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PALAIS-SMALE CONDITION FOR THE STRONGLY DEFINITE FUNCTIONAL

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Abstract. Let Ω be a bounded subset of R^n with smooth boundary and H be a Sobolev space $W_0^{1,2}(\Omega)$. Let $I \in C^{1,1}$ be a strongly definite functional defined on a Hilbert space H. We investigate the conditions on which the functional I satisfies the Palais-Smale condition. Palais-Smale condition is important for determining the critical points for I by applying the critical point theory.

1. Introduction

Let Ω be a bounded subset of R^n with smooth boundary. Let L be an elliptic linear differential operator defined by

$$-L = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}.$$

Let H be a Sobolev space $W_0^{1,2}(\Omega)$ with the norm

$$\|u\| = [\int_\Omega Lu \cdot u dx]^{\frac{1}{2}}.$$

Let I be a strongly definite functional defined on H which is of the form

$$I(u) = \int_\Omega [\frac{1}{2} Lu \cdot u - b(x, u(x))] dx,$$

where $b(x, u(x)) \in C^1(\bar{\Omega} \times H, R)$ is a given function. In this paper we investigate the conditions on which the functional I satisfies the Palais-Smale condition. We say that the functional I satisfies the Palais-Smale condition if for any given number $c \in R$, the sequence $(u_n)_n$ in

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H with $I(u_n) \to c$ and $\nabla I(u_n) \to 0$ possesses a convergent subsequence. Whether I satisfies the Palais-Smale condition or not is important for determining the critical points for I by applying the critical point theory.

Our main results are as follows:

THEOREM 1.1. *Let* $g(x,u) = u_+^p$, $h(x,u) = u_-^p$, *with* $2 < p < 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$, *where* $u_+ = \max\{u,0\}$ *and* $u_- = -\min\{u,0\}$. *Then the functionals*

$$I(u) = \int_\Omega [\frac{1}{2}Lu \cdot u - g(x,u)]dx$$

and

$$K(u) = \int_\Omega [\frac{1}{2}Lu \cdot u - h(x,u)]dx$$

satisfy the Palais-Smale condition.

THEOREM 1.2. *Assume that* $f \in C^1(\bar{\Omega} \times R, R)$ *satisfies the following growth conditions:*
$(f1)$ $f(x,0) = 0$, $f(x,u) > 0$ *if* $u \neq 0$, $\inf_{\substack{x\in\Omega \\ |u|^2=R^2}} f(x,u) > 0$,
$(f2)$ $u \cdot f_u(x,u) \geq pf(x,u)$ $\forall x$, u,
$(f3)$ $|f_u(x,u)| \leq \gamma|u|^\nu$, $\forall x$, u,
where $C > 0$, $2 < p < 2^*$, $2^* = \frac{2n}{n-2}$, $n \geq 3$, $\gamma \geq 0$, $\mu \in]2, 2^*[$, $\nu \leq 2^* - 1 - (2^* - p)(1 - \frac{2^{*\prime}}{2^*})$.
Then the functional

$$J(u) = \int_\Omega [\frac{1}{2}(Lu) \cdot u - f(x,u(x))]dx \tag{1.2}$$

satisfies the Palais-Smale condition.

In section 2 we obtain some results and properties of the linear operator L, and the function f. In section 2 we obtain some result on the corresponding functional $I(u)$ and prove Theorem 1.1. In section 3 we obtain some results and properties of the function f and the corresponding functional $J(u)$, and prove Theorem 1.2.

REMARK 1.1. *We note that the function* $a(x,u) = |u|^p$, *with* $2 < p < 2^*$ *and* Ω *bounded subset of* R^n, *satisfies the conditions* $(f1)$-$(f3)$. *Then the functional on* H

$$A(u) = \int_\Omega [\frac{1}{2}Lu \cdot u - f(x,u)]dx$$

satisfies the Palais-Smale condition.

Remark 1.2. *Let* $u_+ = \max\{u, 0\}$ *and* $u_- = -\min\{u, 0\}$. *Although the functions* $g(x, u) = u_+^p$, $h(x, u) = u_-^p$, *with* $2 < p < 2^*$ *and* Ω *bounded subset of* R^n, *do not satisfy the conditions* $(f2)$, *the functionals*

$$I(u) = \int_\Omega [\frac{1}{2} Lu \cdot u - g(x, u)]dx$$

and

$$K(u) = \int_\Omega [\frac{1}{2} Lu \cdot u - h(x, u)]dx$$

satisfy the Palais-Smale condition.

2. Proof of Theorem 1.1

First we shall prove that the functional

$$I(u) = \int_\Omega [\frac{1}{2} Lu \cdot u - u_+^p)]dx, \quad 2 < p < 2^*, \ 2^* = \frac{2n}{n-2}, \ n \geq 3.$$

satisfy the Palais-Smale condition. The eigenvalue problem $-Lu = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$ has infinitely many eigenvalues λ_k, $k \geq 1$ with $\lambda_1 < \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$, and infinitely many eigenfunctions ϕ_k be the eigenfunction belonging to the eigenvalue λ_k, $k \geq 1$. We need the following proposition for applying the critical point theory:

Proposition 2.1. *The functional* $I(u)$ *is continuous, Fréchet differentiable in* H, *with Fréchet derivative*

$$\nabla I(u)v = \int_\Omega [Lu \cdot v - p u_+^{p-1} \cdot v]dx.$$

Moreover $\nabla I \in C$. *That is* $I \in C^1$.

Proof. First we prove that $I(u)$ is continuous at u. For $u, v \in H$,

$$|I(u+v) - I(u)|$$
$$= |\frac{1}{2}\int_\Omega (Lu + Lv) \cdot (u+v)dx - \int_\Omega (u+v)_+^p dx - \frac{1}{2}\int_\Omega Lu \cdot u dx + \int_\Omega u_+^p dx|$$
$$= |\frac{1}{2}\int_\Omega (Lu \cdot v + Lv \cdot u + Lv \cdot v)dx - \int_\Omega (u+v)_+^p dx - u_+^p)dx|.$$

Let $u = \sum h_n\phi_n$, $v = \sum k_n\phi_n$. Then we have

$$|\int_\Omega Lu \cdot v dx| = |\sum \lambda_n h_n k_n| \leq \|u\| \cdot \|v\|,$$

$$|\int_\Omega Lv\cdot udx| = |\sum \lambda_n k_n h_n| \le \|u\|\cdot\|v\|,$$

$$|\int_\Omega Lv\cdot vdx| = |\sum \lambda_n k_n k_n| \le \|v\|^2,$$

from which we have

$$|\frac{1}{2}\int_\Omega (Lu\cdot v + Lv\cdot u + Lv\cdot v)dx| \le \|u\|\cdot\|v\| + \|v\|^2. \tag{2.1}$$

On the other hand

$$||(u+v)_+|^p - |u_+|^p| \le C_1|u_+^{p-1}||v| + R_2(|u_+|,|v_+|)$$

and hence we have

$$|\int_\Omega (|(u+v)_+|^2 - |u_+|^2)dx| \le 2\|u_+\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)} \le 2\|u\|\cdot\|v\| + \|v\|^2, \tag{2.2}$$

$$\begin{aligned}|\int_\Omega (|(u+v)_+|^p - |u_+|^p)dx| &\le C_1\|u_+^{p-1}\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + R_2(\|u\|_{L^2(\Omega)}, \|v\|_{L^2(\Omega)}) \\ &\le C_2\|u_+^{p-1}\|\|v\| + R_2(\|u\|,\|v\|. \end{aligned} \tag{2.3}$$

Combining (2.1) with (2.2) and (2.3), we have

$$|I(u+v) - I(u)| = o(\|v\|^2)$$

from which we can conclude that $I(u)$ is continuous at u. Next we prove that $I(u)$ is *Fréchet* differentiable in H. For $u, v \in H$,

$$\begin{aligned}&|I(u+v) - I(u) - \nabla I(u)v| \\ &= |\frac{1}{2}\int_\Omega (Lu+Lv)\cdot(u+v)dx - \int_\Omega (u+v)_+^p dx \\ &- \frac{1}{2}\int_\Omega (Lu)\cdot udx + \int_\Omega u_+^p dx - \int_\Omega (Lu - pu_+^{p-1})\cdot vdx| \\ &= |\int_\Omega [\frac{1}{2}(Lv)\cdot v - (u+v)_+^p + u_+^p + pu_+^{p-1}v]dx|.\end{aligned}$$

Combining (2.1) with (2.2) and (2.3), we have that

$$|I(u+v) - I(u) - \nabla I(u)v| = O(\|v\|^2). \tag{2.4}$$

Thus $I(u)$ is *Fréchet* differentiable in H. Similarly, it is easily checked that $I \in C^1$. □

Proof of Theorem 1.1

Let $c \in R$ and $(u_n)_n$ be a sequence such that

$$u_n \in H,\ \forall n,\ I(u_n) \to c,\ \nabla I(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then we have

$$\langle \nabla I(u_n), \hat{u_n}\rangle = \frac{2I(u_n)}{\|u_n\|} - \frac{\int_\Omega p(u_n)_+^{p-1} \cdot u_n dx}{\|u_n\|}$$

$$+\frac{2\int_\Omega (u_n)_+^p dx}{\|u_n\|} \longrightarrow 0.$$

Hence

$$\frac{\int_\Omega [p(u_n)_+^{p-1} \cdot u_n - 2(u_n)_+^p] dx}{\|u_n\|} \longrightarrow 0.$$

Thus there exists a constant $M > 0$ such that

$$\begin{aligned}
M >& |\int_\Omega [p(u_n)_+^{p-1} \cdot u_n) - 2(u_n)_+^p] dx| \\
&\geq |\int_\Omega [p(u_n)_+^{p-1} \cdot u_n - 2(u_n)_+^p] dx| \\
&\geq \int_\Omega [|p(u_n)_+^{p-1}||u_n| - 2|(u_n)_+^p|] dx \\
&\geq \int_\Omega [|p(u_n)_+^{p-1}||(u_n)_+| - 2|(u_n)_+^p|] dx \\
&= \int_\Omega [|p(u_n)_+^p| - 2|(u_n)_+^p|] dx \\
&= (p-2)\int_\Omega |(u_n)_+|^p dx = (p-2)\|(u_n)_+\|^p_{L^p(\Omega)}.
\end{aligned}$$

Thus

$$\begin{aligned}
0 \longleftarrow & \frac{|\int_\Omega [p(u_n)_+^{p-1} \cdot u_n - 2(u_n)_+^p)] dx|}{\|u_n\|} \\
& \geq (p-2)\frac{\|(u_n)_+\|^p_{L^p(\Omega)}}{\|u_n\|}.
\end{aligned}$$

Since $p > 2$,

$$\frac{\|(u_n)_+\|^p_{L^p(\Omega)}}{\|u_n\|} \quad \text{converges .}$$

On the other hand

$$\|p(u_n)^{p-1}_+\| \leq C_1\|(u_n)^{p-1}_+\|_{L^{2^{*'}}(\Omega)}$$

for suitable constant C_1. Then we have

$$\|\frac{p(u_n)^{p-1}_+}{\|u_n\|}\| \leq C_1\|\frac{(u_n)^{p-1}_+}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)}$$

If $p \geq 2^{*'}(p-1)$, then by the $H\ddot{o}lder's$ inequality, it is easily checked that $\|\frac{(u_n)^{p-1}_+}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)}$ can be estimated in terms of $\frac{\|(u_n)_+\|^p_{L^p(\Omega)}}{\|u_n\|}$. If $p \leq 2^{*'}(p-1)$, then by the standard interpolation inequalities, $\|\frac{(u_n)^{p-1}_+}{\|u_n\|}\|_{L^{2^{*'}}(\Omega)} \leq C_2(\frac{\|(u_n)_+\|^p_{L^p(\Omega)}}{\|u_n\|})^{\frac{(p-1)\alpha}{p}}\|(u_n)_+\|^{\beta}$ for some constant C_2, where $\alpha > 0$ is such that $\frac{\alpha}{p} + \frac{1-\alpha}{2^*} = \frac{1}{2^{*'}}$ and $\beta = (1-\alpha)(p-1) - 1 - \frac{(p-1)\alpha}{p}$. Since $p - 1 \leq 2^* - 1 - (2^* - p)(1 - \frac{2^{*'}}{2^*})$, $\beta < 0$. Thus we have

$$\|\frac{p(u_n)^{p-1}_+}{\|u_n\|}\| \leq C_2(\frac{\|(u_n)_+\|^p_{L^p(\Omega)}}{\|u_n\|})^{\frac{(p-1)\alpha}{p}}\|(u_n)_+\|^{\beta}$$

for a constant C_2. Since $\frac{\|(u_n)_+\|^p}{\|u_n\|}$ converges and $\beta < 0$,

$$\frac{p(u_n)^{p-1}_+}{\|u_n\|} \quad \text{converges.} \tag{2.5}$$

By (2.5) and the boundedness of $\hat{u_n}$,

$$\langle\frac{p(u_n)^{p-1}_+}{\|u_n\|}, \hat{u_n}\rangle \quad \text{converges.}$$

Thus by (2.5), we have

$$\langle\frac{p(u_n)^{p-1}_+}{\|u_n\|}, \hat{u_n}\rangle = \int_\Omega \frac{p(u_n)^{p-1}_+}{\|u_n\|} \cdot \hat{u_n}$$

$$= \frac{\frac{\int_\Omega (p(u_n)^{p-1}_+)\cdot u_n}{\|u_n\|}}{\|u_n\|} \longrightarrow 0.$$

Thus $\hat{u_n} \rightharpoonup 0$. We get

$$\frac{\nabla I(u_n)}{\|u_n\|} = L\hat{u_n} - \frac{p(u_n)_+^{p-1}}{\|u_n\|} \longrightarrow 0.$$

By (2.5), $L\hat{u_n}$ converges. Since $(\hat{u_n})_n$ is bounded and the operator of L^{-1} is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup 0$, we get $\hat{u_n} \to 0$, which is a contradiction to the fact that $\|\hat{u_n}\| = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$. We claim that $u_n \to u$ strongly. We have that

$$\langle \nabla I(u_n), u_n \rangle = (\|u_n\|^2 - \int_\Omega [p(u_n)_+^{p-1} u_n] dx) \longrightarrow 0.$$

Since $\int_\Omega [p(u_n)_+^{p-1} u_n] dx \longrightarrow \int_\Omega [p u_-^{p-1} u] dx$, $\|u_n\|^2$ converge. Thus $(u_n)_n$ converges to some u strongly with $\nabla I(u) = \lim \nabla I(u_n) = 0$. Thus we prove the lemma.

For the case $K(u)$, the proof follows arguing as in the case $I(u)$.

3. Proof of Theorem 1.2

We need some lemmas:

Lemma 3.1. *Assume that f satisfies the conditions (f1)-(f3). Then there exist $a_0 > 0$, $b_0 \in R$ such that*

$$f(x,u) \geq a_0|u|^p - b_0, \qquad \forall x, u. \tag{3.1}$$

Proof. Let u be such that $|u|^2 \geq R^2$. Let us set $\varphi(\xi) = f(x, \xi u)$ for $\xi \geq 1$. Then

$$\varphi(\xi)' = u \cdot f_u(x, \xi u) \geq \frac{\mu}{\xi}\varphi(\xi).$$

Multiplying by ξ^{-p}, we get

$$(\xi^{-p}\varphi(\xi))' \geq 0,$$

hence $\varphi(\xi) \geq \varphi(1)\xi^p$ for $\xi \geq 1$. Thus we have

$$f(x,u) \geq f\big(x, \frac{R|u|}{\sqrt{|u|^2}}\big)\big(\frac{\sqrt{|u|^2}}{R}\big)^p \geq c_0\big(\frac{\sqrt{|u|^2}}{R}\big)^p$$

$$\geq a_0|u|^p - b_0, \text{ for some } a_0, b_0,$$

where $c_0 = \inf\{f(x,u) \mid |u|^2 = R^2\}$. □

Lemma 3.2. *Assume that f satisfies the conditions (f1)-(f3). Then if $\|u_n\| \to +\infty$ and*

$$\frac{\int_\Omega u_n \cdot f_u(x,u_n)dx - 2\int_\Omega f(x,u_n)dx}{\|u_n\|} \to 0,$$

then there exist $(u_{h_n})_n$ and $w \in H$ such that

$$\frac{\text{grad}(\int_\Omega f(x,u_{h_n})dx)}{\|u_{h_n}\|} \to w \text{ and } \frac{u_{h_n}}{\|u_{h_n}\|} \rightharpoonup 0.$$

Proof. By (f2) and Lemma 3.1, for $u \in H$,

$$\int_\Omega [u \cdot f_u(x,u)]dx - 2\int_\Omega f(x,u)dx \geq$$

$$(p-2)\int_\Omega f(x,u)dx \geq (p-2)(a_0\|u\|^p_{L^p} - b_1).$$

By (f3),

$$\|\text{grad}(\int_\Omega f(x,u)dx)\| \leq C'\||u|^\nu\|_{L^{2^{*\prime}}}$$

for suitable constant C'. To get the conclusion it suffices to estimate $\|\frac{|u|^\nu}{\|u\|}\|_{L^{2^{*\prime}}}$ in terms of $\frac{\|u\|^p_{L^p}}{\|u\|}$. If $p \geq 2^{*\prime}\nu$, then this is an consequence of *Hölder* inequality. Next we consider the case $p < 2^{*\prime}\nu$. By the assumptions p and ν,

$$\nu \leq 2^* - 1 - (2^* - p)(1 - \frac{2^{*\prime}}{2^*}). \tag{3.2}$$

By the standard interpolation arguments, it follows that $\|\frac{|u|^\nu}{\|u\|}\|_{L^{2^{*\prime}}} \leq C\big(\frac{\|u\|^p_{L^p}}{\|u\|}\big)^{\frac{\nu\alpha}{p}}\|u\|^\beta$, where α is such that $\frac{\alpha}{p} + \frac{1-\alpha}{2^*} = \frac{1}{2^{*\prime}\nu}$ $(\alpha > 0)$ and $\beta = (1-\alpha)\nu - 1 - \frac{\nu\alpha}{p}$. By (3.2), $\beta \leq 0$. Thus we prove the lemma. □

By (f3), the functional $I(u)$ is well-defined and continuous on H.

Proposition 3.1. *Assume that the conditions $(f1)$-$(f3)$ hold. Then the functional $J(u)$ is continuous, Fréchet differentiable in H with Fréch−et derivative*

$$\nabla J(u)v = \int_\Omega [(Lu)\cdot v - f_u(x,u)\cdot v]dx.$$

Moreover $\nabla J \in C$. That is $J \in C^1$.

Proof. First we shall prove that $J(u)$ is continuous at u. For $u, v \in H$,

$$|J(u+v) - J(u)|$$
$$= |\frac{1}{2}\int_\Omega (Lu + Lv)\cdot(u+v)dx - \int_\Omega f(x,u+v)dx$$
$$-\frac{1}{2}\int_\Omega (Lu)\cdot udx + \int_\Omega f(x,u)dx|$$
$$= |\frac{1}{2}\int_\Omega [(Lu\cdot v + Lv\cdot u + Lv\cdot v)dx - \int_\Omega (f(x,u+v) - f(x,u))dx|.$$

Since $f \in C^1(\bar{\Omega}\times H, R)$, we have

$$|\int_\Omega [f(x,u+v) - f(x,u)]dx| \leq |\int_\Omega [f_u(x,u)\cdot v + o(|v|)]dx| = O(|v|). \quad (3.3)$$

Thus we have

$$|J(u+v) - J(u)| = O(|v|^2),$$

So $J(u)$ is continuous at u in H. Next we shall prove that $J(u)$ is *Fréchet* differentiable in H. For $u, v \in H$,

$$|J(u+v) - J(u) - \nabla J(u)v|$$
$$= |\frac{1}{2}\int_\Omega (Lu + Lv)\cdot(u+v)dx - \int_\Omega f(x,u+v)dx$$
$$-\frac{1}{2}\int_\Omega (Lu)\cdot udx + \int_\Omega f(x,u)dx - \int_\Omega (Lu - f_u(x,u))\cdot vdx|$$
$$= |\frac{1}{2}\int_\Omega [Lu\cdot v + Lv\cdot u + Lv\cdot v]dx$$
$$-\int_\Omega [f(x,u+v) - f(x,u)]dx - \int_\Omega [(Lu - f_u(x,u))\cdot v]dx|.$$

By (3.3), we have

$$|J(u+v) - J(u) - \nabla J(u)v| = O(|v|^2).$$

Similarly, it is easily checked that $J \in C^1$. □

Proof of Theorem 1.2

From now on we shall prove that J satisfies Palais-Smale condition under the assumptions (f1)-(f3). Assume that the (f1)-(f3) hold. Let $c \in R$ and $(u_n)_n$ be a sequence in H such that

$$J(u_n) \to c, \quad \nabla J(u_n) \to 0.$$

We claim that $(u_n)_n$ is bounded. By contradiction we suppose that $\|u_n\| \to +\infty$ and set $\hat{u_n} = \frac{u_n}{\|u_n\|}$. Then

$$\langle \nabla J(u_n), \hat{u_n} \rangle = 2\frac{J(u_n)}{\|u_n\|} -$$

$$\frac{\int_\Omega f_u(x, u_n) \cdot u_n dx - 2\int_\Omega f(x, u_n)dx}{\|u_n\|} \longrightarrow 0.$$

Hence

$$\frac{\int_\Omega f_u(x, u_n) \cdot u_n dx - 2\int_\Omega f(x, u_n)dx}{\|u_n\|} \longrightarrow 0.$$

By Lemma 3.2,

$$\frac{\text{grad} \int_\Omega f(x, u_n)dx}{\|u_n\|} \qquad \text{converges}$$

and $\hat{u_n} \rightharpoonup 0$. We get

$$\frac{\nabla J(u_n)}{\|u_n\|} = L\hat{u_n} - \frac{\text{grad}(\int_\Omega f(x, u_n)dx)}{\|u_n\|} \longrightarrow 0,$$

so $L\hat{u_n}$ converges. Since $(\hat{u_n})_n$ is bounded and the operator of L^{-1} is a compact mapping, up to subsequence, $(\hat{u_n})_n$ has a limit. Since $\hat{u_n} \rightharpoonup 0$, we get $\hat{u_n} \to 0$, which is a contradiction to the fact that $\|\hat{u_n}\| = 1$. Thus $(u_n)_n$ is bounded. We can now suppose that $u_n \rightharpoonup u$ for some $u \in H$. Since the mapping $u \mapsto \text{grad}(\int_\Omega f(x, u)dx)$ is a compact mapping, $\text{grad}(\int_\Omega f(x, u_n)dx) \longrightarrow \text{grad}(\int_\Omega f(x, u)dx)$. Thus Lu_n converges. Since the operator of L^{-1} is a compact operator and $(u_n)_n$ is bounded, we deduce that, up to a subsequence, $(u_n)_n$ converges to some u strongly with $\nabla J(u) = \lim \nabla J(u_n) = 0$. Thus we prove the lemma.

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