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THE IDEMPOTENT RELATION AND THE PROOF OF URYSOHN'S LEMMA

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Abstract. The Urysohn's lemma which is crucial tool for the study of the metrization problem is proved in the sense of set-theoretic concept, namely, by the idempotent relation defined on a given topology.

1. Introduction

It is well known in the set-theoretic topology that the lemma of Urysohn is a crucial tool to study the metrization or pseudometrization problems. The basic ideas of both distance functions which induce the given topology are to define a local distance function. In concern with a psuedometrization problem the local pseudometric is defined as follows.

If U is an open set of a topological space (X, τ) and f is a continuous function from X into $\mathbb{R}$ with the zero set $X \setminus U$, i.e. $f(X \setminus U) = \{0\}$, then the function $d_f : X \times X \to \mathbb{R}^+ \cup \{0\}, (x, y) \mapsto |f(x) - f(y)|$ is a pseudometric.

The Urysohn's lemma guaranties that if a given topological space is T_4 space, there exists a function f mentioned above. But in conventional literatures([1],[2],[3]) the constructing process of a continuous function with the given closed sets is not so easy to see because of the complicate structural relations between the sets of the space. In this sense we will describe a method for the proof of the lemma introducing a set-theoretic concept, namely an idempotent relation on a given topology, which makes possible to establish the clear relationship between the ordered sets $\mathbb{Q} \bigcap [0, 1]$ relative to $\leq$ and the given topology relative to the idempotent relation $\triangleright$.

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For this purpose the three steps are given in following ways.

1. step. Establishment of the idempotent relation on a topology τ.

2. step. Defining a function which preserves the relations between an ordered structure of $\mathbb{Q}\bigcap[0,1]$ and a set X with an idempotent relation.

3. step. Construction of a continuous function on a T_4 space with idempotent relation on τ.

2. Idempotent relation

DEFINITION 2.1. *A relation $\mathcal{R}$ is said to be idempotent if $\mathcal{R}\cdot\mathcal{R}=\mathcal{R}$.*

REMARK 2.2. *The equation $\mathcal{R}\cdot\mathcal{R}=\mathcal{R}$ is including two statements, $\mathcal{R}\cdot\mathcal{R}\subset\mathcal{R}$ and $\mathcal{R}\cdot\mathcal{R}\supset\mathcal{R}$.*

$\mathcal{R}\cdot\mathcal{R}\subset\mathcal{R}$ means that $\mathcal{R}$ is transitive and $\mathcal{R}\cdot\mathcal{R}\supset\mathcal{R}$ means that $\mathcal{R}$ satisfies the following intermediate condition, namely, for all x,y with $(x,y)\in\mathcal{R}$ there exists z with $(x,z)\in\mathcal{R}$ and $(z,y)\in\mathcal{R}$.

EXAMPLE 2.3. (a) *Every equivalent relation on a set is idempotent.*

(b) *Every ordering on a set is idempotent.*

(c) *The relation '$\leq$' on $\mathbb{Q}$ is idempotent.*

(d) *The relation '$<$' on a finite subset of $\mathbb{R}$ with at least two elements is not idempotent.*

Proof. Since (b),(c),(d) are obvious, we prove only (a). Let $\mathcal{R}$ be an equivalent relation on set X. We show that $\mathcal{R}\cdot\mathcal{R}=\mathcal{R}$

"$\subseteq$" : Let $x,y\in X$ and $(x,y)\in\mathcal{R}$. Since $\mathcal{R}$ is reflexive, $(y,y)\in\mathcal{R}$. Hence by the transitivity of $\mathcal{R}$ $(x,y)\in\mathcal{R}\cdot\mathcal{R}$.

"$\supseteq$" : Let $(x,y)\in\mathcal{R}\cdot\mathcal{R}$. Then there is $z\in X$ such that $(x,z),(z,y)\in\mathcal{R}$. By the transitivity of $\mathcal{R}$ $(x,y)\in\mathcal{R}$. □

LEMMA 2.4. *Let $\mathcal{R}$ be an idempotent relation on a set X. Let E be a finite subset of $\mathbb{R}$ which has at least 2 elements and $\phi:E\longrightarrow X$ a function which preserves the relation '$<$' on E in X i.e., if $x,y\in E$ with $x<y$ then $(\phi(x),\phi(y))\in\mathcal{R}$.*

Let $z\in\mathbb{R}\backslash E$ with $\min E<z<\max E$. Then there is a function $\bar{\phi}:E\cup\{z\}\rightarrow X$ such that $\bar{\phi}_{|E}=\phi$ which preserves the relation $<$ on $E\cup\{z\}$ in X.

Proof. Let

$$l:=\max(E\cap(-\infty,z)),\ r:=\min(E\cap(z,\infty)).$$

Then $l < z < r$ and $x \leq l$ or $x \geq r$ for all $x \in E$. By assumption we have $(\phi(l), \phi(r)) \in \mathcal{R}$. Since $\mathcal{R}$ is an idempotent relation on X, there is $c \in X$ such that

$$(\phi(l), c) \in \mathcal{R} \text{ and } (c, \phi(r)) \in \mathcal{R}.$$

Define

$$\bar{\phi} := E \cup \{z\} \longrightarrow X, x \mapsto \begin{cases} \phi(x) \text{ if } x \neq z \\ c \text{ if } x = z \end{cases}$$

Then it is obvious that $\bar{\phi}_{|E} = \phi$. It remains to show that $(\bar{\phi}(x), \bar{\phi}(y)) \in \mathcal{R}$ for all $x, y \in E \cup \{z\}$ with $x < y$.

1. case. $x, y \in E$. Since $\bar{\phi}_{|E} = \phi$, $(\bar{\phi}(x), \bar{\phi}(y)) = (\phi(x), \phi(y)) \in \mathcal{R}$.

2. case. $x = z,\ y \in E$. Then $x < r \leq y$.

Hence

$$(\bar{\phi}(x), \bar{\phi}(r)) = (c, \phi(r)) \in \mathcal{R}$$

and

$$(\bar{\phi}(r), \bar{\phi}(y)) = (\phi(r), \phi(y)) \in \mathcal{R}.$$

Since $\mathcal{R}$ is an idempotent relation, $(\bar{\phi}(x), \bar{\phi}(y)) \in \mathcal{R}$. The other cases $x \in E,\ y = z$ could be analogously shown. □

Theorem 2.5. *Let $\mathcal{R}$ be an idempotent relation on a set X. Let $x, y \in X$ with $(x, y) \in \mathcal{R}$. Then there is a function $\phi : [0,1] \cap \mathbb{Q} \to X$ which preserves the relation '$<$' on $\mathbb{Q}$ such that $\phi(0) = x$ and $\phi(1) = y$.*

Proof. Let $\mathcal{R}$ be an idempotent relation on X and $x, y \in X$ such that $(x, y) \in \mathcal{R}$. Let

$$q : \mathbb{N} \cup \{0\} \to [0,1] \cap \mathbb{Q}$$

be a bijection with $q_0 := 0, q_1 := 1$. Let $E_n := \{q_0, q_1, q_2, \cdots, q_n\}$ for every $n \in \mathbb{N}$. Define $\phi_1 : E_1 \to X, q_0 = 0 \mapsto x, q_1 = 1 \mapsto y$. Then ϕ_1 preserves the relation $<$ in X. Besides, it holds that for every $n \in \mathbb{N}$,

$$q_{n+1} \in ([0,1] \cap \mathbb{Q}) \backslash E_n$$

and

$$0 = \min E_n < q_{n+1} < \max E_n = 1.$$

Then by lemma 2.1 a function $\phi_n : E_n \to X$ preserving $<$ on E_n in X can be extended to ϕ_{n+1} on $E_{n+1} = E_n \cup \{q_{n+1}\}$. By the successive process starting from E_1 we can define $\phi_n : E_n \to X$ such that

$$\phi_m = \phi_n|_{E_m} \text{ for all } m, n \in \mathbb{N} \cup \{0\}, m \leq n$$

and it preserves $<$ in X. Let $\phi = \bigcup_{n\in\mathbb{N}} \phi_n$. This is a function from $[0,1]\cap\mathbb{Q} = \bigcup_{n\in\mathbb{N}} E_n$ into X with $\phi(0)=x,\ \phi(1)=y$ and it preserves $<$ in X. □

Definition 2.6. *Let (X,τ) be a topological space and $U,V\in\tau$. We say that U is strongly contained in V denoted by $(U,V)\in\triangleright$ if $\overline{U}\subseteq V$.*

Theorem 2.7. *In every T_4 space (X,τ) $\triangleright$ is an idempotent relation on τ.*

Proof. Let (X,τ) be a T_4 space. It is to show that $\triangleright = \triangleright\cdot\triangleright$.

" $\subseteq$ " : Let $U,V\in\tau$ and $(U,V)\in\triangleright$. Then $\overline{U}\cap X\backslash V=\emptyset$. Since (X,τ) is a T_4 space, there are $W,W'\in\tau$ such that $\overline{U}\subseteq W$ and $X\backslash V\subseteq W'$ and $W\cap W'=\emptyset$. Then $X\backslash W'\subseteq V$ and $W\subseteq X\backslash W'$. Hence $\overline{U}\subseteq X\backslash W'$ and $\overline{X\backslash W'}\subseteq V$. Thus $(U,V)\in\triangleright\cdot\triangleright$.

" $\supseteq$ " : Let $U,V\in\tau$ and $(U,V)\in\triangleright\cdot\triangleright$. Then there is $W\in\tau$ such that $(U,W)\in\triangleright$ and $(W,V)\in\triangleright$. Then $\overline{U}\subseteq W$ and $\overline{W}\subseteq V$. Thus $(U,V)\in\triangleright$. □

3. The Urysohn's lemma.

In this section we will give the main proof of Urysohn's lemma with the theories of the idempotent relation investigated above.

Lemma 3.1. *(Urysohn) Let A,B be closed and disjoint subsets of a T_4 space (X,τ). Then there is a continuous function $f: X\to[0,1]$ such that $f(A)=\{0\}, f(B)=\{1\}$.*

Proof. First we define a function f from (X,τ) into $[0,1]$ as follows. By T_4 property of (X,τ) there are $U,V\in\tau$ such that

$$A\subseteq U, B\subseteq V \text{ and } U\cap V=\emptyset.$$

Since $U\subseteq X\backslash V\subseteq X\backslash B$ and $X\backslash V$ is closed, $\overline{U}\subseteq X\backslash B$. By the definition 2.6 and theorem 2.7, $(U,X\backslash B)\in\triangleright$ where $\triangleright$ is an idempotent relation on τ. By theorem 2.5, there is a function

$$\phi:[0,1]\cap\mathbb{Q}\to\tau, q\mapsto U_q, U_0=U,\ U_1=X\backslash B$$

and

$$(U_q,U_r)\in\triangleright \text{ for all } q,r\in[0,1]\cap\mathbb{Q}, q<r.$$

For every $x \in X$, put

$$K(x) := \{q | q \in [0,1] \cap \mathbb{Q}, x \in U_q\} \cup \{1\}.$$

and define

$$f : X \to [0,1], x \mapsto \inf K(x).$$

As a next step we show that

(a) $r \in K(x)$ implies $f(x) \leq r$ and

(b) $r \notin K(x)$ implies $r \leq f(x)$ for all $r \in [0,1] \cap \mathbb{Q}$ and $x \in X$.

Since (a) is obvious, just (b) will be proved.

(b) Suppose that $r > f(x)$. Then there is $q \in K(x)$ such that $(U_q, U_r) \in \triangleright$. It means that

$$x \in U_q \subseteq \overline{U_q} \subseteq U_r.$$

Thus $x \in U_r$ and $r \in K(x)$ which is contrary to assumption. Finally it remains to show that

(i) $f(a) = 0$ for all $a \in A$.

(ii) $f(b) = 1$ for all $b \in B$.

(iii) f is continuous.

(i) Let $a \in A$. By $a \in A \subseteq U = U_0$, $f(a) = 0$.

(ii) Let $b \in B$. Then $b \notin X \backslash B = U_1$. Since $U_q \subseteq U_1$ for all $q \in [0,1] \cap \mathbb{Q}$ with $q < 1, b \notin U_q$ for all $q < 1$. Hence $f(b) = 1$.

(iii) Let $x \in X$. It is to show that for given $\epsilon > 0$ there is $V \in \tau, x \in V$ such that $|f(x) - f(y)| < \epsilon$ for all $y \in V$.

1. case : $f(x) = 1$. Choose $q \in [0,1] \cap \mathbb{Q}$ such that $1 - \epsilon < q < 1$. Then by (a)

$$\frac{q+1}{2} < 1 = f(x) \text{ and } \frac{q+1}{2} \notin K(x), \text{i.e., } x \notin U_{\frac{q+1}{2}}.$$

By $q < \frac{q+1}{2}, (U_q, U_{\frac{q+1}{2}}) \in \triangleright$. Hence

$$\overline{U}_q \subseteq U_{\frac{q+1}{2}}.$$

Then $x \notin \overline{U_q}$. Put

$$V := X \backslash \overline{U_q}.$$

Then $x \in V$ and $V \in \tau$. Let $y \in V$. Then $y \notin \overline{U_q}$ and also $y \notin U_q$. Thus $q \notin K(y)$ and $q \leq f(y)$ by (b). It follows that

$$|f(x) - f(y)| = 1 - f(y) \leq 1 - q < \epsilon.$$

2. case : $f(x) = 0$. Choose $q \in [0,1] \cap \mathbb{Q}$ with $0 < q < \epsilon$. Then $f(x) = 0 < q$ and $q \in K(x)$ by (b), i.e., $x \in U_q$ Choose $V := U_q$. Then

$x \in V, V \in \tau$. Let $y \in V$. Then $q \in K(y)$ and by (a) $f(y) \leq q$ so that we have

$$|f(x) - f(y)| = f(y) \leq q < \epsilon.$$

3. case : $0 < f(x) < 1$. Choose $l', l, r \in (0,1) \cap \mathbb{Q}$ with

$$f(x) - \frac{\epsilon}{2} < l < l' < f(x) < r < f(x) + \frac{\epsilon}{2}.$$

Since $l' < f(x)$, $l' \notin K(x)$ by (a), so that $x \notin U_{l'}$. By $l < l'$, $\overline{U_l} < U_{l'}$ so that $x \notin \overline{U_l}$. Since $f(x) < r$, $r \in K(x)$ by (b) so that $x \in U_r$ (since $r \neq 1$).

Choose $V := U_r \backslash \overline{U_l}$. Then $x \in U_r \backslash \overline{U_l}$ and $V \in \tau$. Let $y \in V$. Then $y \in U_r$ and also $r \in K(y)$, i.e., $f(y) \leq r$ by (a).

By $y \notin \overline{U_l}$, we have $y \notin U_l$. Hence $l \leq f(y)$ by (b). It holds that

$$l \leq f(x) \leq r, \ l \leq f(y) \leq r$$

from which follows

$$|f(x) - f(y)| \leq r - l < \epsilon.$$

□

Lemma 3.2. *(Urysohn) Let (X, τ) be a topological space. If to every two closed and disjoint subsets A, B there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}, f(B) = \{1\}$ then (X, τ) is a T_4 space.*

Proof. Let $A, B \subseteq X$ be closed and disjoint. It is to show that there are $U, V \in \tau$ such that

$$A \subseteq U, B \subseteq V \text{ and } U \cap V = \emptyset.$$

Let $f : X \to [0,1]$ be a continuous function such that $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$. Put

$$U := f^{-1}([0, \frac{1}{2})), V := f^{-1}((\frac{1}{2}, 1]).$$

Then U, V are open as the inverse images of open sets on $[0,1]$ and $A \subseteq U, B \subseteq V$. Moreover

$$U \cap V = f^{-1}([0, \frac{1}{2})) \cap f^{-1}((\frac{1}{2}, 1]) = f^{-1}([0, \frac{1}{2}) \cap (\frac{1}{2}, 1]) = f^{-1}(\emptyset) = \emptyset.$$

□

From lemma 3.1 and lemma 3.2 we have the main lemma of Urysohn.

LEMMA 3.3. *(Urysohn) A topological space is T_4 space if and only if to every two disjoint and closed subsets A, B of X there is a continuous function $f : X \to [0,1]$ such that $f(A) = \{0\},\ f(B) = \{1\}$.*

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