J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 3 (August 2008), Pages 217–220

# NONPARAMETRIC MINIMAL SURFACE AND HARMONIC MAPPING

Sook Heui Jun

Abstract. In this paper, we investigate the harmonic mappings that arise in connection with Scherk's surface and helicoid.

## 1. Introduction

Let $\mathbb{D}$ be a domain in $\mathbb{C}$. A continuous function $f = u + iv$ defined in $\mathbb{D}$ is harmonic if $u$ and $v$ are real harmonic in $\mathbb{D}$. In any simply connected subdomain of $\mathbb{D}$ we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic and $\overline{g}$ denotes the function $z \longmapsto \overline{g(z)}$. A result of Lewy [2] shows that the harmonic mapping $f = h + \overline{g}$ is locally one-to-one and orientation-preserving if and only if $|g'(z)| < |h'(z)|$. We call such mappings locally univalent, and we say $f$ is univalent in $\mathbb{D}$ if $f$ is one-to-one and orientation-preserving in $\mathbb{D}$.

Let $G(z)$ be a meromorphic function in the unit disk $D = \{z : |z| < 1\}$ and $F(z)$ an analytic function in $D$ having the property that it vanishes only at the poles of $G$, and the order of its zero at such a point is exactly twice the order of the pole of $G$. Then $S$ is a regular minimal surface if and only if $S$ admits an isothermal parametric representation of the form $S = \{(u(z), v(z), \phi(z)) : z \in D\}$, where

$$\begin{aligned} u &= \frac{1}{2} Re\left\{\int_0^z F(1 - G^2)\, dz\right\}, \\ v &= -\frac{1}{2} Im\left\{\int_0^z F(1 + G^2)\, dz\right\}, \\ \phi &= Re\left\{\int_0^z FG\, dz\right\}. \end{aligned} \tag{1.1}$$

Received by the editors August 20, 2007. Accepted July 27, 2008.
2000 *Mathematics Subject Classification.* 30C45, 53A10, 30C50.
*Key words and phrases.* harmonic mapping, minimal surface.
This work was supported by a special research grant from Seoul Women's University (2008).

In addition, the coordinate functions of such representation are harmonic since $S$ is a minimal surface. Therefore, the projection of such representation onto the $uv$-plane defines a harmonic mapping $f(z) = u + iv$ on $D$ [3].

In this paper, we first find the nonparametric minimal surfaces which are constructed by choices of $F(z)$ and $G(z)$. These surfaces are helicoids produced by $F(z) = \frac{2z^2}{(1-z^2)^2}$ and $G(z) = \pm\frac{i}{z}$, and Scherk's surfaces produced by $F(z) = \frac{2z^2}{z^4-1}$ and $G(z) = \pm\frac{1}{z}$. And then we investigate properties of the harmonic mappings

$$f_1(z) = \frac{z}{2(1-z^2)} + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \overline{\left(\frac{z}{2(1-z^2)} - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)\right)}$$

and

$$f_2(z) = -\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \overline{\left(-\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)\right)}$$

that arise in connection with these minimal surfaces which are constructed by $F(z)$ and $G(z)$.

## 2. Properties of the Harmonic Mapping

The representations of the form (1.1) were first given by Enneper and Weierstrass, and have played a major role in the theory of minimal surfaces. One obvious example is to take $F(z) = 1$ and $G(z) = z$ which lead to the surface known as Enneper's surface.

Now let's consider the functions $F(z) = \frac{2z^2}{(1-z^2)^2}$, $G(z) = \pm\frac{i}{z}$ which satisfy the conditions for giving the representation (1.1). Corresponding regular minimal surfaces $S_1 = \{(u(z), v(z), \phi(z)) : z \in D\}$ are given by

$$(2.1)\qquad \begin{aligned} u &= Re\left\{\int_0^z \frac{z^2+1}{(1-z^2)^2}\,dz\right\} = Re\left\{\frac{z}{1-z^2}\right\},\\ v &= Im\left\{\int_0^z \frac{1}{1-z^2}\,dz\right\} = Im\left\{\frac{1}{2}\log\left(\frac{1+z}{1-z}\right)\right\},\\ \phi &= Re\left\{\int_0^z \frac{\pm 2iz}{(1-z^2)^2}\,dz\right\} = \pm Im\left\{\frac{z^2}{1-z^2}\right\}.\end{aligned}$$

The associated harmonic mapping is

$$(2.2)\quad f_1(z) = u + iv = \frac{z}{2(1-z^2)} + \frac{1}{4}log\left(\frac{1+z}{1-z}\right) + \overline{\left(\frac{z}{2(1-z^2)} - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)\right)}.$$

Let $\frac{1+z}{1-z} = Re^{i\theta}$. Then $R > 0$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, and $z = \frac{Re^{i\theta}-1}{Re^{i\theta}+1}$ because $\frac{1+z}{1-z}$ is Möbius transformation from $D$ onto the right half plane. Apply $z = \frac{Re^{i\theta}-1}{Re^{i\theta}+1}$ into (2.1). Then we get the followings;

$$u = \frac{1}{4}\left(R - \frac{1}{R}\right)\cos\theta,\ v = \frac{\theta}{2},\ \phi = \pm\frac{1}{4}\left(R - \frac{1}{R}\right)\sin\theta.$$

It is evident that $u$ varies from $-\infty$ to $\infty$ on each horizontal line $v =$ constant, and therefore, the minimal surfaces $S_1$ lie over all of $\Omega_1 = \{w = u + iv : |v| < \pi/4\}$. The nonparametric forms of these minimal surfaces $S_1$ are given by the nonparametric equations

$$\phi(u, v) = \pm u\ \tan(2v)$$

on $\Omega_1$. This tells us that our minimal surfaces $S_1$ are helicoids. Note that the associated harmonic mapping $f_1$ in (2.2) also maps the unit disk $D$ onto the strip-domain $\Omega_1$.

Similarly, the choices $F(z) = \frac{2z^2}{z^4-1}$, $G(z) = \pm\frac{1}{z}$ produce the regular minimal surfaces $S_2 = \{(u(z), v(z), \phi(z)) : z \in D\}$ which are given by

(2.3)

$$u = \frac{1}{2}Im\left\{\log\left(\frac{1+iz}{1-iz}\right)\right\},\ v = \frac{1}{2}Im\left\{\log\left(\frac{1+z}{1-z}\right)\right\},\ \phi = \pm\frac{1}{2}Re\left\{\log\left(\frac{1+z^2}{1-z^2}\right)\right\}.$$

The nonparametric forms of these minimal surfaces $S_2$, called Scherk's surfaces, are given by the equations

$$\phi(u, v) = \pm\frac{1}{2}\log\left(\frac{\cos(2v)}{\cos(2u)}\right)$$

on $\Omega_2 = \{w = u + iv : |u| < \pi/4,\ |v| < \pi/4\}$, and the associated harmonic mapping

$$f_2(z) = u + iv = -\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) + \frac{1}{4}\log\left(\frac{1+z}{1-z}\right) + \overline{\left(-\frac{i}{4}\log\left(\frac{1+iz}{1-iz}\right) - \frac{1}{4}\log\left(\frac{1+z}{1-z}\right)\right)} \tag{2.4}$$

maps the unit disk $D$ onto the square-domain $\Omega_2$.

In the following theorem, we will show that these harmonic mappings $f_1$ and $f_2$ are univalent in $D$.

**Theorem 1.** *The harmonic mappings $f_k = h_k + \overline{g}_k$ in* (2.2) *and* (2.4) *are univalent.*

*Proof.* Since the Jacobian of $f_k$, $J(f_k) = |h_k'(z)|^2 - |g_k'(z)|^2$, is positive in $D$, $f_k$ is locally one-to-one and orientation preserving, that is locally univalent. The analytic mapping $W = h_k(z) - g_k(z) = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ defined in $D$ is obviously one-to-one, and conformal since $h_k'(z) - g_k'(z) = \frac{1}{1-z^2} \neq 0$ in $D$. In addition, the conformal univalent mapping $W = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ maps $D$ onto the strip-domain $\Omega_1$. Let $z = z(W)$ be an inverse mapping of $W = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$. Then

$$f_k(z(W)) = W + 2Re\{g_k(z(W))\} = W + \psi_k(W)$$

is locally one-to-one. If $f_k(z_1) = f_k(z_2)$ with $z_1 \neq z_2$, then writing $z_1 = z(W_1)$, $z_2 = z(W_2)$ we have $W_1 + \psi_k(W_1) = W_2 + \psi_k(W_2)$ with $W_1 = u_1 + iv_1 \neq u_2 + iv_2 = W_2$. This implies that $v_1 = v_2 = v_0$ and $u_1 + \psi_k(u_1 + iv_0) = u_2 + \psi_k(u_2 + iv_0)$. The continuous real-valued function $H_k(u) = u + \psi_k(u + iv_0)$, which is defined on $(-\infty, \infty)$ since $W = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$ maps $D$ onto the domain $\Omega_1$, is not strictly monotonic and therefore not locally one-to-one. Thus $W + \psi_k(W) = f_k(z(W))$ is not locally one-to-one. Therefore $f_k$ is one-to-one, i.e., univalent. This completes the proof of the theorem. □

## REFERENCES

1. Lars. V. Ahlfors: *Complex Analysis.* McGraw-Hill, 1966.
2. H. Lewy: On the non-vanishing of the Jacobian in certain one-to-one mappings. *Bull. Amer. Math. Soc.* **42** (1936), 689-692.
3. R. Osserman: *A survey of minimal surfaces.* Dover, 1986.
4. M. Spivak: *A Comprehensive Introduction to Differential Geometry.* Perish, 1979.

DEPARTMENT OF MATHEMATICS, SEOUL WOMEN'S UNIVERSITY, SEOUL 139-774, KOREA
*Email address*: `shjun@swu.ac.kr`