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SOME RESULTS ON π-REGULARITY AND πS-UNITALITY

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Abstract. In this paper, we begin with to show the characterization of regularity and S-unitality in near-rings, also consider their application.

Next, we introduce more general concepts of regularity and S-unitality, that is, π-regularity and πS-unitality and then give some examples in near-rings, also investigate their characterization and properties.

1. Introduction

The concept of Von Neumann regularity of near-rings have been studied by many authors Beidleman, Choudhari, Goyal, Heatherly, Hongan, Ligh, Mason and Murty. Their main results are suggested in the book of Pilz [12].

The Von Neumann regularity of rings and its generalization were studied by Fisher, Snider, Hirano, Tominaga, Savaga, Li, Schein and Ohori. In 1985, Ohori investigated the characterization of π-regularity and strong π-regularity of rings.

A near-ring R is an algebraic system $(R,\ +,\ \cdot)$ with two binary operations $+$ and $\cdot$ such that $(R,\ +)$ is a group (not necessarily abelian) with neutral element 0, $(R,\ \cdot)$ is a semigroup and $(a+b)c = ac+bc$ for all a, b, c in R. If R has a unity 1, then R is called *unitary*. A near-ring R with the extra axiom $a0 = 0$ for all $a \in R$ is said to be *zero symmetric*. An element d in R is called *distributive* if $d(a+b) = da+db$ for all a and b in R.

We will use the following notations: Given a near-ring R, $R_0 = \{a \in R \mid a0 = 0\}$ which is called the *zero symmetric part* of R, $R_c = \{a \in$

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$R \mid a0 = a\}$ which is called the *constant part* of R. The set of all distributive elements in R is denoted by R_d.

Obviously, we see that R_0 and R_c are subnear-rings of R, but R_d is a semigroup under multiplication. Clearly, near-ring R is zero symmetric, in case $R = R_0$ also, in case $R = R_c$, R is called a *constant* near-ring and in case $R = R_d$, R is called a *distributive* near-ring.

For notation and basic results, we shall refer to Pilz [12].

2. Results

For a near-ring R, an element $a \in R$ is called *nilpotent* if there exists a positive integer n such that $a^n = 0$. Also, a subset $S \subset R$ is called *nilpotent* if there exists a positive integer n such that $S^n = 0$ and $S \subset R$ is called *nil* if every element in S is nilpotent, which are introduced in [12]. Clearly, every nilpotent subset of R is nil.

A (two-sided) *R-subgroup* of R is a subset H of R such that (i) $(H, +)$ is a subgroup of $(R, +)$, (ii) $RH \subset H$ and (iii) $HR \subset H$. If H satisfies (i) and (ii) then it is called a *left R-subgroup* of R. If H satisfies (i) and (iii) then it is called a *right R-subgroup* of R. In case, $(H, +)$ is normal in above, we say that *normal R-subgroup*, *normal left R-subgroup* and *normal right R-subgroup* instead of R-subgroup, left R-subgroup and right R-subgroup, respectively. Note that normal right R-subgroups of R are the same of right ideals of R.

Also, a subset H of R together with (i) $RH \subset H$ and (ii) $HR \subset H$ is called an *R-subset* of R. If this H satisfies (i) then it is called a *left R-subset* of R, and H satisfies (ii) then it is called a *right R-subset* of R.

Also, we say that R is *reduced* if R has no nonzero nilpotent elements, that is, for each a in R, $a^n = 0$, for some positive integer n implies $a = 0$. McCoy proved that R is reduced iff for each a in R, $a^2 = 0$ implies $a = 0$.

A near-ring R is called *(Von Neumann) regular* if for any element $a \in R$, there exists an element x in R such that $a = axa$. Such an element a is called *regular.*

A near-ring R is called *left S-unital (resp. right S-unital)* if for each a in R, $a \in Ra$ (resp. $a \in aR$), such an element a is called *left S-unital (resp. right S-unital).*

R is called *S-unital*, if R is both left S-unital and right S-unital. Every near-ring with left identity or identity is clearly left S-unital. Also every regular near-ring is S-unital.

We shall use the phrase "$\forall a \in R$, $\exists e^2 = e \in R$" instead of "for every element a in R, there exists some element $e^2 = e$ in R" for convenience in the following.

Now, we begin with to show the characterization of regularity and S-unitality in near-rings, also consider their application.

Proposition 1. *Let R be a near-ring. Then R is regular if and only if R has the condition "$\forall a \in R$, $\exists e^2 = e \in R$ such that $Ra = Re$" and R is left S-unital .*

Proof. Suppose that R is regular. Then for any $a \in R$, there exists $x \in R$ such that $a = axa$. Since xa and ax are idempotents in R, taking $xa = e$, $Ra = Raxa = Rae \subset Re$ and $Re = Rxa \subset Ra$. Hence $Ra = Re$. Obviously, R is left S-unital .

Conversely, assume that R has the given condition "$\forall a \in R$, $\exists e^2 = e \in R$ such that $Ra = Re$" and R is left S-unital. Then S-unitality implies that $a \in Ra = Re$, so that there exists $y \in R$ such that $a = ye$. From this condition, we see that $e = ee \in Re = Ra$, so that there exists $x \in R$ such that $e = xa$. Thus we obtain that $a = ye = yee = yexa = axa$. Consequently, R is regular. □

Corollary 2 [1], [8]. *Let R be a near-ring with identity. Then R is regular if and only if R has the condition "$\forall a \in R$, $\exists e^2 = e \in R$ such that $Ra = Re$".*

The following statements are an application of Proposition 1.

Proposition 3. *Every regular near-ring R has no non-zero nil left R-subset.*

Proof. Let R be a regular near-ring and K be a nil left R-subset of R. It suffices to show that $K = \{0\}$. Indeed, let $a \in K$. Since R is regular, R has the condition "$\exists e^2 = e \in R$ such that $Ra = Re$" and R is left S-unital, by Proposition 1. Since K is a left R-subset, we have that $a \in Ra \subset K$. On the other hand, since K is nil, there exists positive integer m, such that $a^m = 0$.

Next, from the condition $e = ee \in Re = Ra \subset K$, also there exists positive integer n, such that $e = e^n = 0$. From the above two conditions, we have $a \in R0$, so that $a = r0$ for some $r \in R$. Consequently, $a = r0 = (r0)^m = a^m = 0$. That is, $K = \{0\}$. □

Corollary 4 [1]. *Every regular near-ring R with identity has no non-zero nil left R-subgroup.*

From now on, we introduce more general concepts of regularity and S-unitality and then give some examples in near-rings, also investigate their characterization and properties.

A near-ring R is said to be *π-regular* if for each element $a \in R$, there exists a positive integer n such that a^n is a regular element, that is, $a^n = a^n x a^n$, for some $x \in R$. Such an element a is called *π-regular*.

Every regular near-ring is π-regular, but not conversely as following examples.

Examples 5.

(1) *Let $R = \{0, a, b, c\}$ be an additive Klein 4-group. This is a near-ring with the following multiplication table (p. 408 [12]):*

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	a	c	b
c	0	a	b	c

This near-ring R is a zero-symmetric near-ring with identity c. Moreover, R is π-regular, but not regular. Indeed, $0 = 0a0$, $a^2 = a^2ba^2$, $b^4 = b^4ab^4$, $c^2 = c^2cc^2$, but a is not a regular element.

(2) *Let $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ be an additive group of integers modulo 4 and define multiplication as follows:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	3	0	1
2	0	2	0	2
3	0	1	0	3

This near-ring R is a zero-symmetric near-ring without identity. Moreover, R is π-regular, but not regular. Indeed, $0 = 0a0$, $a^2 = a^2ba^2$, $b^4 = b^4ab^4$, $c^2 = c^2cc^2$, but a is not a regular element.

Finally, we can define a general concept of left S-unitality.

A near-ring R is called *left πS-unital (resp. right πS-unital)* if for each a in R, there exists a positive integer n such that a^n is a S-unital element, that is, $a^n \in Ra^n$ (resp. $a^n \in a^nR$), such an element a is called *left πS-unital (resp. right πS-unital)*.

R is called *πS-unital*, if R is both left πS-unital and right πS-unital.

Also, every left S-unital (resp. right S-unital) near-ring is left πS-unital (resp. right πS-unital), but not conversely as following remarks.

Remarks 6. *In Examples 5 (1), clearly, R is a left S-unital near-ring. But in Examples 5 (1), R is left πS-unital, indeed, $0 = 1 \cdot 0 = 2 \cdot 0 = 3 \cdot 0 \in R0$, $1 = 3 \cdot 1 \in R1$, $2^2 = 0 = 0 \cdot 2^2 \in R2^2$ and $3 = 3 \cdot 3 \in R3$. But this near-ring R is not S-unital, because 2 is not a left S-unital element.*

The statements Proposition 1 and Corollary 2 can be extended on π-regular and left πS-unital near-rings as following.

Theorem 7. *Let R be a near-ring. Then R is π-regular if and only if R has the condition "$\forall a \in R$, $\exists e^2 = e \in R$ and $\exists n \in Z^+$ such that $Ra^n = Re$", and R is left πS-unital.*

Proof. Suppose that R is π-regular. Then for any $a \in R$, there exist a positive integer n and $x \in R$ such that $a^n = a^nxa^n$. This equality implies that $a^n \in Ra^n$. Hence R is left πS-unital.

Next, since xa^n and a^nx are idempotent elements in R, putting $xa^n = e$, $Ra^n = Ra^nxa^n \subset Rxa^n = Re$ and $Re = Rxa^n \subset Ra^n$. Hence $Ra^n = Re$.

Conversely, assume that R has the given condition "$\forall a \in R$, $\exists e^2 = e \in R$ and $\exists n \in Z^+$ such that $Ra^n = Re$", and R is left πS-unital. Then the πS-unitality implies that $a^n \in Ra^n = Re$, so that there exists $y \in R$ such that $a^n = ye$.....(1). On the other hand, we see that $e = ee \in Re = Ra^n$, so that there exists $x \in R$ such that $e = xa^n$.....(2). From this two conditions (1) and (2), we obtain that $a^n = ye = yee = yexa^n = a^n xa^n$. Therefore, R is a πS-regular near-ring. □

Corollary 8. *Let R be a near-ring with identity. Then R is π-regular if and only if R has the condition "$\forall a \in R$, $\exists e^2 = e \in R$ and $\exists n \in Z^+$ such that $Ra^n = Re$".*

Remarks 9. *Proposition 3 and Corollary 4 do not hold in π-regular near-rings. In Examples 5 (1), $\{0, a\}$ is a non-zero R-subgroup which is nil.*

For any near-ring R, the center of R is denoted by the set

$$Z(R) = \{x \in R \mid ax = xa, \forall a \in R\}.$$

Note that when R is distributive, that is, $R = R_d$, $Z(R)$ is a subnear-ring of R. In Appendix of (pp. 421-424 [12]), we can find some distributive π-regular near-rings which are not additive abelian .

Theorem 10. *The center of a distributive π-regular near-ring is also π-regular.*

Proof. Let R be a distributive π-regular near-ring, and let $a \in Z(R)$. Then $\exists x \in R$ and $\exists n \in Z^+$ such that $a^n = a^n xa^n$. From this equality, we have that $a^n = a^n xa^n = a^n xa^n xa^n$. We will show that $xa^n x \in Z(R)$. Then our claim is done. Indeed, let $t \in R$. Since $a \in Z(R)$, also $a^n \in Z(R)$. Thus we can deduce that

$$\begin{aligned} t(a^n x) &= (ta^n)x = (a^n t)x = a^n(tx) = a^n xa^n(tx) \\ &= a^n x(tx)a^n = a^n(xtx)a^n \end{aligned}$$

and

$$\begin{aligned} (a^n x)t &= (xa^n)t = x(a^n t) = x(ta^n) = xt(a^n xa^n) \\ &= (xta^n)xa^n = a^n(xtx)a^n. \end{aligned}$$

Hence $a^n x \in Z(R)$. Similarly, we can obtain that $xa^n \in Z(R)$.
Thus,

$$t(xa^n x) = t(a^n xx) = (ta^n x)x = (a^n xt)x = x(a^n t)x$$

and

$$(xa^n x)t = x(a^n x)t = xt(a^n x) = x(ta^n)x = x(a^n t)x.$$

This implies that $t(xa^n x) = (xa^n x)t$, that is, $xa^n x \in Z(R)$. Hence $Z(R)$ is π-regular. □

COROLLARY 11. *The center of a distributive regular near-ring is also regular.*

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