = constant. Using (13) in (9) we get

$$\bar{\xi} = \frac{1}{c}\frac{1 - f\kappa_2}{\kappa_1}\xi - \frac{f}{c}N. \tag{15}$$

Differentiating (15) with respect to p and using the Frenet equation (4), we have

$$\frac{d\bar{p}}{dp}\bar{\kappa}_1 \bar{W}_1 = \frac{1}{c}\frac{d}{dp}\left(\frac{1 - f\kappa_2}{\kappa_1}\right)\xi + \frac{1}{c}(1 - 2f\kappa_2)\,W_1 - \frac{f}{c}\kappa_3\,W_2. \tag{16}$$

Taking the scalar product of (16) with W_2, N and $\bar{W}_1$ and using (7), we obtain

$$\kappa_3 = 0, \qquad \frac{1 - f\kappa_2}{\kappa_1} = b, \qquad \frac{d\bar{p}}{dp}\bar{\kappa}_1 = \frac{1}{c}(2f\kappa_2 - 1) \tag{17}$$

respectively, where b is a constant. If $b \neq 0$, then $1 - f\kappa_2 = b\kappa_1$. Thus, from (10), we have $\kappa_1 = 0$. It is contraction to $\kappa_1 \neq 0$. This implies $b = 0$. Consequently, we have $1 - f\kappa_2 = 0$, this implies $\kappa_2 = 1/f$ = non-zero constant. Using this fact, the third equation of (17) reduces $\bar{\kappa}_1 \frac{d\bar{p}}{dp} = \frac{1}{c}$. From this and (14), we have $c = d$. Thus $\bar{\kappa}_1\,\kappa_1$ = positive constant.

Conversely, assume that C is a null curve such that $\kappa_3 = 0$ and κ_2 = non-zero constant and the product of the first curvatures satisfies $\bar{\kappa}_1 \kappa_1 = \frac{1}{a^2}$, where a is a non-zero constant, then define a new curve $\bar{C}$ by

$$\bar{C}(\bar{p}(p)) = C(p) + \frac{1}{\kappa_2} W_1(p). \tag{18}$$

Differentiating (18) with respect to p and using the Frenet equations (4), we get

$$\frac{d\bar{p}}{dp} \bar{\xi} = -\frac{\kappa_1}{\kappa_2} N. \tag{19}$$

From (19), since $\kappa_1 \neq 0$, we have $\frac{d\bar{p}}{dp} \neq 0$. Thus $\bar{\xi} = \rho N$, where $\rho = -\frac{\kappa_1}{\kappa_2}\frac{dp}{d\bar{p}} \neq 0$ and $< \bar{\xi}, \bar{\xi} >= \rho^2 < N, N >= 0$, that is, $\bar{C}$ is also null curve. Differentiating (19) with respect to p and using the Frenet equation (4) with $\kappa_3 = 0$, we get

$$\frac{d^2\bar{p}}{dp^2} \bar{\xi} + \left(\frac{d\bar{p}}{dp}\right)^2 \bar{\kappa}_1 \bar{W}_1 = -\frac{\kappa_1'}{\kappa_2} N - \kappa_1 W_1. \tag{20}$$

Taking the norm of both sides in (20), we have $\left(\frac{d\bar{p}}{dp}\right)^2 \bar{\kappa}_1 = \pm \kappa_1$. Since $\kappa_1 \neq 0$, we have $\bar{\kappa}_1 \neq 0$ and $\frac{1}{\bar{\kappa}_1} = a^2 \kappa_1$. Thus $\frac{d\bar{p}}{dp} = \pm a \kappa_1$ and $\rho = \mp \frac{1}{a \kappa_2}$ = non-zero constant. Thus, differentiating $\bar{\xi} = \rho N$ with respect to p and using (4) with $\kappa_3 = 0$, we get

$$\frac{d\bar{p}}{dp} \bar{\kappa}_1 \bar{W}_1 = \rho \kappa_2 W_1. \tag{21}$$

Consequently, the null curve $\bar{C}$ is a Bertrand mate of C. Thus we have

Theorem 2. *A non-geodesic null curve in a Lorentz manifold is a null Bertrand curve if and only if κ_2 is a non-zero constant and $\kappa_3 = 0$. The product of the first curvatures of Bertrand pair is positive constant.*

The null Cartan curve is a special case of null curve such that $\kappa_1 = 1$ and $\kappa_{i+1} = \tau_i$ for $i\,(1 \leq i \leq m)$ in (4). Thus we have

Theorem 3. *A null Cartan curve in a Lorentz manifold is a null Bertrand curve if and only if τ_1 is a non-zero constant and $\tau_2 = 0$.*

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