Korean J. Math. **17** (2009), No. 2, pp. 147–154

A CHARACTERIZATION OF THE RIORDAN BELL SUBGROUP BY C-SEQUENCES

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Abstract. In this paper, we consider a new sequence called by the C-sequence of the Riordan array. It allows us to find a simple proof for several combinatorial identities. Further, we prove that a C-sequence characterizes Bell subgroup of the Riordan group.

1. Introduction

A Riordan array introduced by Shapiro et al. in [4] is defined by two formal power series $g(z) = \sum_{n\geq 0} g_n z^n$ and $f(z) = \sum_{n\geq 1} f_n z^n$ where $g_0 \neq 0$ and $f_1 \neq 0$. An infinite lower triangular matrix D is called a *Riordan array* if its k-th column generating function(GF) is $g(z)(f(z))^k$ for $k = 0, 1, 2, \ldots$. As usual, we write $D = (g(z), f(z))$. If we define the multiplication $*$ in the set $\mathcal{R}$ of all Riordan arrays by

$$(g(z), f(z)) * (h(z), \ell(z)) = (g(z)h(f(z)), \ell(f(z))),$$

then the set $\mathcal{R}$ forms a group. We call $\mathcal{R}$ the *Riordan group*. Obviously, the identity element of the Riordan group is $I = (1, z)$, the usual identity matrix, and the inverse of $(g(z), f(z))$ is $(\frac{1}{g(\bar{f}(z))}, \bar{f}(z))$, where $\bar{f}(z)$ is the compositional inverse of $f(z)$. That is, $f(\bar{f}(z)) = \bar{f}(f(z)) = z$.

It is well known [2] that every element of a Riordan array can be expressed as a linear combination of the elements in the preceding row. That is, there exist unique sequences $A = (a_0, a_1, a_2, \ldots)$ and $Z = (z_0, z_1, z_2, \ldots)$ with $a_0 \neq 0$, $z_0 \neq 0$ such that

(i) $d_{n+1,k+1} = \sum_{j=0}^{\infty} a_j d_{n,k+j}$, $(k, n = 0, 1 \ldots)$,
(ii) $d_{n+1,0} = \sum_{j=0}^{\infty} z_j d_{n,j}$, $(n = 0, 1, \ldots)$.

Received March 9, 2009. Revised April 22, 2009.
2000 Mathematics Subject Classification: 05A15.
Key words and phrases: Riordan matrix, C-sequences, Bell subgroup.

Such sequences are called by the *A-sequence* and the *Z-sequence* of D, respectively. Let $A(z)$ and $Z(z)$ be the generating function of A-sequence and Z-sequence of $D=(g(z),f(z))$, then we have

$$f(z)=zA(f(z)) \text{ and } g(z)=\frac{g_0}{1-zZ(f(z))}. \tag{1}$$

Recently, Cheon et al.[1] showed that, if Riordan array D has A-sequence $(a_0,a_1,a_2,\ldots)$ and Z-sequence $(z_0,z_1,z_2,\ldots)$ then the Stieltjes transform $S_D := D^{-1}\overline{D}$ of D has the following forms :

$$S_D=\begin{bmatrix} z_0 & a_0 & 0 & 0 & 0 \\ z_1 & a_1 & a_0 & 0 & 0 \\ z_2 & a_2 & a_1 & a_0 & 0 \\ z_3 & a_3 & a_2 & a_1 & a_0 \\ & & \cdots & & \end{bmatrix}$$

where $\overline{D}=D(1|*)$.

In this paper, we define a new sequence called by *C-sequence*, and we obtain a characterization of the Bell subgroup by a C-sequence. Finally, by using the concept of C-sequences we obtain a simple proof for the following identities :

(i) $B=1+2zCB$

(ii) $C=\frac{F}{1-zF}$

where C, B, and F are generating functions for the Catalan numbers, the Central binomial numbers, and the Fine numbers. In [3], (i) and (ii) are obtained from the product of Riordan arrays and their generating functions.

2. C-sequence

Let $D=(g(z),f(z))$ be a Riordan array. We say that D has a C-sequence GF $C(z)=\sum_{n\geq 0} c_n z^n$ if

$$d_{n+1,k}=d_{n,k-1}+\sum_{i\geq 0} c_i d_{n-i,k} \text{ for } n,k=0,1,2,\ldots \tag{2}$$

where $d_{n,-1}=0$.

For the convenience, we assume that $g(0)=1$ and $f'(0)=1$ for a Riordan array $D=(g(z),f(z))$.

LEMMA 2.1. *Let $D = (g(z), f(z))$ be a Riordan array with a C-sequence GF $C(z)$. Then $f(z)$ and $C(z)$ has the following relation :*

$$f(z) = \frac{z}{1 - zC(z)}. \tag{3}$$

Proof. Since $C(z) = \sum_{n\geq 0} c_n z^n$, the k-th column generating function of D can be represented by (2) as follow :

$$g(z)(f(z))^k = zg(z)(f(z))^{k-1} + c_0 zg(z)(f(z))^k + c_1 z^2 g(z)(f(z))^k + \cdots .$$

By a simple computation, we have $f(z) = z + zf(z)C(z)$. Thus

$$f(z) = \frac{z}{1 - zC(z)},$$

which completes the proof. □

THEOREM 2.2. *Let $D = (g(z), f(z))$ be a Riordan array with a C-sequence GF $C(z)$. If $A_1(z)$ is a generating function for A-sequence of D^{-1} then*

$$A_1(z) = 1 - zC(z).$$

Proof. By Lemma 2.1, we immediately obtain

$$A_1(z) = \frac{z}{f(z)} = \frac{z}{\frac{z}{1-zC(z)}} = 1 - zC(z).$$

□

For some Riordan array $D = [d_{n,k}]_{n,k\geq 0}$, the C-Stieltjes Transform C_D is defined by the solution of

$$C_D D = \overline{D} - D', \tag{4}$$

where $\overline{D} = D(1|*)$, $D' = \left[d'_{n,k}\right]_{n,k\geq 0}$ such that

$$d'_{n,k} = d_{n,k-1} \quad \text{for} \quad k \geq 1 \quad \text{and} \quad d'_{n,0} = 0.$$

Since D is nonsingular, it follows from (4) that

$$C_D = \left(\overline{D} - D'\right) D^{-1}. \tag{5}$$

THEOREM 2.3. *Let D be a Riordan array. Then D has a C-sequence GF $C(z) = \sum_{n\geq 0} c_n z^n$ if and only if C_D has the following form:*

$$C_D = \begin{bmatrix} c_0 & 0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & 0 \\ & & \cdots & & \end{bmatrix}.$$

Proof. Note that D has a C-sequence GF $C(z) = \sum_{n\geq 0} c_n z^n$ if and only if (2) holds for all $n, k \geq 0$. Hence it can be rearranged in the following matrix form :

$$\begin{bmatrix} d_{1,0} & 1 & 0 & 0 & 0 \\ d_{2,0} & d_{2,1} & 1 & 0 & 0 \\ d_{3,0} & d_{3,1} & d_{3,2} & 1 & 0 \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} & 1 \\ & & \cdots & & \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & d_{1,0} & 1 & 0 & 0 \\ 0 & d_{2,0} & d_{2,1} & 1 & 0 \\ 0 & d_{3,0} & d_{3,1} & d_{3,2} & 1 \\ & & \cdots & & \end{bmatrix} + \begin{bmatrix} c_0 & 0 & 0 & 0 & 0 \\ c_1 & c_0 & 0 & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & 0 \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ d_{1,0} & 1 & 0 & 0 & 0 \\ d_{2,0} & d_{2,1} & 1 & 0 & 0 \\ d_{3,0} & d_{3,1} & d_{3,2} & 1 & 0 \\ & & \cdots & & \end{bmatrix}.$$

The above matrix equation implies that $\overline{D} = D' + C_D D$. Hence the proof is completed. □

3. A characterization of the Riordan Bell subgroup

The *Bell subgroup* [3] is the set of all Riordan arrays of the form $D = (\frac{f(z)}{z}, f(z))$. We call the *Bell matrix* the Riordan array in the Bell subgroup. We now characterize the Bell subgroup by a C-sequence.

THEOREM 3.1. *Let $D = (g(z), f(z))$ be a Riordan array. Then D is a Bell matrix if and only if D has a C-sequence.*

Proof. Let $D = (g(z), f(z))$ be a Riordan array with C-sequence GF $C(z) = \sum_{n\geq 0} c_n z^n$.
By Lemma 2.1, we have

$$f(z) = \frac{z}{1 - zC(z)}.$$

Now consider (2) for $k = 0$. There exists

$$d_{n+1,0} = \sum_{i\geq 0} c_i d_{n-i,0}.$$

Since $g(0) = 1$, the 0-th column generating function of D can be represented as :

$$g(z) = 1 + c_0 z g(z) + c_1 z^2 g(z) + \cdots .$$

By a simple computation, we have

$$g(z) = 1 + zg(z)C(z). \tag{6}$$

Thus

$$g(z) = \frac{1}{1 - zC(z)} = \frac{f(z)}{z},$$

which implies that D is a Bell matrix. Conversely, let $D = (\frac{f(z)}{z}, f(z))$. Then there exist $C(z)$ such that

$$C(z) = \frac{f(z) - z}{zf(z)}. \tag{7}$$

By (7), $f(z) = z + zC(z)f(z)$, it implies D has the C-sequence GF $C(z)$. Therefore the proof is completed. □

Theorem 3.2. *Let $D_1 = (\frac{f(z)}{z}, f(z)), D_2 = (\frac{F(z)}{z}, F(z))$ be Riordan arrays with C-sequences GF $C_1(z)$ and $C_2(z)$, respectively. Then $D_1 D_2$ has the C-sequence GF*

$$C(z) = C_1(z) + C_2(f(z)).$$

Proof. By the matrix multiplication of Riordan arrays, we have

$$D_1 D_2 = (\frac{f(z)}{z}, f(z))(\frac{F(z)}{z}, F(z)) = (\frac{F(f(z))}{z}, F(f(z))).$$

By Lemma 2.1, we have

$$f(z) = \frac{z}{1 - zC_1(z)} \text{ and } F(z) = \frac{z}{1 - zC_2(z)}.$$

Hence,

$$\begin{aligned} F(f(z)) &= \frac{f(z)}{1-f(z)C_2(f(z))} = \frac{\frac{z}{1-zC_1(z)}}{1-\left(\frac{z}{1-zC_1(z)}\right)C_2(f(z))} \\ &= \frac{z}{1-z\left(C_1(z)+C_2(f(z))\right)}. \end{aligned}$$

It follows that $C(z) = C_1(z) + C_2(f(z))$. The proof is completed. □

4. Applications of C-sequence

In this section, we obtain a simple proof for two well-known identities related to the Catalan numbers. Let

$$(8) \quad C = \frac{1-\sqrt{1-4z}}{2z}, \; B = \frac{1}{\sqrt{1-4z}}, \text{ and } F = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$$

be generating functions for the Catalan, Central binomial, and Fine numbers, respectively. The concept of C-sequence allows us to obtain a simple proof for (i) and (ii).

Theorem 4.1. *Let C, B, and F be generating functions given in (8). Then*

(i) $B = 1 + 2zCB$
(ii) $C = \frac{F}{1-zF}$

Proof. (i) Consider the Riordan array

$$D = \left(\frac{1}{\sqrt{1-4z}}, \frac{z}{\sqrt{1-4z}}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 & 0 \\ 20 & 16 & 6 & 1 & 0 & 0 \\ 70 & 64 & 30 & 8 & 1 & 0 \\ 252 & 256 & 140 & 48 & 10 & 1 \\ & & \cdots & & & \end{bmatrix}.$$

Since D is the Bell matrix, by Theorem 3.1 D has a C-sequence. Hence by Lemma 2.1, we have

$$(9) \qquad C(z) = \frac{f(z)-z}{zf(z)} = \frac{zB-z}{z^2B} = \frac{1-\sqrt{1-4z}}{z} = 2C.$$

Since $g(z) = B$, it immediately follows from (6) and (9) that

$$B = 1 + 2zCB.$$

(ii) We now consider the Riordan array

$$D = (1 + zC, z + z^2C) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 5 & 6 & 6 & 4 & 1 & 0 \\ 14 & 15 & 13 & 10 & 5 & 1 \\ & & \cdots & & & \end{bmatrix}.$$

It is easy to show that F is the generating function for the C-sequence of D. By Lemma 2.1, we have

$$C(z) = \frac{f(z) - z}{zf(z)} = \frac{(z + z^2C) - z}{z(z + z^2C)} = \frac{C}{1 + zC} = \frac{1}{z}\frac{1 - \sqrt{1 - 4z}}{3 - \sqrt{1 - 4z}} = F.$$

Hence it follow from (6) that

$$1 + zC = 1 + zF(1 + zC),$$

which implies

$$C = \frac{F}{1 - zF}.$$

The proof is completed. □

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