Korean J. Math. **17** (2009), No. 4, pp. 495–505

EXISTENCE OF NONNEGATIVE SOLUTIONS FOR BOUNDARY VALUE PROBLEMS

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Abstract. By means of Green function and fixed point theorem related with cone theoretic method we show that there exist multiple nonnegative solutions of a Dirichlet problem

$$-\big[\,p(t)x'(t)\,\big]' = \lambda q(t)f(x(t)), \quad t \in I = [0,\, T]$$
$$x(0) = 0 = x(T),$$

and a mixed problem

$$-\big[\,p(t)x'(t)\,\big]' = \mu q(t)f(x(t)), \quad t \in I = [0,\, T]$$
$$x'(0) = 0 = x(T),$$

where λ and μ are positive parameters.

1. Introduction

Let λ and μ be positive parameters. The purpose of this paper is to study the existence of multiple nonnegative solutions of a Dirichlet boundary value problem of the type :

$$-\big[\,p(t)x'(t)\,\big]' = \lambda q(t)f(x(t)), \quad t \in I = [0,\, T], \tag{1}$$
$$x(0) = 0 = x(T), \tag{2}$$

where

(A) $p(t)$ is positive and continuous,
(B) $q : I \to [0, \infty)$ is continuous and not identically zero on any subinterval of I,
(C) $f : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing,

Received October 20, 2008. Revised December 3, 2008.
2000 Mathematics Subject Classification: 34C10, 34C15.
Key words and phrases: Green function, Dirichlet problem, mixed problem, fixed point theorem, nonlinear differential equation, cone.
This research was supported by Hallym University Fund, 2008(HRF-2008-022).

(D) $\lim_{x\to+0} f(x)$ exists and there exist $\gamma_1,\ \gamma_2$ such that $0 < \gamma_1 \le \frac{f(x)}{x} \le \gamma_2$ for large x,

and a mixed problem of the type :

$$
\begin{aligned}
-\big[\,p(t)x'(t)\,\big]' &= \mu q(t) f(x(t)), \quad t \in I = [0,\, T] \\
x'(0) &= 0 = x(T)
\end{aligned} \tag{3}
$$

with conditions (A), (B), (C) and

(E) $\lim_{x\to+0} \dfrac{f(x)}{x}$ exists and there exist positive constants $\beta,\ \gamma$ such that $f(x) \ge \beta$ if $x \ge \gamma$.

The above problems arise from many branches of the applied mathematics, for instance, positive, radially symmetric solutions of nonlinear partial differential equations. The cone method is a very useful tool to investigate the existence of positive solutions of differential equations. During last years the study of the existence of positive solutions for boundary value problems by means of cone theoretic techniques has evolved very extensively[1, 3-5, 7-10]. Among others many works have been devoted to study for finding positive solutions of such problems by means of fixed point theorems. Erbe and Wang[4] obtained positive solutions belonging to a cone, and lying in an annular region. The method of [5] were extended to higher order boundary value problems in [3]. In the case of $p(t) \equiv 1$ consider the following equation

$$
\begin{aligned}
-x''(t) &= \lambda q(t) f(x(t)), \qquad t \in I = [0,\, T] \\
x(0) &= 0 = x(T).
\end{aligned} \tag{4}
$$

The existence or nonexistence of solutions and multiplicity of solutions for the equation (4) has been studied and obtained many results. See [6, 7, 11] and reference therein. In this paper we show that the equations (1)–(2) have at least two or three solutions by means of Green function and fixed point theorem.

Put $\psi(t) = \int_0^t \frac{ds}{p(s)}$ and consider the following conditions :

(H1) There exists $\tau \in (0,\, T/2)$ such that $\int_{T-\tau}^{T} \frac{ds}{p(s)} \ge \tau\,\psi(T)$ and $\int_0^{\tau} \frac{ds}{p(s)} \ge \tau\,\psi(T)$.

(H2) $\int_0^{T/2} \frac{ds}{p(s)} \le \frac{1}{2}\psi(T)$.

Remark 1.1. In the case of $0 < T < \frac{1}{4}$, $p(t) = \frac{T^3}{6t^2-6Tt+2T^2}$ satisfies the condition (H1) and (H2). If $p(t) \equiv 1$ and $T = 1$, the equalities in the conditions both (H1) and (H2) are valid.

2. Preliminaries

We will apply the fixed point theorem to a completely continuous integral operator whose kernel is $G(t,s)$ called the Green function of

$$-\big[p(t)x'(t)\big]' = 0,$$
$$x(0) = 0 = x(T).$$

Then $G(t,s)$ is explicitly given by

$$G(t,s) = \begin{cases} \psi(s)\big(\psi(T) - \psi(t)\big)/\psi(T), & 0 \le s \le t \le T, \\ \psi(t)\big(\psi(T) - \psi(s)\big)/\psi(T), & 0 \le t \le s \le T. \end{cases}$$

By means of condition (H1) we obtain an inequality similar to that of E. R. Kaufmann and N. Kosmatov[8]

Lemma 2.1. *Assume that $\tau \in (0, T/2)$ is the number satisfying the condition (H1). For $0 \le s \le T$ we obtain*

$$G(s,s) \ge G(t,s) > 0 \text{ for } t \in (0,\, T) \tag{5}$$

$$G(t,s) \ge \tau\, G(s,s) \text{ for } t \in [\tau,\, T-\tau]. \tag{6}$$

Proof. We note that $\psi(t)$ is increasing in I. The inequality (5) is clear. Let $t \in [\tau,\, T-\tau]$. If $0 \le s \le t \le T$, by (H1) we obtain $\psi(t) \le \psi(T-\tau) = \psi(T) - \{\psi(T) - \psi(T-\tau)\} \le (1-\tau)\psi(T)$. So (6) is valid for $s \le t$. If $T \ge s \ge t \ge 0$, we have $\psi(t) \ge \psi(\tau) \ge \tau\psi(T) \ge \tau\psi(s)$. Thus (6) is valid for $s \ge t$. □

We note that the fixed points of the operator

$$T_\lambda x(t) = \lambda \int_0^T G(t,s)q(s)f(x(s))\,ds \tag{7}$$

are solutions of (1)–(2). We employ a cone in a Banach space X to establish the existence of fixed points of (7). We note that

$$||T_\lambda x(t)|| \le \lambda \int_0^T G(s,s)q(s)f(x(s))\,ds. \tag{8}$$

Definition. Let X be a Banach space and Ω a closed convex set $\subset X$. We call $\Omega \neq \{0\}$ a cone if

(i) $\tau\,\Omega \subset \Omega$ for all $\tau \geq 0$.
(ii) $\Omega \cap (-\Omega) = \{0\}$.

Proposition 2.2. [2] *Let X be a Banach space with norm $||\cdot||$, $\Omega \subset X$ a cone, $\varphi : \Omega \to [0, \infty)$ continuous and concave, $\varphi(x) \leq ||x||$ on Ω, $T : \overline{\Omega}_r \to \Omega$ compact where $\overline{\Omega}_r = \Omega \cap \overline{B_r(0)}$. Suppose also that there exist $0 < \sigma < \rho < r$ such that*

(i) $\{\, x \,|\, \varphi(x) > \rho\} \cap \overline{\Omega}_r \neq \phi$,
(ii) $\varphi(Tx) > \rho$ *on* $\{\, x \,|\, \varphi(x) \geq \rho\} \cap \overline{\Omega}_r$,
(iii) $||Tx|| < \sigma$ *on* $\overline{\Omega}_\sigma$.

Then we have

(a) *T has at least two fixed points if $\varphi(Tx) > \dfrac{\rho}{r}||Tx||$ on $\{x \in \overline{\Omega}_r \,|\, ||Tx|| > r\}$.*
(b) *T has at least three fixed points if $T(\overline{\Omega}_r) \subset \overline{\Omega}_r$.*

3. Main results

Let $X = \mathcal{C}(I)$ be the space of continuous functions $x : I \to R$ with the norm $||x(t)|| = \max_{t\in I} |x(t)|$. We define a cone Ω by

$$\Omega = \{x \in \mathcal{C}(I) \,|\, x(t) \geq 0 \text{ for all } t \in I\}$$

It is not difficult to show that the operator $T_\lambda : \overline{\Omega}_r \to \Omega$ defined by (7) is compact where $\Omega_r = \Omega \cap \overline{B_r(0)}$ and $r > 0$(see p54 in [12]). If we set

$$\varphi(x) = \min_{t\in[\tau,\, T-\tau]} x(t)$$

for $x \in \Omega$ it is obvious that $\varphi : \Omega \to [0, \infty)$ is continuous, concave and $\varphi(x) \leq ||x||$. Then (i) of Proposition 2.2 holds for any r and ρ satisfying $0 < \rho < r$.

3.1. The Dirichlet boundary value problem.

In this section we assume that $\tau \in (0, T/2)$ is a fixed number satisfying the condition (H1). We consider the Dirichlet boundary value problem (1)–(2) with conditions (A), (B), (C) and (D).

THEOREM 3.1. *There exist positive constants σ, λ_0 such that*

$$||T_{\lambda_0}x|| < \sigma \text{ for } x \in \overline{\Omega}_\sigma$$

Proof. Put $\lim_{x\to+0} f(x) < A$. There exists $\delta_A > 0$ such that

$$0 \le x \le \delta_A \quad \Rightarrow \quad 0 \le f(x) < A$$

Then taking $x \in \overline{\Omega}$ with $||x|| \le \delta_A$ we have by (5)

$$T_\lambda x(t) < \lambda\, A \int_0^T G(s,s)q(s)\, ds.$$

Put $\sigma = \delta_A$. If we choose $0 < \lambda_0$ so that

$$\lambda_0\, A \int_0^T G(s,s)q(s)\, ds < \sigma, \tag{9}$$

the proof is then complete. □

REMARK 3.2. Even if the condition $\lim_{x\to+0} f(x)$ is replaced with $\lim_{x\to+0} \dfrac{f(x)}{x}$ Theorem 3.1 is valid.

THEOREM 3.3. *Assume that σ and λ_0 are the numbers determined in Theorem 3.1 and that the inequality*

$$\gamma_1 > \frac{1}{\lambda'\,\tau \int_\tau^{T-\tau} G(s,s)q(s)\, ds} \tag{10}$$

is valid for some λ' satisfying $\lambda' < \lambda_0$. Then there exist positive constants ρ, and r with $\sigma < \rho < r$ such that

$$\varphi(T_{\lambda'}x) > \rho \text{ on } \{\, x \,|\, \phi(x) \ge \rho\} \cap \overline{\Omega}_r.$$

Proof. There exist positive numbers B_1 and δ_{B_1} such that $\frac{f(x)}{x} > B_1 > \frac{1}{\lambda'\,\tau \int_\tau^{T-\tau} G(s,s)q(s)\, ds}$ for x with $x \ge \delta_{B_1}$. It follows that

$$\lambda'\,\tau\, B_1 \int_\tau^{T-\tau} G(s,s)q(s)\, ds > 1. \tag{11}$$

and that $x \ge \delta_{B_1}$ implies $f(x) > B_1 x$. Then taking $x \in \Omega$ with $\varphi(x) \ge \delta_{B_1}$, we have by (6)

$$\varphi(T_{\lambda'}x) > \lambda'\,\tau\, B_1 \int_\tau^{T-\tau} G(s,s)q(s)\, ds \cdot \varphi(x). \tag{12}$$

Put $\rho = \max\{2\sigma, \delta_{B_1}\}$ and $r = k\rho,\ k > 1$. Here k will be determined in Theorem 3.4. Therefore our theorem follows. □

THEOREM 3.4. *Let the assumptions of theorem 3.3 be valid. Then there exist positive constants λ_1, λ_2 and r such that the equation (1)–(2) with $\lambda \in [\lambda_1, \lambda_2]$ has at least two solutions in $\overline{\Omega}_r$.*

Proof. For τ' with $0 < \tau' < \min\{1, \tau\}$ we take $r = \rho/\tau'$ in the proof of Theorem 3.3. From (9), (10) and (11) there exist λ_1, λ_2 satisfying $0 < \lambda_1 < \lambda' < \lambda_0 < \lambda_2$ such that we have

$$\lambda_1 \, \tau \, B_1 \int_{\tau}^{T-\tau} G(s,s)q(s)\,ds > 1 \tag{13}$$

$$\lambda_2 \, A \int_0^T G(s,s)q(s)\,ds < \sigma. \tag{14}$$

where λ_0 and λ' are numbers determined in Theorem 3.1 and Theorem 3.3, respectively. We note that both (13) and (14) are valid for λ satisfying $\lambda \in [\lambda_1, \lambda_2]$. Consider the operator (7) with $\lambda \in [\lambda_1, \lambda_2]$. Let $||T_\lambda x|| > r$ and $x \in \overline{\Omega}_r$. We have then by (6)

$$\varphi(T_\lambda x) \geq \lambda \, \tau' \int_0^T G(s,s)q(s)f(x(s))\,ds$$

because of $\tau' < \tau$. Thus our theorem follows from (8) and (a) of Proposition 2.2. □

THEOREM 3.5. *Let λ_1 and λ_2 be the numbers determined in Theorem 3.4. Assume that the assumptions of theorem 3.3 hold and that*

$$\gamma_2 < \frac{1}{\lambda_2 \int_0^T G(s,s)q(s)\,ds}. \tag{15}$$

Then there exist positive constants λ_3 and R satisfying $\lambda_2 < \lambda_3$, $r \leq R$ respectively, such that the equation (1)–(2) with $\lambda \in [\lambda_1, \lambda_3]$ has at least three solutions in $\overline{\Omega}_R$

Proof. From (D) and (15) there exist positive constants λ_3, B_2 and δ_{B_2} such that $\lambda_2 < \lambda_3$ and $\frac{f(x)}{x} < B_2 < \frac{1}{\lambda_3 \int_0^T G(s,s)q(s)\,ds}$ for x with $x > \delta_{B_2}$. Then $x > \delta_{B_2}$ implies

$$\lambda_3 \int_0^T G(s,s)q(s)\,ds \cdot f(x) < x. \tag{16}$$

Take R so large that $R \geq \max\{r,\, \delta_{B_2}\}$. Let $||x|| \leq R$ for $x \in \Omega$. Then since $\max_{x\in[0,\,R]} f(x) = \max_{x\in[\delta_{B_2},\,R]} f(x)$, we obtain

$$\lambda_3 \int_0^T G(s,s)q(s)\,ds \cdot \max_{x\in[0,\,R]} f(x) \leq R \text{ for } ||x|| \leq R. \tag{17}$$

Consider the operator (7) with $\lambda \in [\lambda_1,\, \lambda_3]$. It is clear that $T_\lambda(\overline{\Omega}_R) \subset \overline{\Omega}_R$. So the proof is complete. □

Remark 3.6. If $\lambda'\tau \leq \lambda_2$, (10) and (15) are not compatible.

3.2. The mixed problem.

In this section we assume that the condition (H2) is valid and that τ with $0 < \tau < \min\{1/2,\, T/2\}$ is a fixed number. We consider the mixed problem (3) with conditions (A), (B),(C) and (E). The Green function $K(t,s)$ of

$$\begin{aligned} -\big[\,p(t)x'(t)\,\big]' &= 0 \\ x'(0) = 0 &= x(T) \end{aligned} \tag{18}$$

is given by

$$K(t,s) = \begin{cases} \psi(T) - \psi(t), & 0 \leq s \leq t \leq T, \\ \psi(T) - \psi(s), & 0 \leq t \leq s \leq T. \end{cases}$$

We note that $K(\cdot, s)$ is nonnegative and nonincreasing for every fixed s. If we set

$$\phi(x) = \min_{t\in[0,\,\tau]} x(t)$$

for $x \in \Omega$, it is obvious that $\phi : \Omega \to [0, \infty)$ is continuous, concave and $\phi(x) \leq ||x||$. Let $S_\mu x(t)$ be denoted by

$$S_\mu x(t) = \mu \int_0^T K(t,s)q(s)f(x(s))\,ds. \tag{19}$$

Then standard arguments show that the map $S_\mu : \overline{\Omega}_r \to \Omega$ is compact for any r with $0 < r$.

Lemma 3.7. *For $0 \leq s \leq T$ the inequalities*

$$\begin{aligned} K(s,s) \geq K(t,s) &\geq 0 \quad \text{for } t \in I, \\ K(t,s) &\geq \tau\, K(s,s) \quad \text{for } t \in [0,\, \tau], \end{aligned} \tag{20}$$

are valid.

Proof. Since by (H2) $(1-\tau)\psi(T) - \int_0^\tau \frac{ds}{p(s)}) \geq (\frac{1}{2} - \tau)\psi(T) > 0$ for $\tau \in (0, T/2)$, our lemma follows. □

THEOREM 3.8. *There exist positive constants ρ and μ_0 such that*

$$\phi(S_{\mu_0}x) > \rho \text{ on } \{\, x \,|\, \phi(x) \geq \rho\} \cap \overline{\Omega}_r$$

where r with $\rho < r$ will be determined in Theorem 3.10.

Proof. Taking into account of the condition (E), we put $\rho = \gamma$ and let $\phi(x) \geq \rho$. It follows from (20) that

$$\begin{aligned}\phi(S_\mu x) &\geq \mu \min_{t\in[0,\,\tau]} \int_0^T K(t,s)q(s)f(x(s))\,ds,\\ &\geq \mu\tau \int_0^\tau K(s,s)q(s)\,ds \cdot \beta.\end{aligned}$$

We choose μ_0 satisfying

$$\rho' = \mu_0\tau \int_0^\tau K(s,s)q(s)\,ds \cdot \beta \tag{21}$$

for some ρ' satisfying $\rho < \rho'$. Therefore our theorem is proved. □

THEOREM 3.9. *Let ρ and μ_0 be the numbers determined in Theorem 3.8. Assume that the inequality*

$$\lim_{x\to+0} \frac{f(x)}{x} < \frac{1}{\mu' \int_0^T K(s,s)q(s)\,ds} \tag{22}$$

is valid for some μ' satisfying $\mu' > \mu_0$. There exist positive constants σ, μ_1 and μ_2 satisfying $\sigma < \rho$ and $\mu_1 < \mu_0 < \mu' < \mu_2$, respectively, such that the operator (19) with $\mu \in [\mu_1, \mu_2]$ satisfies

$$||S_\mu x|| < \sigma \text{ for } x \in \overline{\Omega}_\sigma. \tag{23}$$

Proof. There exists α such that

$$\lim_{x\to+0} \frac{f(x)}{x} < \alpha < \frac{1}{\mu' \int_0^T K(s,s)q(s)\,ds}. \tag{24}$$

Thus it follows that there exists $\mu_2 > 0$ satisfying $\mu' < \mu_2$ such that

$$\alpha\mu_2 \int_0^T K(s,s)q(s)\,ds < 1 \tag{25}$$

Similarly from (21) there exists $\mu_1 > 0$ with $\mu_1 < \mu_0$ such that

$$\rho < \mu_1 \tau \int_0^\tau K(s,s)q(s)\,ds \cdot \beta. \tag{26}$$

On the other hand, in view of (24) there exists $\delta_\alpha > 0$ such that

$$0 < x \le \delta_\alpha \quad \Rightarrow \quad f(x) < \alpha x.$$

Taking $\sigma = \delta_\alpha < \rho$, we obtain (23) for the operator (19) with $\mu \in [\mu_1, \mu_2]$. So the proof is complete. □

THEOREM 3.10. *Let μ_1 and μ_2 be the numbers determined in Theorem 3.9. Assume that the assumption of theorem 3.9 is valid. Then there exists r with $r > \rho$ such that the equation (3) with $\mu \in [\mu_1, \mu_2]$ has at least two solutions in $\overline{\Omega}_r$.*

Proof. Let $\mu_1 \le \mu \le \mu_2$. Since $S_\mu x(t)$ is nonincreasing in $[0, \tau]$, we obtain

$$\begin{aligned}\phi(S_\mu x) &= S_\mu x(\tau)\\ &= \mu\Big\{\int_0^\tau (\psi(T)-\psi(\tau))q(s)f(x(s))\,ds + \int_\tau^T (\psi(T)-\psi(s))q(s)f(x(s))\,ds\Big\}\\ &= -\mu\int_0^\tau \{\psi(\tau)-\psi(s)\}q(s)f(x(s))\,ds + S_\mu x(0).\end{aligned}$$

By (H2) it follows that $0 \le \psi(\tau) - \psi(s) < \frac{\psi(T)}{2}$ for $0 \le s \le \tau$. Thus we have

$$\begin{aligned}\phi(S_\mu x) &> -\frac{\mu\psi(T)}{2}\int_0^\tau q(s)f(x(s))\,ds + S_\mu x(0)\\ &= \Big\{1 - \frac{\mu\psi(T)\int_0^\tau q(s)f(x(s))\,ds}{2S_\mu x(0)}\Big\}S_\mu x(0).\end{aligned}$$

On the other hand, since $\psi(s) < \frac{\psi(T)}{2}$ for $0 \le s \le \tau < T/2$, it is obvious that

$$S_\mu x(0) > \frac{\mu\psi(T)}{2}\int_0^\tau q(s)f(x(s))\,ds.$$

Put $\theta = \Big\{1 - \frac{\mu\psi(T)\int_0^\tau q(s)f(x(s))\,ds}{2S_\mu x(0)}\Big\}^{-1}$ and $r = \theta\rho$. Then $x \in \overline{\Omega}_r$ and $||S_\mu x|| > r$ imply

$$\phi(S_\mu x) > \frac{S_\mu x(0)}{\theta} = \frac{\rho}{r}\,S_\mu x(0). \tag{27}$$

We note that $||S_\mu x|| = S_\mu x(0)$. So the proof is complete. □

THEOREM 3.11. *Let μ_1 and μ_2 be the numbers determined in Theorem 3.9. Assume that the assumption of theorem 3.9 is valid. Then there exist positive constants μ_3 and R satisfying $\mu_2 < \mu_3$ and $r \le R$, respectively, such that the equation (3) with $\mu \in [\mu_1,\, \mu_3]$ has at least three solutions in $\overline{\Omega}_R$ where r is the number determined in Theorem 3.10.*

Proof. First, assume that the function f is bounded above by $M > 0$. Take μ_3 such that $\mu_2 < \mu_3$ and $r \le \mu_3 M \int_0^T K(s,s)q(s)\,ds$. Put $R = \mu_3 M \int_0^T K(s,s)q(s)\,ds$ and let $||x|| \le R$. Then for any $\mu \in [\mu_1,\, \mu_3]$ we have $S_\mu(\overline{\Omega}_R) \subset \overline{\Omega}_R$. Thus (3) has at least three solutions in $\overline{\Omega}_R$. Next, assume that the function f is unbounded. Choose a large number R with $R \ge r$ and consider a function $f^* = f \cdot \chi_{[0,R]}$ where $\chi_{[0,R]}(t) = 1$ if $x \in [0, R]$, 0 otherwise. Let

$$S_\mu^* x(t) = \mu \int_0^T K(t,s)q(s)f^*(x(s))\,ds. \tag{28}$$

Since then by (C) f^* is bounded above, $S_\mu^* x$ has at least three fixed points. Let us show that all three fixed points lie in $\overline{\Omega}_R$. If $S_\mu^* x = x$, $x(t)$ is nonincreasing. Assume that $R < ||x||$. There exists a $t_0 \in I$ such that $x(t_0) = R$ and $x(t) \ge R$ for $t \in [0, t_0]$. From the equation (3) we obtain

$$x(t) = \mu \int_t^T \frac{1}{p(s)} \int_0^s q(u)f^*(x(u))\,duds. \tag{29}$$

Then it follows that

$$\begin{aligned} x(0) &= \mu \int_{t_0}^T \frac{1}{p(s)} \int_0^s q(u)f^*(x(u))\,duds \\ &\quad + \mu \int_0^{t_0} \frac{1}{p(s)} \int_0^s q(u)f^*(x(u))\,duds \\ &= x(t_0). \end{aligned}$$

So we have $x(t) \le ||x|| = x(0) = R$ for $t \in I$, which proves our theorem. □

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