J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 3 (August 2008), Pages 221–227

RESULTS ON STRONG GENERALIZED NEIGHBORHOOD SPACES

WON KEUN MIN

ABSTRACT. We introduce and study the new concepts of interior and closure operators on strong generalized neighborhood spaces. Also we introduce and investigate the concept of sgn-continuity on SGNS.

1. INTRODUCTION

Császár introduced the notions of generalized neighborhood systems and generalized topological spaces [1]. The author introduced the strong generalized neighborhood systems [3] which is a generalization of neighborhood systems on a nonempty set.

The strong generalized neighborhood system induces a strong generalized neighborhood space(briefly SGNS) which implies a generalized neighborhood space. In this paper, we introduce the new concepts of interior and closure operators on SGNS's and we characterize properties of the operators by using the convergence of m-families on SGNS's. Also we introduce the concept of sgn-continuity on SGNS's and we investigate characterizations for the sgn-continuity by using the new interior and closure operators defined on SGNS's.

2. PRELIMINARIES

Let X be a nonempty set and $\psi : X \to \exp(\exp(X))$ satisfy $x \in V$ for $V \in \psi(x)$. Then $V \in \psi(x)$ is called a *generalized neighborhood* of $x \in X$ and ψ is called a *generalized neighborhood system* [1] on X.

Received by the editors September 3, 2007. Revised April 14, 2008. Accepted July 27, 2008.
2000 *Mathematics Subject Classification.* 54A10, 54A20, 54D10, 54D30.
Key words and phrases. strong generalized neighborhood space, sg-open set, sgn-continuous, sgn-open function, m-family.
This work was supported by a grant from Research Institute for Basic Science at Kangwon National University.

Definition 2.1 ([3]). Let $\psi : X \to \exp(\exp(X))$. Then ψ is called a *strong generalized neighborhood system* on X if it satisfies the following:

(1) $x \in V$ for $V \in \psi(x)$;

(2) for $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then the pair (X, ψ) is called a *strong generalized neighborhood space* (briefly SGNS) on X. Then $V \in \psi(x)$ is called a *strong generalized neighborhood* of $x \in X$.

Definition 2.2 ([3]). Let (X, ψ) be an SGNS on X and $A \subseteq X$. Then the interior and closure of A on ψ (denoted by $\iota_\psi(A)$, $\gamma_\psi(A)$, respectively) are defined as following:

$\iota_\psi(A) = \{x \in A : \text{there exists } V \in \psi(x) \text{ such that } V \subseteq A\}$;

$\gamma_\psi(A) = \{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}$.

Definition 2.3 ([3]). Let (X, ψ) be an SGNS on X and $G \subseteq X$. Then G is called an *sg_ψ-open* set if for each $x \in G$, there is $V \in \psi(x)$ such that $V \subseteq G$.

Let us denote $sg_\psi(X)$ the collection of all *sg_ψ-open* sets on an SGNS (X, ψ). The complements of *sg_ψ-open* sets are called *sg_ψ-closed* sets. The *sg_ψ-interior* of A (denoted by $i_{g_\psi}(A)$) is the union of all $G \subseteq A$, $G \in sg_\psi(X)$, and the *sg_ψ-closure* of A (denoted by $c_{g_\psi}(A)$) is the intersection of all sg_ψ-closed sets containing A.

Definition 2.4 ([3]). Let (X, ψ) be an SGNS on X.

(1) The empty set is an sg_ψ-open set;

(2) The intersection of two sg_ψ-open sets is an sg_ψ-open set;

(3) The arbitrary union of sg_ψ-open sets is an sg_ψ-open set.

Definition 2.5 ([3]). Let (X, ψ) be an SGNS on X and $A \subseteq X$. Then $\iota_\psi(A) = A$ iff A is sg_ψ-open.

Definition 2.6 ([3]). Let (X, ψ) and (Y, ϕ) be two SGNS's. Then $f : X \to Y$ is said to be

(1) *sg-continuous* if for every $A \in sg_\phi(Y)$, $f^{-1}(A)$ is in $sg_\psi(X)$,

(2) *(ψ, ϕ)-continuous* if for $x \in X$ and $V \in \phi(f(x))$, there is $U \in \psi(x)$ such that $f(U) \subset V$.

For a nonempty set X, a collection $\mathbf{H}$ of subsets of X is called an *m-family* [2] on X if $\cap\mathbf{H} \neq \emptyset$. Let ψ be a generalized neighborhood system on X and let $\mathbf{H}$ be an m-family on X. Then we say that an m-family $\mathbf{H}$ converges to $x \in X$ if $\mathbf{H}$ is finer than $\psi(x)$ i.e., $\psi(x) \subseteq \mathbf{H}$.

3. Main Results

Definition 3.1. Let (X, ψ) be an SGNS and $A \subseteq X$.

(a) $I_\psi^*(A) = \{x \in A : A \in \psi(x)\}$.

(b) $cl_\psi^*(A) = \{x \in X : X - A \notin \psi(x)\}$.

Theorem 3.2. *Let (X, ψ) be an SGNS and $A, B \subseteq X$.*

(a) $I_\psi^*(A) \subseteq A$.

(b) $I_\psi^*(A) \cap I_\psi^*(B) \subseteq I_\psi^*(A \cap B)$.

(c) $I_\psi^*(A) = X - cl_\psi^*(X - A)$.

(d) $I_\psi^*(A) \subseteq \iota_\psi(A)$.

Proof. (a) Obvious.

(b) Let $x \in I_\psi^*(A) \cap I_\psi^*(B)$; then $A, B \in \psi(x)$. From the property of strong generalized neighborhood, it follows that $A \cap B \in \psi(x)$. Hence $x \in I_\psi^*(A \cap B)$.

(c) Let $x \in I_\psi^*(A)$ for $A \subseteq X$; then $A = X - (X - A) \in \psi(x)$ and by Definition 3.1, $x \notin cl_\psi^*(X - A)$. Thus we have $x \in X - cl_\psi^*(X - A)$.

The converse is obvious.

(d) Obvious. □

Example 3.3. Let $X = \{a, b, c\}$ and $A = \{a, b\}$. Consider $\psi(a) = \{\{a, b\}\}$, $\psi(b) = \{\{b\}\}$ and $\psi(c) = \emptyset$. Then ψ is a strong generalized neighborhood system. Now we get the following results:

(a) Since $I_\psi^*(A) = \{a\}$ and $\iota_\psi(A) = \{a, b\}$, so $I_\psi^*(A) \neq \iota_\psi(A)$.

(b) Let $B = X$; then $A \subseteq B$ but since $I_\psi^*(B) = \emptyset$, $I_\psi^*(A) \nsubseteq I_\psi^*(B)$.

(c) Since $I_\psi^*(A \cap B) = \{a\}$ and $I_\psi^*(A) \cap I_\psi^*(B) = \emptyset$, $I_\psi^*(A \cap B) \nsubseteq I_\psi^*(A) \cap I_\psi^*(B)$.

(d) Since $I_\psi^*(A) = \{a\}$ and $I_\psi^*(I_\psi^*(A)) = \emptyset$, $I_\psi^*(I_\psi^*(A)) \neq I_\psi^*(A)$.

Theorem 3.4. *Let (X, ψ) be an SGNS and $A \subset X$.*

(a) $A \subseteq cl_\psi^*(A)$.

(b)$cl_\psi^*(A \cup B) \subseteq cl_\psi^*(A) \cup cl_\psi^*(B)$.

(c) $cl_\psi^*(A) = X - I_\psi^*(X - A)$.

(d) $\gamma_\psi(A) \subseteq cl_\psi^*(A)$.

Proof. (a), (c) and (d) are obvious.

(b) Let $x \notin cl_\psi^*(A) \cup cl_\psi^*(B)$; $X - A \in \psi(x)$ and $X - B \in \psi(x)$. From the property of strong generalized neighborhood, it follows that $(X - A) \cap (X - B) \in \psi(x)$. Hence $x \notin cl_\psi^*(A \cup B)$.

The converse is obvious. □

Remark 3.5. From Example 3.3 and Theorem 3.4, we can explain the following statements are not true:

(a) $cl^*_\psi(A) = \gamma_\psi(A)$ for every $A \subseteq X$.

(b) For every $A, B \subseteq X$, if $A \subseteq B$, then $cl^*_\psi(A) \subseteq cl^*_\psi(B)$.

(c) $cl^*_\psi(A) \cup cl^*_\psi(B) \subseteq cl^*_\psi(A \cup B)$ for every $A, B \subseteq X$.

(d) $cl^*_\psi(cl^*_\psi(A)) = cl^*_\psi(A)$ for every $A \subseteq X$.

Theorem 3.6. *Let (X, ψ) be an SGNS and $A \subseteq X$.*

(1) If $I^*_\psi(A) = A$, then A is sg_ψ-open.

(2) If $cl^*_\psi(A) = A$, then A is sg_ψ-closed.

Proof. (1) If $I^*_\psi(A) = A$, then by Theorem 3.2(d), $\iota_\psi(A) = A$. From Theorem 2.5, A is sg_ψ-open.

(2) From Theorem 3.4, it is obvious. □

Example 3.7. Let $X = \{a, b, c, d\}$. Consider an SGNS ψ defined as the following: $\psi(a) = \{\{a, b\}\}$, $\psi(b) = \{\{b, c\}\}$, $\psi(c) = \{\{b, c\}\}$ and $\psi(d) = \emptyset$. Let $A = \{a, b, c\}$; then A is sg_ψ-open but $I^*_\psi(A) = \emptyset$, that is, $I^*_\psi(A) \neq A$.

Theorem 3.8. *Let (X, ψ) be an SGNS and let $\mathbf{B} = \{A \subseteq X : I^*_\psi(A) = A\}$. Then $\Psi_{I^*} = \{\cup\sigma : \sigma \subseteq \mathbf{B}\}$ is coarser than the collection $sg_\psi(X)$ of all sg_ψ-open sets on an SGNS.*

Proof. From Theorem 3.2 and definition of Ψ_{I^*}, we have the following:

(1) The empty set is in Ψ_{I^*};

(2) The intersection of any two elements in Ψ_{I^*} is in Ψ_{I^*};

(3) The arbitrary union of elements in Ψ_{I^*} is in Ψ_{I^*}. Thus from Theorem 2.4 and Theorem 3.6, it follows the collection Ψ_{I^*} is coarser than $sg_\psi(X)$. □

Theorem 3.9. *Let (X, ψ) be an SGNS and $A \subseteq X$.*

(a) $I^*_\psi(A) = \{x \in A : A \in \mathbf{H}$, *for every m-family* $\mathbf{H}$ *converging to* $x\}$.

(b) $cl^*_\psi(A) = \{x \in X :$ *there exists an m-family* $\mathbf{H}$ *such that* $\mathbf{H}$ *converges to* x *and* $X - A \notin \mathbf{H}\}$.

Proof. (a) Let $x \in I^*_\psi(A)$ and an m-family $\mathbf{H}$ converge to x. Then from definition of convergence of m-family, it follows $A \in \psi(x) \subseteq \mathbf{H}$.

Suppose that for every m-family $\mathbf{H}$ converging to x, $A \in \mathbf{H}$. Then since clearly $\psi(x)$ converges to x, by hypothesis, $A \in \psi(x)$, so that $x \in I^*_\psi(A)$.

(b) Let $x \in cl^*_\psi(A)$; then $X - A \notin \psi(x)$. We take $\mathbf{H} = \psi(x)$, then $\mathbf{H}$ satisfies the condition.

For the converse, let $\mathbf{H}$ be an m-family converging to x and $X - A \notin \mathbf{H}$; then since $\psi(x)$ is contained in $\mathbf{H}$, $X - A \notin \psi(x)$, so that $x \in cl^*_\psi(A)$. □

Definition 3.10. Let $f : (X, \psi) \to (Y, \phi)$ be a function between SGNS's (X, ψ) and (Y, ϕ). Then f is called *sgn-continuous* if for every $A \in \phi(f(x))$, $f^{-1}(A)$ is in $\psi(x)$. □

Every sgn-continuous function is (ψ, ϕ)-continuous but the converse may not be true as the following:

Example 3.11. Let $X = \{a, b, c\}$. Consider two SGNS's ψ and ϕ on X defined as the following: $\psi(a) = \{\{a\}, \{a, b\}\}$, $\psi(b) = \{\{b\}\}$, $\psi(c) = \{X\}$, $\phi(a) = \{\{a\}, \{a, b\}\}$, $\phi(b) = \{\{a, b\}\}$ and $\phi(c) = \{X\}$.

Let $f : (X, \psi) \to (X, \phi)$ be a function defined by $f(x) = x$, for $x \in X$. Then f is (ψ, ϕ)-continuous, but not sgn-continuous.

We get the following implications:

continuous $\Rightarrow$ sgn-continuous $\Rightarrow$ (ψ, ϕ)-continuous $\Rightarrow$ sg-continuous

Theorem 3.12. *Let $f : (X, \psi) \to (Y, \phi)$ be a function between SGNS's (X, ψ) and (Y, ϕ). Then the following are equivalent:*

(a) *f is sgn-continuous;*

(b) *$f^{-1}(I^*_\phi(B)) \subseteq I^*_\psi(f^{-1}(B))$ for $B \subseteq Y$;*

(c) *$cl^*_\psi(f^{-1}(B)) \subseteq f^{-1}(cl^*_\phi(B))$ for $B \subseteq Y$.*

Proof. (a) $\Rightarrow$ (b) Suppose f is sgn-continuous and $x \in f^{-1}(I^*_\phi(B))$; then $A \in \phi(f(x))$. $f^{-1}(B) \in \psi(x)$ follows from the sgn-continuity, so that $x \in I^*_\psi(f^{-1}(B))$.

(b) $\Rightarrow$ (a) It is obtained by Definition 3.1.

(b) $\Leftrightarrow$ (c) It is obvious by Theorem 3.2 and Theorem 3.4. □

Theorem 3.13. *Let $f : (X, \psi) \to (Y, \phi)$ be a bijective function between SGNS's (X, ψ) and (Y, ϕ). Then f is sgn-continuous iff $f(cl^*_\psi(A)) \subseteq cl^*_\phi(f(A))$ for $A \subseteq X$.*

Proof. Suppose f is sgn-continuous and $A \subseteq X$. From Theorem 3.12, it follows $cl^*_\psi(f^{-1}(f(A)) \subseteq f^{-1}(cl^*_\phi(f(A)))$. Since f is injective, $cl^*_\psi(A) \subseteq f^{-1}(cl^*_\phi(f(A)))$.

Suppose $f(cl^*_\psi(A)) \subseteq cl^*_\phi(f(A))$ for $A \subseteq X$. For $B \subseteq Y$, by hypothesis and surjectivity, $f(cl^*_\psi(f^{-1}(B)) \subseteq cl^*_\phi(f(f^{-1}(B))) = cl^*_\phi(B)$. Hence from Theorem 3.12, it follows f is sgn-continuous. □

Theorem 3.14. *Let $f : (X, \psi) \to (Y, \phi)$ be a bijective function between SGNS's (X, ψ) and (Y, ϕ). Then f is sgn-continuous iff for an m-family* $\mathbf{H}$ *converging to $x \in X$, $f(\mathbf{H})$ converges to $f(x)$.*

Proof. Suppose f is sgn-continuous and $\mathbf{H}$ is an m-family converging to $x \in X$. It is obvious $f(\mathbf{H}) = \{f(F) : F \in \mathbf{H}\}$ is an m-family on Y. By hypothesis and surjectivity, we get $\phi(f(x)) \subseteq f(\psi(x)) \subseteq f(\mathbf{H})$, so that $f(\mathbf{H})$ converges to $f(x)$.

For the converse, let $G \in \phi(f(x))$ for $G \subseteq Y$. Clearly since $\psi(x)$ converges to x, by hypothesis, we get $\phi(f(x)) \subseteq f(\psi(x))$ for $x \in X$. Since f is injective. $f^{-1}(G) \in \psi(x)$, so that f is sgn-continuous. □

Definition 3.15. Let $f : (X, \psi) \to (Y, \phi)$ be a function between SGNS's (X, ψ) and (Y, ϕ). Then f is said to be *sgn-open* if for $x \in X$ and for every $A \in \psi(x)$, $f(A) \in \phi(f(x))$.

Theorem 3.16. *Let $f : (X, \psi) \to (Y, \phi)$ be a function between two SGNS's (X, ψ) and (Y, ϕ). Then the following are equivalent:*

(a) *f is sgn-open.*

(b) *$f(I^*_\psi(A)) \subseteq I^*_\phi(f(A))$ for $A \subseteq X$.*

Proof. (a) ⇒ (b) Suppose that f is sgn-open and $y \in f(I^*_\psi(A))$. Then there exists $x \in I^*_\psi(A)$ such that $f(x) = y$, and so $A \in \psi(x)$. Since f is sgn-open, $f(A) \in \phi(f(x))$, so that we have $y \in I^*_\phi(f(A))$.

(b) ⇒ (a) For the converse, let $A \in \psi(x)$; then by hypothesis $f(x) \in f(I^*_\psi(A)) \subseteq I^*_\phi(f(A))$. Hence we have $f(A) \in \phi(f(x))$. □

Theorem 3.17. *Let $f : (X, \psi) \to (Y, \phi)$ be a bijection between SGNS's (X, ψ) and (Y, ϕ). Then f is sgn-open iff $I^*_\psi(f^{-1}(B)) \subseteq f^{-1}(I^*_\phi(B))$ for $B \subseteq Y$.*

Proof. Suppose f is sgn-open and $x \in I^*_\psi(f^{-1}(B))$ for $B \subseteq Y$; then $f^{-1}(B) \in \psi(x)$. Since f is surjective, by hypothesis, $B \in \phi(f(x))$ and $f(x) \in I^*_\phi(B)$. Thus we get $x \in f^{-1}(I^*_\phi(B))$.

For the converse, suppose $I^*_\psi(f^{-1}(B)) \subseteq f^{-1}(I^*_\phi(B))$ for $B \subseteq Y$ and $A \in \psi(x)$; then $x \in I^*_\psi(A)$. Since f is injective, $x \in I^*_\psi f^{-1}(f(A))$ and by hypothesis we get $x \in f^{-1}(I^*_\phi(f(A)))$, so that $f(A) \in I^*_\phi(f(x))$. □

Acknowledgements. The author thanks the referee for careful reading and helpful suggestions and comments.

References

1. Á. Csázár: Generalized topology, Generalized continuity. *Acta Math. Hungar.* **96** (2002), 351-357.
2. W.K. Min: Some results on generalized topological spaces and generalized systems. *Acta Math. Hungar.* **108** (2005), no.1-2, 171-181.
3. ______ : On strong generalized neighborhood systems and sg-open sets. *Commun. Korean Math. Soc.* **23** (2008), no. 1, 125-131.

Department of Mathematics, Kangwon National University, Chuncheon 200-701, Korea
Email address: wkmin@cc.kangwon.ac.kr