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ON THE STRUCTURE OF MINIMAL SUBMANIFOLDS IN A RIEMANNIAN MANIFOLD OF NON-NEGATIVE CURVATURE

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Abstract. Let M^n be a complete oriented non-compact minimally immersed submanifold in a complete Riemannian manifold N^{n+p} of non-negative curvature. We prove that if M is super-stable, then there are no non-trivial L^2 harmonic one forms on M. This is a generalization of the main result in [8].

1. Introduction

One of the ways to study non-compact Riemannian manifolds is to use harmonic function theory in the aspect of analysis. It is well known that in order to study structures of topology and curvature of non-compact Riemannian manifolds, harmonic function theory plays an important role (cf. [6], [7] and references are therein). On the other hand, the Hodge theory shows that harmonic differential forms are important in the topology of compact Riemannian manifolds. In case of non-compact manifolds, the Hodge theory does not work well anymore. However L^2-Hodge theory remains valid in non-compact manifolds as classical Hodge theory works well in the compact case (cf. [2]).

A minimal submanifold M is a critical point of the volume functional, while M is said to be stable if the second variation of its volume is always non-negative for any normal deformation with compact support. Since each component function of a minimal immersion in Euclidean space is a harmonic function, any minimally immersed submanifold in Euclidean space becomes automatically non-compact by the Hopf lemma. In this direction related with stable minimal hypersurfaces, there are some known results. Miyaoka ([8]) proved that if M is a complete oriented stable minimal hypersurface in a Riemannian manifold of non-negative curvature, then there are no non-trivial L^2 harmonic one forms

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on M. The first author ([14]) proved that if M is a complete oriented minimal hypersurface in Euclidean space and the total scalar curvature on M is less than $n/2$ power of the reciprocal of the Sobolev constant, then there are no non-trivial L^2 harmonic one forms on M. As a corollary, it is proved that such a manifold M should be a hyperplane.

In higher codimensional case not hypersurfaces, one can ask whether the same property on stable minimal submanifolds holds or not. In higher codimensional case, there is a little more difficulty in analyzing L^2 harmonic differential forms because there are no known effective methods in the aspect of variation. Recently, in [9], Seo proved that if M^n is an n-dimensional complete minimal submanifold in $\mathbb{R}^{n+p}$ and with flat normal bundle, then M is an n-dimensional plane. In this paper, we deal with the existence of L^2 harmonic one forms on manifolds with higher codimension and prove that if M^n is a complete non-compact super-stable minimal submanifold in a complete Riemannian manifold N^{n+p} with non-negative curvature, then there are no non-trivial L^2 harmonic one forms on M. This is a generalization of known results for stable minimal hypersurfaces.

Theorem 1.1. *Let M^n be a complete oriented non-compact super-stable minimal submanifold in a complete Riemannian manifold N^{n+p} of non-negative sectional curvature, then there are no non-trivial L^2 harmonic one forms on M.*

Corollary 1.2. *Let M^n be a complete oriented non-compact super-stable minimal submanifold in $\mathbb{R}^{n+p}$, then there are no non-trivial L^2 harmonic one forms on M.*

2. Preliminaries

Let M^n be a complete non-compact minimally immersed submanifold in a complete Riemannian manifold N^{n+p} of dimension $n+p$. It follows from the Gauss equation ([4]) that

$$\overline{\mathrm{Ric}}(X,X) - \mathrm{Ric}(X,X) = \sum_{i=1}^{p} |A_{\nu_i}X|^2 + \sum_{i=1}^{p} \langle \overline{R}(X,\nu_i)\nu_i, X\rangle, \tag{2.1}$$

where $\overline{\mathrm{Ric}}$ and Ric are the Ricci curvatures of the ambient space N and M, respectively, and $\{\nu_1, \nu_2, \dots, \nu_p\}$ is a local orthonormal frame on the normal bundle of M in N. Here A_{ν_i} is the second fundamental form defined by

$$A_{\nu_i}X = -\left(\overline{\nabla}_X \nu_i\right)^{\mathsf{T}}, \tag{2.2}$$

the tangent component of the covariant derivative. In particular, in case of hypersurfaces, the equation (2.1) becomes

$$\overline{\mathrm{Ric}}(X,X) - \mathrm{Ric}(X,X) = |AX|^2 + \langle \overline{R}(X,\nu)\nu, X\rangle,$$

where ν is the unit normal vector field on M and $A = A_\nu$.

Recall that a minimal submanifold is stable if the second variation of its volume is always non-negative for any normal variation with compact support. In case $p = 1$, i.e., if M is a hypersurface, this is equivalent that for any function $\phi \in C_0^1(M)$,

$$\int_M |\nabla\phi|^2 - (\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2)\phi^2 \geq 0, \tag{2.3}$$

where ν is an unit normal vector of M and A denotes the second fundamental form of M. However if $p > 1$, the stability is not equivalent to the inequality (2.3) anymore. Instead in case $N = \mathbb{R}^{n+p}$, for example, it follows from [10] and $\overline{\mathrm{Ric}} \equiv 0$ that if M^n is a stable minimal submanifold in a Riemannian manifold $\mathbb{R}^{n+p}$, then for any normal vector field ν and any function $\phi \in C_0^1(M)$,

$$\left.\frac{d^2\mathcal{A}}{dt^2}\right|_{t=0} \geq \int_M |\nabla\phi|^2 - |A_\nu|^2\phi^2, \tag{2.4}$$

where $\mathcal{A}$ is the volume functional with normal variation $\phi\nu$ and A_ν is defined as in the equation (2.2)

Based on the inequality (2.4), Q. Wang ([12]) introduced the concept of super-stability for minimal submanifolds in Euclidean space. Motivated by this concept, we define the notion of super-stability for minimal submanifolds in a Riemannian manifold N^{n+p} as follows.

Definition 2.1. We call a minimal submanifold M^n in N^{n+p} super-stable if for any $\psi \in C_0^1(M)$ and a normal vector ν to M,

$$\int_M |\nabla\psi|^2 - (\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2)\psi^2 \geq 0.$$

Assume $p \geq 1$ and let $e_1, \ldots, e_n, \nu_1, \ldots, \nu_p$ be a (local) orthonormal frame in N^{n+p} such that, restricted to M, the vectors $e_1, \ldots, e_n$ are tangent to M and $\nu_1, \ldots, \nu_p$ are normal to M. Then the square norm of the second fundamental form, $|A|^2$, of M is defined by

$$|A|^2 = \sum_{i=1}^p |A_{\nu_i}|^2 = \sum_{i=1}^p \sum_{j=1}^n \langle A_{\nu_i}(e_j), e_j\rangle^2.$$

Note that the definition of super-stability is exactly same as that of stability in case $p = 1$.

Now let ω be an L^2 harmonic one form on a minimally immersed submanifold M in N^{n+p}. This means

$$\Delta\omega = -(d\delta + \delta d)\omega = 0 \quad \text{and} \quad \int_M \omega \wedge *\omega = \int_M |\omega|^2 dv < \infty,$$

where $*$ denotes the Hodge star operator and dv is the volume form on M. If ω is a harmonic one form, then its dual $\omega^\sharp$ is a harmonic vector field on M in the following sense: if we choose a local frame $e_1, \ldots, e_n$ such that $\nabla_{e_i} e_j = 0$ at a point and $\omega^\sharp = \omega^i e_i$, then $\nabla_{e_i}\omega^j = \nabla_{e_j}\omega^i$ and $\nabla_{e_i}\omega^i = 0$ at the point. Note that $|\omega| = |\omega^\sharp|$. Thus from now we use confused notation for ω and $\omega^\sharp$.

The following type of Bochner formula is well known:

$$\frac{1}{2}\Delta|\omega|^2 = |\nabla\omega|^2 + \mathrm{Ric}(\omega,\omega). \tag{2.5}$$

Also we have for the Laplacian on functions

$$\Delta|\omega|^2 = 2\left(|\omega|\Delta|\omega| + |\nabla|\omega||^2\right). \tag{2.6}$$

Since $|\nabla|\omega|| \le |\nabla\omega|$ by Kato's inequality, it follows from (2.5) and (2.6) that

$$|\omega|\Delta|\omega| \ge \mathrm{Ric}(\omega,\omega). \tag{2.7}$$

3. Proofs of the theorems

In a non-compact complete Riemannian manifold, the volume growth or finiteness of the total volume plays an important role in the structure of manifolds. For instance, if M^n is a minimal hypersurface in $\mathbb{R}^{n+1}$, then it follows from [1] that M has an Euclidean volume growth. More precisely, if x is a point on M, then

$$\frac{\mathrm{vol}(B_x(r))}{r^n} \ge \omega_n,$$

where $B_x(r)$ is a geodesic ball in M, of radius r and centered at x, and ω_n is the volume of the unit ball in $\mathbb{R}^{n+1}$. In particular the volume of M is infinite and this fact implies also non-existence of L^2 harmonic one forms on M if M is stable.

In case that the codimension is greater than one, the similar property does hold for a non-compact super-stable minimal submanifold in a Riemannian manifold of non-negative sectional curvature. A complete Riemannian manifold M is said to be non-parabolic if it admits a positive Green's function. Otherwise, M is said to be parabolic. M is non-parabolic if and only if it admits a non-constant positive superharmonic function. Also if M is non-parabolic, then for any point x and a geodesic ball $B_x(r)$ on M,

$$\int_1^\infty \frac{t}{\mathrm{vol}(B_x(t))}dt < \infty. \tag{3.1}$$

In particular, such a manifold satisfying (3.1) has infinite volume.

Lemma 3.1. *Let M^n be a complete non-compact super-stable minimal submanifold in a complete Riemannian manifold N^{n+p} of non-negative sectional curvature, then M^n has infinite volume.*

Proof. It follows from (3.1) that M has infinite volume if M is non-parabolic. Thus we may assume that M is parabolic. By using the same arguments as in [3] together with the condition that M is super-stable, we can show that there exists a positive function u satisfying the equation

$$\Delta u + (\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2)u = 0$$

on M. Since $\overline{\mathrm{Ric}}(\nu,\nu) \ge 0$ and $|A|^2 \ge 0$, u is superharmonic. Consequently it follows from parabolicity of M that u is constant and so we obtain that $A = 0$.

By the Gauss equation, the sectional curvature of M, K_M, is non-negative. So it follows from [13] that the volume of M is infinite. Hence in any case, M has infinite volume. □

Next if M is a complete minimal submanifold in a complete Riemannian manifold, then the Ricci curvature of M is deeply related with the second fundamental form A of M. For example if M is a complete immersed minimal hypersurface in $\mathbb{R}^{n+1}$, then for any tangent vector field X on M, it follows from the Gauss equation that $\mathrm{Ric}(X,X) = -|AX|^2$. In fact, P. F. Leung obtained a lower bound estimate for the Ricci curvature of submanifolds minimally immersed in Riemannian manifold of non-negative sectional curvature which is a little sharp.

Lemma 3.2 ([5]). *Let M^n be a complete immersed minimal submanifold in N^{n+p} of non-negative sectional curvature. Then the Ricci curvature of M satisfies*

$$\mathrm{Ric}(M) \geq -\frac{n-1}{n}|A|^2.$$

Now we are ready to prove our main theorem.

Theorem 3.3. *Let M^n be a complete oriented non-compact super-stable minimal submanifold in a complete Riemannian manifold N^{n+p} of non-negative sectional curvature, then there are no non-trivial L^2 harmonic one forms on M.*

Proof. Assume ω is an L^2 harmonic one form on M. It follows from the inequality (2.7) and Lemma 3.2 that

$$|\omega|\Delta|\omega| \geq -\frac{n-1}{n}|\omega|^2|A|^2. \tag{3.2}$$

Fix a point of M and for $r > 0$, choose a cut-off function ψ satisfying the following properties:

(i) $0 \leq \psi \leq 1$,
(ii) $\psi|_{B(r)} \equiv 1$, $\psi|_{M-B(2r)} = 0$,
(iii) $|\nabla\psi| \leq \dfrac{2}{r}$.

Multiplying both sides by ψ^2 and integrating over M, we have

$$\int_M \psi^2|\omega|\Delta|\omega| \geq -\frac{n-1}{n}\int_M \psi^2|\omega|^2|A|^2.$$

Using integration by parts, we get

$$\begin{aligned} \frac{n-1}{n}\int_M \psi^2|\omega|^2|A|^2 \quad &\geq \quad \int_M \langle\nabla(\psi^2|\omega|), \nabla|\omega|\rangle \\ &= \quad \int_M \psi^2|\nabla|\omega||^2 + 2\psi|\omega|\langle\nabla\psi, \nabla|\omega|\rangle. \end{aligned} \tag{3.3}$$

Since M is super-stable and the sectional curvature of N is non-negative, we have for any $\psi \in C_0^2(M)$

$$\int_M |\nabla\psi|^2 \geq \int_M (\overline{\mathrm{Ric}}(\nu,\nu) + |A|^2)\psi^2 \geq \int_M |A|^2\psi^2$$

for any normal vector field ν. Replacing ψ by $\psi|\omega|$, we have

$$\int_M \psi^2|\nabla|\omega||^2 + |\omega|^2|\nabla\psi|^2 + 2\psi|\omega|\langle\nabla|\omega|, \nabla\psi\rangle \geq \int_M \psi^2|\omega|^2|A|^2. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$-\frac{1}{n}\int_M \psi^2|\nabla|\omega||^2 + \frac{n-1}{n}\int_M |\omega|^2|\nabla\psi|^2 - \frac{2}{n}\int_M \psi|\omega|\langle\nabla|\omega|, \nabla\psi\rangle \geq 0. \tag{3.5}$$

Since for any positive real number ϵ,

$$\begin{aligned} -\frac{1}{n}\int_M 2\psi|\omega|\langle\nabla\psi, \nabla|\omega|\rangle &\leq \frac{1}{n}\int_M 2\psi|\omega||\nabla\psi||\nabla|\omega|| \\ &\leq \frac{1}{n}\left\{\int_M \epsilon\psi^2|\nabla|\omega||^2 + \frac{1}{\epsilon}|\omega|^2|\nabla\psi|^2\right\}, \end{aligned}$$

we have

$$\frac{\epsilon-1}{n}\int_M \psi^2|\nabla|\omega||^2 + \frac{\epsilon(n-1)+1}{n\epsilon}\int_M |\omega|^2|\nabla\psi|^2 \geq 0. \tag{3.6}$$

Choosing $0 < \epsilon < 1$, it follows from the properties of ψ (i), (ii), (iii) that

$$\frac{\epsilon(n-1)+1}{n\epsilon}\cdot\frac{4}{r^2}\int_{B(2r)} |\omega|^2 \geq \frac{1-\epsilon}{n}\int_{B(r)} |\nabla|\omega||^2. \tag{3.7}$$

Letting $r \to \infty$, we obtain from finiteness of L^2 norm of ω

$$0 \leq \frac{1-\epsilon}{n}\int_M |\nabla|\omega||^2 \leq 0.$$

Thus $\nabla|\omega| = 0$ and so $|\omega|$ is a constant.

Since the volume of M is infinity by Lemma 2.2 and ω must vanish. $\square$

Remark 3.4. As in the case of compact manifolds, the space of L^2 harmonic differential forms does satisfy the Poincaré duality. That is, the space of L^2 harmonic k-forms is isomorphic to the space of L^2 harmonic $(n-k)$-forms, where $n = \dim(M)$. Thus it follows from Theorem 3.3 that if M^n is a complete oriented non-compact super-stable minimal submanifold in a complete Riemannian manifold N^{n+p} of non-negative sectional curvature, then there are no non-trivial L^2 harmonic $(n-1)$-forms on M.

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