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THE DENJOY$_*$-STIELTJES EXTENSION OF THE BOCHNER, DUNFORD, PETTIS AND MCSHANE INTEGRALS

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ABSTRACT. In this paper we introduce the concepts of Denjoy$_*$-Stieltjes-Dunford, Denjoy$_*$-Stieltjes-Pettis, Denjoy$_*$-Stieltjes-Bochner and Denjoy$_*$-McShane-Stieltjes integrals of Banach-valued functions using the Denjoy$_*$-Stieltjes integral of real-valued functions and investigate their properties.

1. INTRODUCTION

The Denjoy integral of real-valued functions which is an extension of the Lebesgue integral was studied by some authors ([3], [4], [9]). In [7] we introduced the Denjoy$_*$ integral of real-valued functions. J. L. Gamez and J. Mendoza [2] and R. A. Gordon [3] studied the Denjoy extension of the Bochner, Pettis and Dunford integrals which is defined by the Denjoy integral. J. M. Park and D. H. Lee [8] introduced the concept of Denjoy-McShane integral of Banach-valued functions. In [7] we introduced the concept of Denjoy$_*$-Stieltjes integral which is a generalization of the Denjoy$_*$ integral and obtained some properties of the Denjoy$_*$-Stieltjes integral.

In this paper we deal with the Denjoy$_*$-Stieltjes extension of the Bochner, Pettis, Dunford and McShane integrals. We first define the Denjoy$_*$-Stieltjes-Dunford, Denjoy$_*$-Stieltjes-Pettis, Denjoy$_*$-Stieltjes-Bochner and Denjoy$_*$-McShane-Stieltjes integrals of Banach-valued functions using the Denjoy$_*$-Stieltjes integral of real-valued functions and then investigate their properties.

2. PRELIMINARIES

Throughout this paper, X denotes a real Banach space and X^* its dual. Let

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$$\omega(F, [c,d]) = \sup\{\|F(y) - F(x)\| : c \leq x < y \leq d\}$$

denote the oscillation of the function $F : [a,b] \to X$ on the interval $[c,d]$.

Definition 2.1 ([9]). Let $F : [a,b] \to X$ and let $E \subset [a,b]$.

(a) The function F is AC_* on E if F is bounded on an interval that contains E and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon$$

whenever $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n} (d_i - c_i) < \delta.$$

(b) The function F is ACG_* on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is AC_*.

Definition 2.2 ([7, 9]). Let $F : [a,b] \to X$ and let $t \in (a,b)$. A vector z in X is the *approximate derivative* of F at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{s - t} = z.$$

We will write $F'_{ap}(t) = z$.

A function $f : [a,b] \to \mathbb{R}$ is *Denjoy$_*$ integrable* on $[a,b]$ if there exists an ACG_* function $F : [a,b] \to \mathbb{R}$ such that $F' = f$ almost everywhere on $[a,b]$. In this case, we write

$$(D_*) \int_a^b f = F(b) - F(a).$$

The function f is Denjoy$_*$ integrable on a set $E \subset [a,b]$ if $f\chi_E$ is Denjoy$_*$ integrable on $[a,b]$. In this case, we write

$$(D_*) \int_E f = (D_*) \int_a^b f\chi_E.$$

Definition 2.3 ([5]). A *McShane partition* of $[a,b]$ is a finite collection $\mathcal{P} = \{([c_i, d_i], t_i) : 1 \leq i \leq n\}$ such that $\{[c_i, d_i] : 1 \leq i \leq n\}$ is a non-overlapping family of subintervals of $[a,b]$ covering $[a,b]$ and $t_i \in [a,b]$ for each $i \leq n$. A *gauge* on $[a,b]$

is a function $\delta : [a,b] \to (0,\infty)$. A McShane partition $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is *subordinate* to a gauge δ if

$$[c_i,d_i] \subset (t_i - \delta(t_i), t_i + \delta(t_i))$$

for every $i \le n$. If $f : [a,b] \to X$ and if $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$, we will denote $f(\mathcal{P})$ for

$$\sum_{i=1}^{n} f(t_i)(d_i - c_i).$$

A function $f : [a,b] \to X$ is *McShane integrable* on $[a,b]$, with McShane integral z, if for each $\varepsilon > 0$ there exists a gauge $\delta : [a,b] \to (0,\infty)$ such that

$$\|f(\mathcal{P}) - z\| < \varepsilon$$

whenever $\mathcal{P} = \{([c_i,d_i],t_i) : 1 \le i \le n\}$ is a McShane partition of $[a,b]$ subordinate to δ. In this case, we write

$$(M)\int_a^b f = z.$$

3. The Denjoy$_*$-Stieltjes Extension of the Bochner, Dunford and Pettis Integrals

In this section we introduce the concepts of Denjoy$_*$-Stieltjes-Bochner, Denjoy$_*$-Stieltjes-Pettis and Denjoy$_*$-Stieltjes-Dunford integrals and investigate their properties.

Definition 3.1 ([7]). Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function and let $E \subset [a,b]$.

(a) The function F is α-AC$_*$ on E if F is bounded on an interval that contains E and for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{n} \omega(F, [c_i,d_i]) < \epsilon$$

whenever $\{[c_i,d_i] : 1 \le i \le n\}$ is a finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] < \delta.$$

(b) The function F is α-ACG$_*$ on E if F is continuous on E and E can be expressed as a countable union of sets on each of which F is α-AC$_*$.

Theorem 3.2 ([7]). *Let $F : [a,b] \to X$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then F is AC_* on E if and only if F is α-AC_* on E.*

Proof. Suppose that F is AC$_*$ on E. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that

$$\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon$$

whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is any finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n} (d_i - c_i) < \eta.$$

Since α is a strictly increasing function such that $\alpha \in C^1([a,b])$, there exists $m > 0$ such that

$$|\alpha'(t)| = \alpha'(t) \ge m$$

for all $t \in [a,b]$. Take $\delta = m\eta$. Let $\{[c_i, d_i] : 1 \le i \le n\}$ be any finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] < \delta.$$

Then by the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that

$$\alpha(d_i) - \alpha(c_i) = \alpha'(t_i)(d_i - c_i), \ 1 \le i \le n.$$

So $\alpha(d_i) - \alpha(c_i) \ge m(d_i - c_i), \ 1 \le i \le n$. Hence

$$\sum_{i=1}^{n} (d_i - c_i) \le \frac{1}{m} \sum_{i=1}^{n} [\alpha(d_i) - \alpha(c_i)] \le \frac{1}{m} \cdot \delta = \eta.$$

So

$$\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon.$$

Thus F is α-AC$_*$ on E.

Conversely, suppose that F is α-AC$_*$ on E. Let $\epsilon > 0$ be given. Then there exists $\eta > 0$ such that

$$\sum_{i=1}^{n} \omega(F, [c_i, d_i]) < \epsilon$$

whenever $\{[c_i, d_i] : 1 \le i \le n\}$ is any finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] < \eta.$$

Since $\alpha \in C^1([a,b])$, there exists $M > 0$ such that

$$|\alpha'(t)| \le M$$

for all $t \in [a,b]$. Take $\delta = \frac{\eta}{M}$. Let $\{[c_i, d_i] : 1 \le i \le n\}$ be any finite collection of non-overlapping intervals that have endpoints in E and satisfy

$$\sum_{i=1}^{n}(d_i - c_i) < \delta.$$

Then by the Mean Value Theorem there exists $t_i \in (c_i, d_i)$ such that

$$\alpha(d_i) - \alpha(c_i) = \alpha'(t_i)(d_i - c_l),\ 1 \le i \le n.$$

So

$$\alpha(d_i) - \alpha(c_i) \le M(d_i - c_i),\ 1 \le i \le n.$$

Hence

$$\sum_{i=1}^{n}[\alpha(d_i) - \alpha(c_i)] \le M\sum_{i=1}^{n}(d_i - c_i) < M\delta = \eta.$$

So

$$\sum_{i=1}^{n}\omega(F, [c_i, d_i]) < \epsilon.$$

Thus F is AC$_*$ on E. □

Definition 3.3 ([7]). Let $F : [a,b] \to X$ and let $t \in (a,b)$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A vector $z \in X$ is the *α-approximate derivative* of F at t if there exists a measurable set $E \subset [a,b]$ that has t as a point of density such that

$$\lim_{\substack{s \to t \\ s \in E}} \frac{F(s) - F(t)}{\alpha(s) - \alpha(t)} = z.$$

We will write $F'_{\alpha,ap}(t) = z$.

Note that $F'_{ap}(t) = F'_{\alpha,ap}(t) \cdot \alpha'(t)$ for each $t \in (a,b)$.

Definition 3.4. (a) A function $f : [a,b] \to X$ is *Denjoy$_*$-Dunford integrable* on $[a,b]$ if for each $x^* \in X^*$ the function x^*f is Denjoy$_*$ integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector x_I^{**} in X^{**} such that

$$x_I^{**}(x^*) = (D_*)\int_I x^* f$$

for all $x^* \in X^*$.

(b) A function $f : [a,b] \to X$ is *Denjoy$_*$-Pettis integrable* on $[a,b]$ if f is Denjoy$_*$-Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(c) A function $f : [a,b] \to X$ is *Denjoy$_*$-Bochner integrable* on $[a,b]$ if there exists an ACG$_*$ function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

A function $f : [a,b] \to X$ is integrable in one of the above senses on a set $E \subset [a,b]$ if the function $f\chi_E$ is integrable in that sense on $[a,b]$.

Definition 3.5 ([7]). Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A function $f : [a,b] \to \mathbb{R}$ is *α-Denjoy$_*$-Stieltjes integrable* on $[a,b]$ if there exists an α-ACG$_*$ function $F : [a,b] \to \mathbb{R}$ such that $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$. In this case, we write

$$(D_*S)\int_a^b f d\alpha = F(b) - F(a).$$

The function f is α-Denjoy$_*$-Stieltjes integrable on a set $E \subset [a,b]$ if $f\chi_E$ is α-Denjoy$_*$-Stieltjes integrable on $[a,b]$. In this case, we write

$$(D_*S)\int_E f d\alpha = (D_*S)\int_a^b f\chi_E d\alpha.$$

Theorem 3.6 ([7]). *Let $f : [a,b] \to \mathbb{R}$ and let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then f is α-Denjoy$_*$-Stieltjes integrable on E if and only if $\alpha' f$ is Denjoy$_*$ integrable on E.*

Proof. If f is α-Denjoy$_*$-Stieltjes integrable on E, then there exists an α-ACG$_*$ function $F : [a,b] \to \mathbb{R}$ such that $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.2, F is an ACG$_*$ function on $[a,b]$ such that $F' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. Hence $\alpha' f\chi_E$ is Denjoy$_*$ integrable on $[a,b]$. Thus $\alpha' f$ is Denjoy$_*$ integrable on E.

Conversely, if $\alpha' f$ is Denjoy$_*$ integrable on E, then there exists an ACG$_*$ function $F : [a,b] \to \mathbb{R}$ on $[a,b]$ such that $F' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.2, F is an α-ACG$_*$ function on $[a,b]$ such that $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. Hence $f\chi_E$ is α-Denjoy$_*$-Stieltjes integrable on $[a,b]$. Thus f is α-Denjoy$_*$-Stieltjes integrable on E. □

Definition 3.7. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$.

(a) $f : [a,b] \to X$ is *α-Denjoy$_*$-Stieltjes-Dunford integrable* on $[a,b]$ if for each $x^* \in X^*$ x^*f is α-Denjoy$_*$-Stieltjes integrable on $[a,b]$ and if for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that

$$x_I^{**}(x^*) = (D_*S) \int_I x^*f \, d\alpha$$

for all $x^* \in X^*$.

(b) $f : [a,b] \to X$ is *α-Denjoy$_*$-Stieltjes-Pettis* integrable on $[a,b]$ if f is α-Denjoy$_*$-Stieltjes -Dunford integrable on $[a,b]$ and if $x_I^{**} \in X$ for every interval I in $[a,b]$.

(c) $f : [a,b] \to X$ is *α-Denjoy$_*$-Stieltjes-Bochner integrable* on $[a,b]$ if there exists an α-ACG$_*$ function $F : [a,b] \to X$ such that F is α-approximately differentiable almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f$ almost everywhere on $[a,b]$.

$f : [a,b] \to X$ is integrable in one of the above senses on a set $E \subset [a,b]$ if $f\chi_E$ is integrable in that sense on $[a,b]$.

Theorem 3.8. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Bochner integrable on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on E.*

Proof. If $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Bochner integrable on E, then there exists an α-ACG$_*$ function $F : [a,b] \to X$ such that F is α-approximately differentiable almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.2, F is ACG$_*$ on $[a,b]$. F is also approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = F'_{\alpha,ap}\alpha' = \alpha' f\chi_E$ almost everywhere on $[a,b]$. Hence $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on E.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on E, then there exists an ACG$_*$ function $F : [a,b] \to X$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = \alpha' f\chi_E$ almost everywhere on $[a,b]$. By Theorem 3.2, F is α-ACG$_*$ on $[a,b]$. F is also α-approximately differentiable almost everywhere on $[a,b]$ and $F'_{\alpha,ap} = \frac{1}{\alpha'}F'_{ap} = f\chi_E$ almost everywhere on $[a,b]$. Hence $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Bochner integrable on E. □

Theorem 3.9. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Dunford integrable on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Dunford integrable on E.*

Proof. If $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Dunford integrable on E, then for each $x^* \in X^*$ x^*f is α-Denjoy$_*$-Stieltjes integrable on E and for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that

$$x_I^{**}(x^*) = (D_*S)\int_I x^* f\chi_E \, d\alpha$$

for all $x^* \in X^*$. By Theorem 3.6, for each $x^* \in X^*$ $\alpha'(x^*f) = x^*(\alpha' f)$ is Denjoy$_*$ integrable on E and

$$x_I^{**}(x^*) = (D_*S)\int_I x^* f\chi_E \, d\alpha = (D_*)\int_I x^*(\alpha' f\chi_E)$$

for all $x^* \in X^*$. Hence $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Dunford integrable on E.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Dunford integrable on E, then for each $x^* \in X^*$ $x^*(\alpha' f) = \alpha'(x^* f)$ is Denjoy$_*$ integrable on E and for every interval I in $[a,b]$ there exists a vector $x_I^{**} \in X^{**}$ such that

$$x_I^{**}(x^*) = (D_*)\int_I x^*(\alpha' f\chi_E)$$

for all $x^* \in X^*$. By Theorem 3.6, for each $x^* \in X^*$ x^*f is α-Denjoy$_*$-Stieltjes integrable on E and

$$x_I^{**}(x^*) = (D_*)\int_I x^*(\alpha' f\chi_E) = (D_*)\int_I \alpha'(x^* f\chi_E) = (D_*S)\int_I x^* f\chi_E \, d\alpha$$

for all $x^* \in X^*$. Hence $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Dunford integrable on E. □

Theorem 3.10. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $E \subset [a,b]$. Then $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Pettis integrable on E if and only if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Pettis integrable on E.*

Proof. The proof is similar to Theorem 3.9. □

4. THE DENJOY$_*$-STIELTJES EXTENSION OF THE MCSHANE INTEGRAL

In this section we introduce the concept of the Denjoy$_*$-McShane-Stieltjes integral and investigate some properties of this integral.

Definition 4.1. A function $f : [a,b] \to X$ is *Denjoy$_*$-McShane integrable* on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$.

In this case, we write

$$(D_*M)\int_a^b f = F(b) - F(a).$$

Definition 4.2. Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. A function $f : [a,b] \to X$ is *α-Denjoy$_*$-McShane-Stieltjes integrable* on $[a,b]$ if there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is α-ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is α-approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$.

In this case, we write

$$(D_*MS)\int_a^b f d\alpha = F(b) - F(a).$$

Theorem 4.3. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. Then $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$ if and only if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is α-ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is α-approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$.

From Theorem 3.2 we have

(i) for each $x^* \in X^*$ x^*F is ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is differentiable almost everywhere on $[a,b]$ and

$$(x^*F)' = (x^*F)'_{ap} = (x^*F)'_{\alpha,ap}\alpha' = (x^*f)\alpha' = x^*(\alpha' f)$$

almost everywhere on $[a,b]$.

Hence $\alpha' f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$.

Conversely, if $\alpha' f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is differentiable almost everywhere on $[a,b]$ and $(x^*F)' = x^*(\alpha' f)$ almost everywhere on $[a,b]$.

From Theorem 3.2 we have

(i) for each $x^* \in X^*$ x^*F is α-ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is α-approximately differentiable almost everywhere on $[a.b]$ and

$$(x^*F)'_{\alpha,ap} = \frac{1}{\alpha'}(x^*F)'_{ap} = \frac{1}{\alpha'}(x^*F)' = \frac{1}{\alpha'}x^*(\alpha' f) = x^*f$$

almost everywhere on $[a,b]$.

Hence $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$. □

Theorem 4.4. *If $f : [a,b] \to X$ is McShane integrable on $[a,b]$, then $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$.*

Proof. Let $f : [a,b] \to X$ be McShane integrable on $[a,b]$. Then for each $x^* \in X^*$ x^*f is McShane integrable on $[a,b]$ and hence x^*f is Lebesgue integrable on $[a,b]$. Let

$$F(t) = (M)\int_a^t f.$$

Then $F : [a,b] \to X$ is continuous on $[a,b]$ by [5, Theorem 8] and for each $x^* \in X^*$

$$x^*F(t) = (M)\int_a^t x^*f = (L)\int_a^t x^*f.$$

Hence x^*F is AC and so x^*F is ACG$_*$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$. Thus $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$. □

We can obtain the following corollary from Theorem 4.3, 4.4.

Corollary 4.5. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If $\alpha' f : [a,b] \to X$ is McShane integrable on $[a,b]$, then $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$.*

Theorem 4.6. *If $f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on $[a,b]$, then $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$.*

Proof. Let $f : [a,b] \to X$ be Denjoy$_*$-Bochner integrable on $[a,b]$. Then there exists an ACG$_*$ function $F : [a,b] \to X$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. For each $x^* \in X$ x^*F is also ACG$_*$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$. Hence f is Denjoy$_*$-McShane integrable on $[a,b]$. □

We can obtain the following corollary from Theorem 4.3, 4.6.

Corollary 4.7. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Bochner integrable on $[a,b]$, then $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$.*

Theorem 4.8. *If $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$, then $f : [a,b] \to X$ is Denjoy$_*$-Pettis integrable on $[a,b]$.*

Proof. Suppose that $f : [a,b] \to X$ is Denjoy$_*$-McShane integrable on $[a,b]$. Let

$$F(t) = (D_*M)\int_a^t f.$$

Since x^*F is ACG$_*$ and $(x^*F)' = x^*f$ almost everywhere on $[a,b]$ for each $x^* \in X^*$, x^*f is Denjoy$_*$ integrable on $[a,b]$ for each $x^* \in X^*$. For every interval $[c,d]$ in $[a,b]$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (D_*)\int_a^d x^*f - (D_*)\int_a^c x^*f \\ &= (D_*)\int_c^d x^*f. \end{aligned}$$

Since $F(d) - F(c) \in X$, $f : [a,b] \to X$ is Denjoy$_*$-Pettis integrable on $[a,b]$. □

We can obtain the following corollary from Theorem 4.3, 4.8.

Corollary 4.9. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$ and let $f : [a,b] \to X$. If $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$, then $\alpha' f : [a,b] \to X$ is Denjoy$_*$-Pettis integrable on $[a,b]$.*

Theorem 4.10. *Let $\alpha : [a,b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a,b])$. If $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$ and $T : X \to Y$ is a bounded linear operator, then $T \circ f : [a,b] \to Y$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$.*

Proof. If $f : [a,b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a,b]$, then there exists a continuous function $F : [a,b] \to X$ such that

(i) for each $x^* \in X^*$ x^*F is α-ACG$_*$ on $[a,b]$ and

(ii) for each $x^* \in X^*$ x^*F is α-approximately differentiable almost everywhere on $[a,b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a,b]$.

Let $G = T \circ F$. Then $G : [a,b] \to Y$ is a continuous function such that

(i) for each $y^* \in Y^*$ $y^*G = y^*(T \circ F) = (y^*T)F$ is α-ACG$_*$ on $[a, b]$ since $y^*T \in X^*$, and

(ii) for each $y^* \in Y^*$ $y^*G = y^*(T \circ F) = (y^*T)F$ is α-approximately differentiable almost everywhere on $[a, b]$ and

$$(y^*G)'_{\alpha,ap} = (y^*(T \circ F))'_{\alpha,ap} = ((y^*T)F)'_{\alpha,ap} = (y^*T)f = y^*(T \circ f)$$

almost everywhere on $[a, b]$ since $y^*T \in X^*$.

Hence $T \circ f : [a, b] \to Y$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a. b]$. □

Theorem 4.11. *Let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$. If $f : [a, b] \to X$ is α-Denjoy$_*$-Stieltjes-Bochner integrable on $[a, b]$, then $f : [a, b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a, b]$.*

Proof. If $f : [a, b] \to X$ is α-Denjoy$_*$-Stieltjes-Bochner integrable on $[a, b]$, then there exists an α-ACG$_*$ function $F : [a, b] \to X$ such that F is α-approximately differentiable almost everywhere on $[a, b]$ and $F'_{\alpha,ap} = f$ almost everywhere on $[a, b]$. It is easy to show that for each $x^* \in X^*$ x^*F is α-ACG$_*$ on $[a, b]$ and x^*F is α-approximately differentiable almost everywhere on $[a, b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a, b]$. Hence $f : [a, b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a, b]$. □

Theorem 4.12. *Let $\alpha : [a, b] \to \mathbb{R}$ be a strictly increasing function such that $\alpha \in C^1([a, b])$. If $f : [a, b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a, b]$, then $f : [a, b] \to X$ is α-Denjoy$_*$-Stieltjes-Pettis integrable on $[a, b]$.*

Proof. Suppose that $f : [a, b] \to X$ is α-Denjoy$_*$-McShane-Stieltjes integrable on $[a, b]$. Let

$$F(t) = (D_*MS) \int_a^t f d\alpha.$$

Since x^*F is α-ACG$_*$ on $[a, b]$ and $(x^*F)'_{\alpha,ap} = x^*f$ almost everywhere on $[a, b]$ for each $x^* \in X^*$, x^*f is α-Denjoy$_*$-Stieltjes integrable on $[a, b]$ for each $x^* \in X^*$. For every interval $[c, d]$ in $[a, b]$ and $x^* \in X^*$, we have

$$\begin{aligned} x^*(F(d) - F(c)) &= x^*F(d) - x^*F(c) \\ &= (D_*S) \int_a^d x^* f d\alpha - (D_*S) \int_a^c x^* f d\alpha \\ &= (D_*S) \int_c^d x^* f d\alpha. \end{aligned}$$

Since $F(d) - F(c) \in X$. $f : [a,b] \to X$ is α-Denjoy$_*$-Stieltjes-Pettis integrable on $[a,b]$. □

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