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REMARKS ON THE SUTURED MANIFOLDS

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Abstract. Gabai′s sutured manifold theory has produced many remarkable results in knot theory. Let M be the compact oriented 3-manifold and (M, γ) be sutured manifold. The aim of this note is to show that there exist a sutured manifold decomposition and a surface of M which defines a sutured manifold decomposition.

1. Introduction

A 3-manifold M is a separable metric space in which every point has a neighborhood homeomorphic to an open set in $R^2 \times [0, \infty)$. A point in M which has no neighborhood homeomorphic to R^3 is called a boundary of M. The set of all boundary points is a surface denoted ∂M.

In this paper M is a compact oriented 3-manifold. If R and S are oriented submanifolds of M, then $[R]$ denotes the homology class which R represents and $N(S)$ denotes the or a product neighborhood of S in M. Furthermore, if $\dim R + \dim S = \dim M$, then $<,>$ denotes their algebraic intersection number. $IntD$ denotes the interior of D. Finally, basic concepts and terminologies concerning 3-manifold can be found in [2] and [4].

In [6] Thurston defines a pseudonorm on $H_2(M, \partial M)$ and $H_2(M)$. Let χ denote the Euler-characteristic. For S a connected orientable compact surface, define
$\chi_-(S) = \max\{0, -\chi(S)\}$. That is, $\chi_-(S)$ is $-\chi(S)$ unless S is a sphere or a disk, in which case $\chi_-(S) = 0$. For S not necessarily connected, define $\chi_-(S) = \sum\{\chi_-(S_i) | S_i \textit{ a component of } S\}$. If S has no component S_i with $\chi(S_i) \leq 0$, then $\chi_-(S) = 0$. Equivalently, $-\chi_-(S)$ is the sum of the Euler-characteristics of the nonsimply connected components of S.

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Let N be a subsurface of ∂M for M. For a homology class $z \in H_2(M,N)$, the Thurston norm of the class z is defined by

$$x(z) = \min\{\chi_-(S) | (S, \partial S) \subset (M,N) with [S, \partial S] = z\}.$$

Let S be an oriented surface properly embedded in M with $\partial S \subset N$. Then S is norm minimizing in $H_2(M,N)$ if S is incompressible in M and $x_-(S) = x([S])$ for [S]$\in H_2(M,N)$.

Let γ be a compact 2-manifold in ∂M . The manifold pair (M,γ) is a sutured manifold if we have the following conditions (i)γ is the union of mutually disjoint annuli and tori. We denoted the union of annuli by $A(\gamma)$ and the union of tori by $T(\gamma)$. (ii) Each component of $A(\gamma)$ contains an oriented core loop, called a suture. We denote the set of sutures by $s(\gamma)$. (iii)$R(\gamma) = \partial M - Int(\gamma)$ is oriented so that each component of $\partial R(\gamma)$ is homologous to a component of $s(\gamma)$ in γ. Define $R_+(\gamma)(or R_-(\gamma))$ to be union of those components of $R(\gamma)$ whose positive normal vectors point outward(or inward) of M.

Sutured manifolds were introduced by David Gabai to study taut foliations on 3-manifolds, and they proved to be powerful tools in 3-dimensional topology. A 3-sutured manifold can be thought of as a 3-manifold with boundary, with some disjoint simple closed curves (annular sutures) drawn on the boundary.

If $S \subset M$ is an oriented compact surface (transverse to γ), we can cut M open along S to get a new sutured manifold (M', γ'). A sutured manifold (M,γ) is taut if M is irreducible and $R(\gamma)$ is norm minimizing in $H_2(M,N)$.

In this paper we state the following results : If (M,γ) is a taut and $H_2(M,N)$ is nontrivial, then there exists a sutured manifold decomposition and a surface of M which defines a sutured manifold decomposition.

2. Main results

Let (M,γ) be a sutured manifold, and S a properly embedded surface in M such that for every component λ of $S \cap \lambda$ one of (1) $\sim$ (3) holds :

(1) λ is a properly embedded nonseparating arc in γ.

(2) λ is a simple closed curve in an annular component A of γ in the same homology class as $A \cap s(\gamma)$.

(3) λ is a homotopically nontrivial curve in a total component T of γ, and if δ is another component of $T \cap S$, then λ and δ represent the

same homology class in $H_1(T)$. Then S defines a sutured manifold decomposition

$$(M,\gamma) \xrightarrow{S} (M^{'},\gamma^{'}) .$$

where $M^{'} = M - IntN(S)$ and $\gamma^{'} = (\gamma \cap M^{'}) \cup N(S^{'}_{+} \cap R_{-}(\gamma)) \cup N(S^{'}_{-} \cap R_{+}(\gamma))$, $R_{+}(\gamma^{'}) = ((R_{+}(\gamma) \cap M^{'}) \cup S^{'}_{+}) - Int(\gamma^{'})$, $R_{-}(\gamma^{'}) = ((R_{-}(\gamma) \cap M^{'}) \cup S^{'}_{-}) - Int(\gamma^{'})$, where $S^{'}_{+}$(*or* $S^{'}_{-}$) is that component of $\partial N(S) \cap M^{'}$ whose normal vector point outward(inward) $M^{'}$. If $(M,\gamma) \xrightarrow{S} (M^{'},\gamma^{'})$ is a sutured manifold decomposition, define

$$S_{+} = S^{'}_{+} \cap R_{+}(\gamma^{'}),\ S_{-} = S^{'}_{-} \cap R_{-}(\gamma^{'}).$$

Lemma 2.1. *In* [6]. *Let (M,γ) be a taut. Let N be the manifold with boundary obtained by doubling M along $R(\gamma)$, and let $z \in H_2(N,\partial N)$. Then there exists an integer $n \geqslant 0$ and a properly embedded oriented surface T such that the following hold.*

(i) $[T] = n[R] + z$ *and T is norm minimizing.*

(ii) *If S is a surface obtained by doing cut and paste surgery to T and either $R_{+}(\gamma)$ or $R_{-}(\gamma)$, then S is norm minimizing, and each component of $S \cap \gamma$ satisfies one of three conditions of the sutured manifold decomposition where (M,γ) is viewed as being embedded in N.*

(iii) *If V is a component of $R(\gamma)$, then no nontrivial subset of $V \cap T$ is homologically trivial in $H_1(V,\partial V)$.*

Lemma 2.2. *In* [5]. *Let* $(M,\gamma) \xrightarrow{D} (M^{'},\gamma^{'})$ *be a decomposition such that either D is a disc and $|D \cap s(\gamma)| = 2$ or D is an annulus with one component of ∂D lying in each of $R_{+}(\gamma)$ and $R_{-}(\gamma)$. Then (M,γ) is taut if and only if $(M^{'},\gamma^{'})$ is taut.*

Suppose (M,γ) is a sutured manifold and $(M,\gamma) \xrightarrow{S} (M^{'},\gamma^{'})$ *is a sutured manifold decomposition. It is well known that if $(M^{'},\gamma^{'})$ is taut, then so is (M,γ).*

Theorem 2.3. *If (M,γ) is a taut and $H_2(M,\partial M)$ is nontrivial, then there exists a decomposition* $(M,\gamma) \xrightarrow{S} (M^{''},\gamma^{''})$ *such that $(M^{''},\gamma^{''})$ is taut, S is connected, and $0 \neq [\partial S] \in H_1(\partial M)$ if $\partial M \neq \emptyset$.*

Proof. If M is closed let S be any norm minimizing surface. If $\partial M \neq \emptyset$, let P be a properly embedded surface in M, N the 3-manifold obtained by doubling M along $R(\gamma)$, and P' the oriented surface in N obtained by doubling P along $\partial P - Int(\gamma)$. Let $z = [P'] \in H_2(N, \partial N)$, T be the properly embedded oriented surface obtained by applying Lemma 2.1 to z , T' be the surface obtained by doing surgery with T and $R(\gamma)$, $S' = T \cap M$, and S a component of S' such that $0 \neq [\partial S] \in H_1(\partial M)$. Consider the decompositions.

$$(M, \gamma) \xrightarrow{\quad S \quad} (M'', \gamma'')$$

$$(M, \gamma) \xrightarrow{\quad S' \quad} (M', \gamma')$$

$$(M, \partial N) \xrightarrow{\quad T' \quad} (N', \delta').$$

It suffices to show (M'', γ'') is taut. There exists a set D of properly embedded pairwise disjoint annuli and discs in N' satisfying the hypotheses of Lemma 2.2 such that the decomposition $(N', \delta') \xrightarrow{\quad D \quad} (N'', \delta'')$ yields (M', γ') as a component of (N'', δ''). By Lemma 2.2, hence (M', γ') is taut. In order to complete the proof we consider the commutative diagram:

$$\begin{array}{ccc} (M, \gamma) & \xrightarrow{\qquad S' \qquad} & (M', \gamma') \\ & {\scriptstyle S} \searrow \qquad \nearrow {\scriptstyle S-S'} & \\ & (M'', \gamma'') & \end{array} .$$

□

THEOREM 2.4. *If* (M, γ) *is a taut and* $\alpha \in H_2(M, \partial M)$ *is nontrivial, then there exists a surface* $(S, \partial S) \subset (M, \partial M)$ *such that* $[S, \partial S] = \alpha$ *and* $(M, \gamma) \xrightarrow{\quad S \quad} (M', \gamma')$ *is a taut decomposition.*

Proof. Let N be the manifold obtained by doubling M along $R(\gamma)$. Let $z \in H_2(N, \partial N)$ be the class obtained by doubling α. Now apply Lemma 2.1 to obtain the norm minimizing surface $T \subset N$ where $z =$

$[T] \in H_2(N, \partial N)$. Note that if δ is a component of $\partial R(\gamma)$, then $|\delta \cap T| = | < \delta, T > |$.

Let S' be a union of components of $T \cap M$ such that S' is nonseparating and $[S', \partial S'] = [T, \partial T] \cap M = \alpha \in H_2(M, \partial M)$.
By the proof of Theorem 2.3 we conclude that the decomposition

$$(M, \gamma) \xrightarrow{S} (M', \gamma')$$

yields a taut. If V is a component of $R(\gamma)$ such that $\partial V \cap S' \neq \emptyset$, then $V \cap S'$ is homologous in $H_1(V, \partial V)$ to a set of arcs λ such that $|\delta \cap \lambda| = | < \delta, \lambda > |$ for each component δ of ∂V. Note that if $\partial V \cap S' \neq \emptyset$, then $V \cap S'$ is homologous to a set λ of parallel coherently oriented closed curves. And we can modify S' near ∂M to find the desired S. □

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