J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 15, Number 3 (August 2008), Pages 343–351

THE AP-DENJOY INTEGRAL OF BANACH-VALUED FUNCTIONS

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ABSTRACT. In this paper, we define the Denjoy and ap-Denjoy integrals of Banach-valued functions, and we investigate some properties of these two integrals. In particular, we show that a Denjoy integrable function is ap-Denjoy integrable.

1. INTRODUCTION AND PRELIMINARIES

The ap-Denjoy integral of real valued functions was introduced in [13]. It is known [13] that the ap-Denjoy integral is equivalent to the ap-Henstock integral.

In this paper, we define the Denjoy integral and ap-Denjoy integrals of Banach-valued functions, and we investigate the relationship of these two integrals.

Throughout this paper, X is a Banach space with dual X^*.

For a measurable set E of real numbers we denote by $|E|$ its Lebesgue measure. Let E be a measurable set and let c be a real number. The *density* of E at c is defined by

$$d_cE = \lim_{h \to 0^+} \frac{|E \cap (c-h, c+h)|}{2h}$$

provided the limit exists. The point c is called *a point of density* of E if $d_cE = 1$. The set E^d represents the set of all points $x \in E$ such that x is a point of density of E. A function $F : [a,b] \to X$ is said to be *approximately differentiable* at $c \in [a,b]$ if there exists a measurable set $E \subseteq [a,b]$ such that $c \in E^d$ and

$$\lim_{\substack{x \to c \\ x \in E}} \frac{F(x) - F(c)}{x - c}$$

exists. The approximate derivative of F at c is denoted by $F'_{ap}(c)$.

Received by the editors May 6, 2008. Accepted July 27, 2008.
2000 *Mathematics Subject Classification.* 28B05, 26A39.
Key words and phrases. Denjoy integral, approximate Lusin function, ap-Denjoy integral.

An *approximate neighborhood*(or ap-nbd) of $x \in [a,b]$ is a measurable set $S_x \subseteq [a,b]$ containing x as a point of density. For every $x \in E \subseteq [a,b]$, choose an ap-nbd $S_x \subseteq [a,b]$ of x. Then we say that $S = \{S_x : x \in E\}$ is a *choice* on E. A tagged interval $(x, [c,d])$ is said to be *subordinate* to the choice $S = \{S_x\}$ if $c, d \in S_x$. Let $\mathcal{P} = \{(x_i, [c_i, d_i]) : 1 \leq i \leq n\}$ be a finite collection of non-overlapping tagged intervals. If $(x_i, [c_i, d_i])$ is subordinate to a choice S for each i, then we say that $\mathcal{P}$ is subordinate to S. Let $E \subseteq [a,b]$. If $\mathcal{P}$ is subordinate to S and each $x_i \in E$, then $\mathcal{P}$ is called E-subordinate to S. If $\mathcal{P}$ is subordinate to S and $[a,b] = \bigcup_{i=1}^{n}[c_i, d_i]$, then we say that $\mathcal{P}$ is a tagged partition of $[a,b]$ that is subordinate to S.

2. The ap-Denjoy Integral of Banach-valued Functions

Definition 2.1. A function $F : [a,b] \to X$ is AC_s on a measurable set $E \subseteq [a,b]$ if for each $\epsilon > 0$ there exist a positive number δ and a choice S on E such that $\|(\mathcal{P})\sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is subordinate to S and satisfies $(P)\sum |I| < \delta$. The function F is ACG_s on E if E can be expressed as a countable union of measurable sets on each of which F is AC_s.

We introduce the notion of the approximate Lusin function. This function is used to define the ap-Denjoy integral. For a function $F : [a,b] \to X$, F can be treated as a function of intervals by defining $F([c,d]) = F(d) - F(c)$.

Definition 2.2. Let $F : [a,b] \to X$ be a function. The function F is *an approximate Lusin function*(or F is an AL function) on $[a,b]$ if for every measutable set $E \subseteq [a,b]$ of measure zero and for every $\epsilon > 0$ there exists a choice S on E such that $\|(\mathcal{P})\sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S.

Theorem 2.3. If $F : [a,b] \to X$ is ACG_s on $[a,b]$, then F is an AL function on $[a,b]$.

Proof. Suppose that $E \subseteq [a,b]$ is a measurable set of measure zero. Let $E = \bigcup_{n=1}^{\infty} E_n$, where $\{E_n\}$ is a sequence of disjoint measurable sets and F is AC_s on each E_n. Let $\epsilon > 0$. For each positive integer n, there exist a choice $S^n = \{S_x^n : x \in E_n\}$ on E_n and a positive number δ_n such that $\|(\mathcal{P})\sum F(I)\| < \epsilon/2^n$ whenever $\mathcal{P}$ is E_n-subordinate to S^n and $(\mathcal{P})\sum |I| < \delta_n$. For each positive integer n, choose an open set O_n such that $E_n \subseteq O_n$ and $|O_n| < \delta_n$. Let $S_x = S_x^n \cap O_n$ for each $x \in E_n$.

Then $S = \{S_x : x \in E\}$ is a choice on E. Suppose that $\mathcal{P}$ is E-subordinate to S. Let $\mathcal{P}_n$ be a subset of $\mathcal{P}$ that has tags in E_n and note that $(\mathcal{P}_n)\sum |I| < |O_n| < \delta_n$. Hence, we have

$$\left\|(\mathcal{P})\sum F(I)\right\| \leq \sum_{n=1}^{\infty} \left\|(\mathcal{P}_n)\sum F(I)\right\| < \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

□

Definition 2.4. A function $f : [a,b] \to X$ is *ap-Denjoy integrable* on $[a,b]$ if there exists an AL function F on $[a,b]$ such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. The function f is ap-Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is ap-Denjoy integrable on $[a,b]$.

If we add the condition $F(a) = 0$, then the function F is unique. We will denote this function $F(x)$ by

$$(AD)\int_a^x f.$$

It is easy to show that if $f : [a,b] \to X$ is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$. This gives rise to an interval function F such that

$$F(I) = (AD)\int_I f$$

for every subinterval $I \subseteq [a,b]$. The function F is called the primitive of f.

From the definition of the ap-Denjoy integral, we get the following theorem.

Theorem 2.5. Let $f : [a,b] \to X$ be ap-Denjoy integrable on $[a,b]$ and let

$$F(x) = (AD)\int_a^x f$$

for each $x \in [a,b]$. Then the function F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$.

Theorem 2.6. Let $f : [a,b] \to X$ and let $c \in (a,b)$.

(a) If f is ap-Denjoy integrable on $[a,b]$, then f is ap-Denjoy integrable on every subinterval of $[a,b]$.

(b) If f is ap-Denjoy integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is ap-Denjoy integrable on $[a,b]$ and

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f.$$

Proof. (a) Let $[c,d]$ be any subinterval on $[a,b]$. Let

$$F(x) = (AD)\int_a^x f$$

for each $x \in [a,b]$. Since F is an AL function on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$, F is an AL function on $[c,d]$ and $F'_{ap} = f$ almost everywhere on $[c,d]$. Hence, f is ap-Denjoy integrable on $[c,d]$.

(b) Since f is ap-Denjoy integrable on each of intervals $[a,c]$ and $[c,b]$, there exist AL functions F and G such that $F'_{ap} = f$ almost everywhere on $[a,c]$ and $G'_{ap} = f$ almost everywhere on $[c,b]$, respectively. Define $H : [a,b] \to X$ by

$$H(x) = \begin{cases} F(x), & \text{if } x \in [a,c]; \\ F(c) + G(x), & \text{if } x \in (c,b]. \end{cases}$$

Then H is an AL function on $[a,b]$ and $H'_{ap} = f$ almost everywhere on $[a,b]$. Hence f is ap-Denjoy integrable on $[a,b]$ and $H(b) = F(c) + G(b)$, i.e.,

$$(AD)\int_a^b f = (AD)\int_a^c f + (AD)\int_c^b f$$

□

We can easily get the following theorem.

Theorem 2.7. Suppose that f and g are ap-Denjoy integrable on $[a,b]$. Then

(a) kf is ap-Denjoy integrable on $[a,b]$ and

$$(AD)\int_a^b kf = k(AD)\int_a^b f$$

for each $k \in \mathbb{R}$,

(b) $f+g$ is ap-Denjoy integrable on $[a,b]$ and

$$(AD)\int_a^b (f+g) = (AD)\int_a^b f + (AD)\int_a^b g.$$

Theorem 2.8. Let $f : [a,b] \to X$ be ap-Denjoy integrable on $[a,b]$. Then for each $x^* \in X^*$ the function x^*f is ap-Denjoy integrable on $[a,b]$ and

$$x^*(AD)\int_a^b f = (AD)\int_a^b x^*f$$

Proof. Let $f : [a,b] \to X$ be ap-Denjoy integrable on $[a,b]$. Then by definition, there exists an AL function F such that F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = f$ almost everywhere on $[a,b]$. Since for each $x^* \in X^*$,

x^*F is an AL function and $(x^*F)'_{ap} = x^*f$ almost everywhere on $[a,b]$, x^*f is ap-Denjoy integrable on $[a,b]$ and

$$x^*F(x) = (AD)\int_a^x x^*f.$$

Hence, for each $x^* \in X^*$

$$x^*(AD)\int_a^b f = x^*F(b) = (AD)\int_a^b x^*f.$$

□

Theorem 2.9. A function $f : [a,b] \to X$ is ap-Denjoy integrable on $[a,b]$ if and only if there exists an ACG_s function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$.

Proof. Suppose that there exists an ACG_s function F on $[a,b]$ such that $F'_{ap} = f$ almost everywhere on $[a,b]$. Then F is an AL function by Theorem 2.3. Hence, f is ap-Denjoy integrable on $[a,b]$.

Conversely, suppose that f is ap-Denjoy integrable on $[a,b]$ and let

$$F(x) = (AD)\int_a^x f$$

for each $x \in [a,b]$. Then F is an AL function such that $F'_{ap} = f$ almost everywhere on $[a,b]$. Let $E = \{x \in [a,b] : F'_{ap}(x) \neq f(x)\}$. Then $|E| = 0$. Since F is an AL function, F is AC_s on E. For each positive integer n, let

$$E_n = \{x \in [a,b] - E : n-1 \leq \|f(x)\| < n\}.$$

Fix n and let $\epsilon > 0$. Since F is approximately differentiable for each $x \in E_n$, there exist a measurable set A_x containing x as a point of density and a positive number δ_x such that

$$\left\|\frac{F(y) - F(x)}{y - x} - f(x)\right\| < \epsilon$$

i.e.,

$$\|F(y) - F(x) - f(x)(y - x)\| < \epsilon|y - x|,$$

if $y \in A_x \cap (x - \delta_x, x + \delta_x)$. For each $x \in E_n$, let

$$S_x = A_x \cap (x - \delta_x, x + \delta_x)$$

Then $S = \{S_x : x \in E_n\}$ is a choice on E_n. Suppose that $\mathcal{P}$ is a finite collection of non-overlapping tagged intervals that is E_n-subordinate to S and satisfies $\mu(\mathcal{P}) < \frac{\epsilon}{n}$.

Then since $\|F(\mathcal{P}) - f(\mathcal{P})\| < \epsilon\mu(\mathcal{P})$, we have

$$\begin{aligned}\|F(\mathcal{P})\| &\leq \|F(\mathcal{P}) - f(\mathcal{P})\| + \|f(\mathcal{P})\| \\ &< \epsilon\mu(\mathcal{P}) + n\mu(\mathcal{P}) \\ &< (b-a+1)\epsilon\end{aligned}$$

Hence, F is AC_s on E_n. Since $[a,b] = [\cup_{n=1}^{\infty} E_n] \cup E$, F is ACG_s on $[a,b]$. □

Theorem 2.10. Let $f : [a,b] \to X$ be ap-Denjoy integrable on $[a,b]$ and let

$$F(x) = (AD)\int_a^x f$$

for each $x \in [a,b]$. Then F is approximately continuous on $[a,b]$.

Proof. From the definition of the ap-Denjoy integral, F is approximately differentiable almost everywhere on $[a,b]$. Let E be the set of all non-approximately differentiable points in $[a,b]$. Then E is a measurable set of measure zero. Since F is approximately continuous on $[a,b] - E$, it is sufficient to show that F is approximately continuous on E. Let $c \in E$ and let $\epsilon > 0$. Since F is an AL function, there exists a choice $S = \{S_x : x \in E\}$ such that $\|(\mathcal{P})\sum F(I)\| < \epsilon$ for every finite collection $\mathcal{P}$ of non-overlapping tagged intervals that is E-subordinate to S. If $x \in S_c \cap (c-\eta, c+\eta)$ for some $\eta > 0$, then the tagged interval $(c, [c,x])$(or$(c, [x,c])$) is E-subordinate to S. Hence, $\|F(x) - F(c)\| = \|F([c,x])\| < \epsilon$. This shows that F is approximately continuous on E. □

Definition 2.11. Let $F : [a,b] \to X$ and let $E \subseteq [a,b]$. The function F is AC_δ on E if for each $\epsilon > 0$ there exist a positive number η and a gauge δ on E such that $\| (\mathcal{P})\Sigma F(I) \| < \epsilon$ whenever $\mathcal{P}$ is E-subordinate to δ and $(\mathcal{P})\Sigma \mid I \mid < \eta$. The function F is ACG_δ on E if E can be written as a countable union of sets on each of which F is AC_δ

It is easy to show that on ACG_δ function on $[a,b]$ is continuous on $[a,b]$.

Definition 2.12. A function $f : [a,b] \to X$ is Denjoy integrable on $[a,b]$ if there exists on ACG_δ function $F : [a,b] \to X$ such that $F' = f$ almost everywhere on $[a,b]$. The function f is Denjoy integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is Denjoy integrable on $[a,b]$.

Theorem 2.13. If a function $f : [a,b] \to X$ is Denjoy integrable on [a,b], then f is ap-Denjoy integrable on $[a,b]$.

Proof. Let $f : [a,b] \to X$ be Denjoy integrable on $[a,b]$. Then by definition, there exists on ACG_δ function $F : [a,b] \to X$ such that $F' = f$ almost everywhere on $[a,b]$. It is easy to show that F is an AL function. The proof is similar to the proof that an ACG_s function is an AL function in Theorem 2.3.

Since $F' = f$ almost everywhere on $[a,b]$, F is approximately differentiable almost everywhere on $[a,b]$ and $F'_{ap} = F' = f$ almost everywhere on $[a,b]$. Hence, f is ap-Denjoy intrgrable on $[a,b]$. □

The following example shows that there exists an ap-Denjoy integrable function that is not Denjoy integrable.

Example 2.14. Let $\{(a_n,b_n)\}$ be a sequence of disjoint open intevals in (a,b) with the following properties;
(1) $b_1 < b$ and $b_{n+1} < b_n$ for all n;
(2) $\{a_n\}$ converges to a;
(3) a is a point of dispersion of

$$O = \bigcup_{n=1}^{\infty}(a_n,b_n).$$

Define $F : [a,b] \to$ R by $F(x) = 0$ all $x \in [a,b] - O$ and

$$F(x) = \sin^2\Big(\frac{x-a_n}{b_n-a_n}\Big)\pi$$

for $x \in (a_n,b_n)$. Then it is easy to show that the function F is differentiable on $(a,b]$ and approximately differentiable at a, but F is not continuous at a. Hence $F' = F'_{ap}$ almost everywhere on $[a,b]$, but F'_{ap} is not Denjoy integrable on $[a,b]$, since F is not continuous on $[a,b]$.

To show that F'_{ap} is ap-Denjoy integrable on $[a,b]$, it is sufficient ao show that F is an AL function on $[a,b]$.Let E be a measurable set in $[a,b]$ of measure zero and let $\epsilon > 0$.

For each positive integer n, choose an open set O_n such that $E \cap [a_n,b_n] \subseteq O_n$ and $| O_n |< (b_n - a_n)\epsilon/\pi 2^{n+1}$.
For each $x \in E$, define

$$S_x = \begin{cases} [a,b] - \cup_{n=1}^{\infty}(a_n,b_n) & \text{if } x = a; \\ (b_{n+1},a_n) & \text{if } b_{n+1} < x < a_n,\ n = 1,2,\cdots; \\ (x-\rho(x,O_n^c), x+\rho(x,O_n^c)) & \text{if } a_n \le x \le b_n,\ n = 1,2,\cdots; \\ (b_1,b] & \text{if } b_1 < x \le b \end{cases}$$

Then $S = \{S_x : x \in E\}$ is a choice on E. Let $\mathcal{P}$ be a finite collection of non-overlapping tagged intervals that is E-subordinate to S. Then we have

$$
\begin{aligned}
(\mathcal{P})\sum \mid F([c,d]) \mid &= \sum_{n=1}^{\infty} \sum_{x\in(b_{n+1},a_n)} \mid F([c,d]) \mid + \sum_{n=1}^{\infty} \sum_{x\in[a_n,b_n]} \mid F([c,d]) \mid \\
&\leq \sum_{n=1}^{\infty} \sum_{x\in[a_n,b_n]} \frac{2\pi(d-c)}{b_n - a_n} \\
&\leq \sum_{n=1}^{\infty} \frac{2\pi}{b_n - a_n} \mid O_n \mid \\
&< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.
\end{aligned}
$$

Hence, F is an AL function on $[a, b]$

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