STABILITY OF HOMOMORPHISMS AND DERIVATIONS IN PROPER JCQ*-TRIPLES ASSOCIATED TO THE PEXIDERIZED CAUCHY TYPE MAPPING

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ABSTRACT. In this paper, we investigate homomorphisms in proper JCQ*-triples and derivations on proper JCQ*-triples associated to the following Pexiderized functional equation

\[ f(x + y + z) = f_0(x) + f_1(y) + f_2(z). \]

This is applied to investigate homomorphisms and derivations in proper JCQ*-triples.

1. Introduction and preliminaries

Ulam [14] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

Let \((G_1, \ast)\) be a group and let \((G_2, \circ, d)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist a \(\delta(\epsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality

\[ d(h(x \ast y), h(x) \circ h(y)) < \delta \]

for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with

\[ d(h(x), H(x)) < \epsilon \]

for all \(x \in G_1\)?

Hyers [7] considered the case of approximately additive mappings \(f : E \to E'\), where \(E\) and \(E'\) are Banach spaces and \(f\) satisfies Hyers inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon \]

for all \(x, y \in E\). It was shown that the limit

\[ L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]

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exists for all \( x \in E \) and that \( L : E \to E' \) is the unique additive mapping satisfying
\[
\|f(x) - L(x)\| \leq \epsilon.
\]

Th. M. Rassias [13] provided a generalization of Hyers' theorem which allows the Cauchy difference to be unbounded.

We recall some basic facts concerning quasi \(*\)-algebras.

**Definition 1.1.** Let \( A \) be a linear space and \( A_0 \) be a \(*\)-algebra contained in \( A \) as a subspace. We say that \( A \) is a **quasi \(*\)-algebra** over \( A_0 \) if

(i) the right and left multiplications of an element of \( A \) and an element of \( A_0 \) are always defined and linear;

(ii) \( x_1(x_2a) = (x_1x_2)a, (ax_1)x_2 = a(x_1x_2) \) and \( x_1(ax_2) = (x_1a)x_2 \) for all \( x_1, x_2 \in A_0 \) and all \( a \in A \);

(iii) an involution \( \ast \), which extends the involution of \( A_0 \), is defined in \( A \) with the property \((ab)^\ast = b^\ast a^\ast\), whenever the multiplication is defined.

Quasi \(*\)-algebras [8, 9] arise in natural way as completions of locally convex \(*\)-algebras whose multiplication is not jointly continuous; in this case one has to deal with topological quasi \(*\)-algebras.

A quasi \(*\)-algebra \((A, A_0)\) is called **topological** if a locally convex topology \( \tau \) on \( A \) is given such that:

(i) the involution \( a \mapsto a^\ast \) is continuous for each \( a \in A \),

(ii) the mappings \( a \mapsto ab \) and \( a \mapsto ba \) are continuous for each \( a \in A \) and \( b \in A_0 \),

(iii) \( A_0 \) is dense in \( A[\tau] \).

Throughout this paper, we suppose that a locally convex quasi \(*\)-algebra \((A, A_0)\) is complete. For an overview on partial \(*\)-algebra and related topics we refer to [1].

In a series of papers [2], [3], [4], [5] many authors have considered a special class of quasi \(*\)-algebras, called proper \(CQ^\ast\)-algebras, which arise as completions of \(C^\ast\)-algebras. They can be introduced in the following way:

**Definition 1.2.** Let \( A \) be a Banach module over the \(C^\ast\)-algebra \( A_0 \) with involution \( \ast \) and \(C^\ast\)-norm \( \| \cdot \|_0 \) such that \( A_0 \subset A \). We say that \((A, A_0)\) is a **proper \(CQ^\ast\)-algebra** if

(i) \( A_0 \) is dense in \( A \) with respect to its norm \( \| \cdot \|_0 \);

(ii) \((ab)^\ast = b^\ast a^\ast\) whenever the multiplication is defined;

(iii) \( \|y\|_0 = \max\{ \sup_{a \in A, \|a\|_0 \leq 1} \|ay\|, \sup_{a \in A, \|a\|_0 \leq 1} \|ya\| \} \) for all \( y \in A_0 \).

A proper \(CQ^\ast\)-algebra \((A, A_0)\) is said to have a unit \( e \) if there exists an element \( e \in A_0 \) such that \( ae = ea = a \) for all \( a \in A \). In this paper we will always assume that the proper \(CQ^\ast\)-algebra under consideration have an identity.

**Definition 1.3.** A proper \(CQ^\ast\)-algebra \((A, A_0)\), endowed with the Jordan triple product
\[
\{z, x, w\} = \frac{1}{2}\{zx^\ast w + wx^\ast z\}
\]
for all \( x \in A \) and all \( z, w \in A_0 \), is called a proper JCQ\(^*\)-triple, and denoted by \( (A, A_0, \{.,.,\}) \).

**Definition 1.4.** Let \( (A, A_0, \{.,.,\}) \) and \( (B, B_0, \{.,.,\}) \) be proper JCQ\(^*\)-triples.

(i) A \( \mathbb{C} \)-linear mapping \( H : A \to B \) is called a proper JCQ\(^*\)-triple homomorphism if \( H(z) \in B_0 \) and \( H(\{z, x, w\}) = \{H(z), H(x), H(w)\} \) for all \( z, w \in A_0 \) and all \( x \in A \).

(ii) A \( \mathbb{C} \)-linear mapping \( \delta : A_0 \to A \) is called a proper JCQ\(^*\)-triple derivation if

\[
\delta(\{w_0, w_1, w_2\}) = \{\delta(w_0), w_1, w_2\} + \{w_0, \delta(w_1), w_2\} + \{w_0, w_1, \delta(w_2)\}
\]

for all \( w_0, w_1, w_2 \in A_0 \).


In this paper, we investigate homomorphisms and derivations in proper JCQ\(^*\)-triples associated to the following Pexiderized Cauchy type functional equation

\[
f(x + y + z) = f_0(x) + f_1(y) + f_2(z).
\]

Throughout this paper, assume that \( k \) is a fixed positive integer.

2. Homomorphisms in proper JCQ\(^*\)-triples

Throughout this section, assume that \( (A, A_0, \{.,.,\}) \) is a proper JCQ\(^*\)-triple with \( C^* \)-norm \( \| \cdot \|_{A_0} \) and norm \( \| \cdot \|_A \), and that \( (B, B_0, \{.,.,\}) \) is a proper JCQ\(^*\)-triple with \( C^* \)-norm \( \| \cdot \|_{B_0} \) and norm \( \| \cdot \|_B \).

**Theorem 2.1.** Let \( \varphi : A \times A \times A \to [0, +\infty) \) be a function such that

\[
\lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n w_0, x, w_2) = 0
\]

for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). Assume that \( f, f_i : A \to B \) \((0 \leq i \leq 2)\) are mappings with \( f(0) = 0 \) and \( f(w), f_0(0), f_2(0) \in B_0 \) for all \( w \in A_0 \) and

\[
\|\mu f(x) - f_0(y) - f_1(z) - f_2(t)\|_B \leq k f\left(\frac{\mu x + y + z + t}{k}\right)\|_B,
\]

\[
\|f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \leq \varphi(w_0, x, w_2)
\]

for all \( \mu \in \mathbb{T} : = \{\mu \in \mathbb{C} : |\mu| = 1\} \), all \( w_0, w_2 \in A_0 \) and all \( x, y, z, t \in A \). Then the mapping \( f : A \to B \) is a proper JCQ\(^*\)-triple homomorphism. Moreover,

\[
f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)
\]

for all \( x \in A \).
Proof. Letting \( \mu = 1, x = y = z = t = 0 \) in (2.2), we get
\[
 f_0(0) + f_1(0) + f_2(0) = 0.
\]
So by letting \( \mu = 1, y = -x, \) and \( z = t = 0 \) in (2.2), we get
\[
 f(x) = f_0(-x) - f_0(0)
\]
for all \( x \in A \). Similarly, we have
\[
 f(x) = f_1(-x) - f_1(0) = f_2(-x) - f_2(0)
\]
for all \( x \in A \). So \( f_0(w), f_2(w) \in B_0 \) for all \( w \in A_0 \).

It follows from (2.2) that
\[
 \| \mu f(x + y) - f(\mu x) - f(\mu y) \|_B = \| \mu f(x + y) - f_0(-\mu x) - f_1(-\mu y) - f_2(0) \|_B = 0
\]
for all \( x, y \in A \) and all \( \mu \in T^1 \). Therefore, the mapping \( f : A \to B \) is additive and \( f(\mu x) = \mu f(x) \) for all \( x \in A \) and all \( \mu \in T^1 \). By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping \( f : A \to B \) is C-linear. If at least \( w_0 = 0 \) or \( x = 0 \) or \( w_2 = 0 \), then \( f(\{w_0, x, w_2\}) = f(0) = 0 \), and hence by (2.1) and (2.3), we have
\[
 \lim_{n \to \infty} \frac{1}{2^n} \| \{ f_0(2^n w_0), f_1(x), f_2(w_2) \} \|_B = 0.
\]
Also, we have
\[
 f(x) = f_i(0) - f_i(x)
\]
for all \( x \in A \) and all \( 0 \leq i \leq 2 \). So (2.3) implies that
\[
 \| f(\{w_0, x, w_2\}) - \{ f(w_0), f(x), f(w_2) \} \|_B = \lim_{n \to \infty} \frac{1}{2^n} \| f(2^n \{w_0, x, w_2\}) - \{ f(2^n w_0), f(x), f(w_2) \} \|_B = \lim_{n \to \infty} \frac{1}{2^n} \| f(2^n \{w_0, x, w_2\}) + \{ f_0(2^n w_0), f_1(x), f_2(w_2) \} \|_B \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n w_0, x, w_2) = 0
\]
for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). So
\[
 f(\{w_0, x, w_2\}) = \{ f(w_0), f(x), f(w_2) \}
\]
for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). Since \( f(w) \in B_0 \) for all \( w \in A_0 \), the mapping \( f : A \to B \) is a proper JCQ*-triple homomorphism, as desired. \( \square \)

Remark 2.2. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions
- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(w_0, 2^n x, w_2) = 0; \)
- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(w_0, x, 2^n w_2) = 0; \)
- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n w_0, 2^n x, 2^n w_2) = 0 \)
for all \( w_0, w_2 \in A_0 \) and all \( x \in A \).
Remark 2.3. We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- $\lim_{n \to \infty} 2^n \varphi(\frac{w_0}{2^n}, x, w_2) = 0$;
- $\lim_{n \to \infty} 2^n \varphi(w_0, \frac{x}{2^n}, w_2) = 0$;
- $\lim_{n \to \infty} 2^n \varphi(w_0, x, \frac{w_2}{2^n}) = 0$;
- $\lim_{n \to \infty} 8^n \varphi(\frac{w_0}{2^n}, \frac{x}{2^n}, \frac{w_2}{2^n}) = 0$.

for all $w_0, w_2 \in A_0$ and all $x \in A$.

Corollary 2.4. Let $\theta, r_i (0 \leq i \leq 2)$ be non-negative real numbers such that $r_0 + r_1 + r_2 \neq 3$ or $r_j \neq 1$ for some $0 \leq j \leq 2$. Suppose that $f, f_i : A \to B (0 \leq i \leq 2)$ are mappings satisfying (2.2) with $f(0) = 0$, and $f(w), f_0(0), f_2(0) \in B_0$ for all $w \in A_0$. Let

$$
\| f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\} \|_B \leq \theta \|\| w_0 \|_A^r_0 + \| x \|_A^r + \| w_2 \|_A^r \|
$$

for all $w_0, w_2 \in A_0$ and all $x \in A$ (by putting $\| . \|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism. Moreover,

$$
f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)
$$

for all $x \in A$.

Proof. It follows from Theorem 2.1 and Remarks 2.2 and 2.3.

Corollary 2.5. Let $\theta, r_i (0 \leq i \leq 2)$ be non-negative real numbers such that $r_j \in [0, 1)$ for some $0 \leq j \leq 2$ or $r_i < 3$ (respectively, $r_i > 3$) for all $0 \leq i \leq 2$. Suppose that $f, f_i : A \to B (0 \leq i \leq 2)$ are mappings satisfying (2.2) with $f(0) = 0$, and $f(w), f_0(0), f_2(0) \in B_0$ for all $w \in A_0$. Let

$$
\| f(\{w_0, x, w_2\}) + \{f_0(w_0), f_1(x), f_2(w_2)\} \|_B \leq \theta (\| w_0 \|_A^r + \| x \|_A^r + \| w_2 \|_A^r)
$$

for all $w_0, w_2 \in A_0$ and all $x \in A$ (by putting $\| . \|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism. Moreover,

$$
f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)
$$

for all $x \in A$.

Proof. The result follows from Theorem 2.1 and Remarks 2.2 and 2.3.

3. Derivations on proper $JCQ^*$-triples

Throughout this section, assume that $(A, A_0, \{., ., .\})$ is a proper $JCQ^*$-triple with $C^*$-norm $\| . \|_A$. We investigate derivations on proper $JCQ^*$-triples.

Theorem 3.1. Let $\varphi : A_0 \times A_0 \times A_0 \to [0, +\infty)$ be a function satisfying (2.1) for all $x, w_0, w_2 \in A_0$. Assume that $f, f_i : A_0 \to A (0 \leq i \leq 2)$ are mappings with $f(0) = 0$. Let

$$
(3.1) \quad \| \mu f(x) - f_0(w_0) - f_1(w_1) - f_2(w_2) \|_A \leq \| k f(\frac{\mu x + w_0 + w_1 + w_2}{k}) \|_A,
$$

for all $x, w_0, w_1, w_2 \in A_0$. Then $f$ is a $\mu$-derivation on $A_0$.
\[ \| f\{w_0, w_1, w_2\} + f_0(w_0), w_1, w_2 \} + \{w_0, f_1(w_1), w_2\} + \{w_0, f_2(w_2)\} \|_A \leq \varphi(w_0, w_1, w_2) \]

for all \( x, w_0, w_1, w_2 \in A_0 \). Then the mapping \( f : A_0 \to A \) is a proper JCQ*-triple derivation. Moreover,

\[ f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x) \]

for all \( x \in A_0 \).

**Proof.** By the same reasoning as in the proof of Theorem 2.1, the mapping \( f : A_0 \to A \) is \( C \)-linear and

\[ f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x) \]

for all \( x \in A_0 \). If at least \( w_0 = 0 \) or \( w_1 = 0 \) or \( w_2 = 0 \), then \( f(\{w_0, w_1, w_2\}) = f(0) = 0 \) and by (2.1) and (3.2), we have

\[ \lim_{n \to \infty} \frac{1}{2^n} \| f(2^n w_0), w_1, w_2 \} + \{2^n w_0, f_1(w_1), w_2\} + \{2^n w_0, f_2(w_2)\} \|_A = 0. \]

So (3.2) implies that

\[ \| f(\{w_0, w_1, w_2\}) - f(w_0), w_1, w_2 \} - \{w_0, f(w_1), w_2\} - \{w_0, w_1, f(w_2)\} \|_A \]

\[ = \lim_{n \to \infty} \frac{1}{2^n} \| f(2^n \{w_0, w_1, w_2\}) + \{f_0(2^n w_0), w_1, w_2\} + \{2^n w_0, f_1(w_1), w_2\} + \{2^n w_0, f_2(w_2)\} \|_A \]

\[ \leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n w_0, w_1, w_2) = 0 \]

for all \( w_0, w_1, w_2 \in A_0 \). Hence

\[ f(\{w_0, w_1, w_2\}) = f(w_0), w_1, w_2 \} + \{w_0, f(w_1), w_2\} + \{w_0, w_1, f(w_2)\} \]

for all \( w_0, w_1, w_2 \in A_0 \).

Therefore the mapping \( f : A_0 \to A \) is a proper JCQ*-triple derivation. \( \Box \)

**Remark 3.2.** We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(w_0, 2^n w_1, 2^n w_2) = 0 \);
- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(w_0, 2^n w_1, w_2) = 0 \);
- \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n w_0, 2^n w_1, 2^n w_2) = 0 \)

for all \( w_0, w_1, w_2 \in A_0 \).

**Remark 3.3.** We can formulate a similar theorem if we replace the condition (2.1) by one of the following conditions

- \( \lim_{n \to \infty} 2^n \varphi(\frac{w_0}{2^n}, w_1, w_2) = 0 \);
- \( \lim_{n \to \infty} 2^n \varphi(w_0, \frac{w_1}{2^n}, w_2) = 0 \);
- \( \lim_{n \to \infty} 2^n \varphi(w_0, w_1, \frac{w_2}{2^n}) = 0 \);
- \( \lim_{n \to \infty} 8^n \varphi(\frac{w_0}{2^n}, \frac{w_1}{2^n}, \frac{w_2}{2^n}) = 0 \)

for all \( w_0, w_1, w_2 \in A_0 \).
Corollary 3.4. Let $\theta, r_i$ $(0 \leq i \leq 2)$ be non-negative real numbers such that $r_0 + r_1 + r_2 \neq 3$ or $r_j \neq 1$ for some $0 \leq j \leq 2$. Suppose that $f, f_i : A_0 \to A$ $(0 \leq i \leq 2)$ are mappings satisfying (3.1) with $f(0) = 0$. Let
\[
\| f(\{w_0, w_1, w_2\}) + \{f_0(w_0), w_1, w_2\} + \{w_0, f_1(w_1), w_2\} \\
+ \{w_0, w_1, f_2(w_2)\} \|_A \leq \theta \| w_0 \|_{A_0}^{r_0} \| w_1 \|_{A_0}^{r_1} \| w_2 \|_{A_0}^{r_2}
\]
for all $w_0, w_1, w_2 \in A_0$ (by putting $\| \cdot \|_{A_0}^{0} = 1$). Then the mapping $f : A_0 \to A$ is a proper $JCQ^*$-triple derivation. Moreover,
\[
f(x) = f_0(0) - f_0(x) + f_1(0) - f_1(x) = f_2(0) - f_2(x)
\]
for all $x \in A$.

Proof. It follows from Theorem 3.1 and Remarks 3.2 and 3.3. \qed

Corollary 3.5. Let $\theta, r_i$ $(0 \leq i \leq 2)$ be non-negative real numbers such that $r_0 \in [0, 1)$ for some $0 \leq j \leq 2$ or $r_i < 3$ (respectively, $r_i > 3$) for all $0 \leq i \leq 2$. Suppose that $f, f_i : A_0 \to A$ $(0 \leq i \leq 2)$ are mappings satisfying (3.1) with $f(0) = 0$. Let
\[
\| f(\{w_0, w_1, w_2\}) + \{f_0(w_0), w_1, w_2\} + \{w_0, f_1(w_1), w_2\} \\
+ \{w_0, w_1, f_2(w_2)\} \|_A \leq \theta(\| w_0 \|_{A_0}^{r_0} + \| w_1 \|_{A_0}^{r_1} + \| w_2 \|_{A_0}^{r_2})
\]
for all $w_0, w_1, w_2 \in A_0$ (by putting $\| \cdot \|_{A_0}^{0} = 1$). Then the mapping $f : A_0 \to A$ is a proper $JCQ^*$-triple derivation. Moreover,
\[
f(x) = f_0(0) - f_0(x) = f_1(0) - f_1(x) = f_2(0) - f_2(x)
\]
for all $x \in A_0$.

Proof. It follows from Theorem 3.1 and Remarks 3.2 and 3.3. \qed

4. Stability of homomorphisms on proper $JCQ^*$-triples

In this section, by using an idea of Găvruta [6], we prove the generalized Hyers-Ulam stability of homomorphisms in proper $JCQ^*$-triples.

Theorem 4.1. Let $\varphi : A \times A \times A \to [0, +\infty)$ be a function such that $\varphi(0, 0, 0) = 0$ and
\[
\lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0,
\]
\[
(4.2) \quad \varphi(x) := \sum_{i=0}^{\infty} \frac{1}{2^i} \left[ \varphi(2^i x, 2^i y, 0) + \varphi(2^i x, 0, 0) + \varphi(0, 2^i x, 0) \right] < \infty
\]
for all $x, y, z \in A$. Suppose that $f, f_i : A \to B$ $(0 \leq i \leq 2)$ are mappings satisfying $f(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$ and all $0 \leq i \leq 2$. Let
\[
(4.3) \quad \| f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z) \|_B \leq \varphi(x, y, z),
\]
\[
(4.4) \quad \| f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2) \|_{B_0} \leq \varphi(w_0, w_1, w_2),
\]
for all \( x \in A \).

**Proof.** Letting \( y = z = 0 \) and \( \mu = 1 \) in (4.3), we get

(4.7) \[ \|f(x) - f_0(x) - f_1(0) - f_2(0)\|_B \leq \varphi(x, 0, 0) \]

for all \( x \in A \). Similarly, we get

(4.8) \[ \|f(y) - f_1(y) - f_0(0) - f_2(0)\|_B \leq \varphi(0, y, 0), \]

(4.9) \[ \|f(z) - f_2(z) - f_0(0) - f_1(0)\|_B \leq \varphi(0, 0, z) \]

for all \( y, z \in A \). Since \( f_0(0) + f_1(0) + f_2(0) = 0 \), by using (4.3), (4.7), (4.8) and (4.9), we get

(4.10) \[ \|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \leq \psi(x, y, z), \]

where

\[ \psi(x, y, z) := \varphi(x, y, z) + \varphi(x, 0, 0) + \varphi(0, y, 0) + \varphi(0, 0, z) \]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \). Letting \( y = x, z = 0 \) and \( \mu = 1 \) in (4.10), we get

(4.11) \[ \|f(2x) - 2f(x)\|_B \leq \psi(x, x, 0) \]

for all \( x \in A \). Replacing \( x \) by \( 2^n x \) in (4.11) and dividing both sides of (4.11) by \( 2^{n+1} \), we get

(4.12) \[ \left\| \frac{f(2^{n+1} x)}{2^{n+1}} - \frac{f(2^n x)}{2^n} \right\|_B \leq \frac{1}{2^{n+1}} \psi(2^n x, 2^n x, 0) \]

for all \( x \in A \) and all non-negative integers \( n \). By (4.12), we have

(4.13) \[ \left\| \frac{f(2^{n+1} x)}{2^{n+1}} - \frac{f(2^m x)}{2^m} \right\|_B \leq \frac{1}{2} \sum_{i=m}^{n} \psi(2^i x, 2^i x, 0) \]

for all \( x \in A \) and all non-negative integers \( n \) and \( m \) with \( n \geq m \). Thus we conclude from (4.2) and (4.13) that the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) is a Cauchy sequence.
in B for all \( x \in A \). Since B is complete, the sequence \( \{ \frac{1}{2^n} f(2^n x) \} \) converges in B for all \( x \in A \). So one can define the mapping \( H : A \to B \) by

\[
H(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) = \lim_{n \to \infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1, 2)
\]

for all \( x \in A \). Letting \( m = 0 \) and passing the limit when \( n \to \infty \) in (4.13), we get (4.6). It follows from (4.1), (4.3) and (4.14) that

\[
\left\| H(\mu x + \mu y) - \mu H(x) - \mu H(y) \right\|_B \\
= \lim_{n \to \infty} \frac{1}{2^n} \left\| f(2^n \mu x + 2^n \mu y) - \mu f_0(2^n x) - \mu f_1(2^n y) \right\|_B \\
\leq \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 0) = 0
\]

for all \( \mu \in T^1 \) and all \( x, y \in A \). Hence

\[ H(\mu x + \mu y) = \mu H(x) + \mu H(y) \]

for all \( \mu \in T^1 \) and all \( x, y \in A \). By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping \( H : A \to B \) is C-linear. It follows from (4.4) that the sequence \( \{ \frac{1}{2^n} f(2^n w) \} \) is a Cauchy sequence in \( B_0 \) for all \( w \in A_0 \). So \( H(w) \in B_0 \) for all \( w \in A_0 \). It follows from (4.1), (4.5) and (4.14) that

\[
\left\| H(\{w_0, x, w_2\}) - \{H(w_0), H(x), H(w_2)\} \right\|_B \\
\leq \lim_{n \to \infty} \frac{1}{8^n} \left\| f(8^n \{w_0, x, w_2\}) - \{f_0(2^n w_0), f_1(2^n x), f_2(2^n w_2)\} \right\|_B \\
\leq \lim_{n \to \infty} \frac{1}{8^n} \varphi(2^n w_0, 2^n x, 2^n w_2) = 0
\]

for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). Hence

\[ H(\{w_0, x, w_1\}) = \{H(w_0), H(x), H(w_2)\} \]

for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). So \( H : A \to B \) is a proper JCQ*-triple homomorphism. Now, we show that \( H \) is unique. Let \( T : A \to B \) be another proper JCQ*-triple homomorphism satisfying (4.6). It follows from (4.2), (4.6) and (4.14) that

\[
\left\| H(x) - T(x) \right\|_B = \lim_{n \to \infty} \frac{1}{2^n} \left\| f(2^n x) - T(2^n x) \right\|_B \\
\leq \frac{1}{2} \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x) = 0
\]

for all \( x \in A \). So \( H = T \).

\[ \square \]

**Theorem 4.2.** Let \( \phi : A \times A \times A \to [0, +\infty) \) be a function such that

\[
\lim_{n \to \infty} 8^n \phi\left(\frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n}\right) = 0,
\]

\[
\phi(x) := \sum_{i=1}^{\infty} 2^i \left[ \phi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0\right) + \phi\left(\frac{x}{2^i}, 0, 0\right) + \phi(0, \frac{x}{2^i}, 0) \right] < \infty
\]
for all \(x, y, z \in A\). Suppose that \(f, f_i : A \to B\) (\(0 \leq i \leq 2\)) are mappings satisfying \(f(0) = f_i(0) = 0\) and \(f(w), f_i(w) \in B_0\) for all \(w \in A_0\) and all \(0 \leq i \leq 2\). Let
\[
\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \leq \phi(x, y, z),
\]
(4.17)
\[
\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_B \leq \phi(w_0, w_1, w_2),
\]
(4.18)
\[
\|f(\{w_0, x, w_1\} - \{f_0(w_0), f_1(x), f_2(w_2)\}\|_B \leq \phi(w_0, x, w_2)
\]
for all \(\mu \in \mathbb{T}^1\), all \(w_0, w_1, w_2 \in A_0\) and all \(x, y, z \in A\). Then there exists a unique proper JCQ*\(\ast\)-triple homomorphism \(H : A \to B\) such that
\[
\|f(x) - H(x)\|_B \leq \frac{1}{2}\widetilde{\phi}(x),
\]
(4.19)
\[
\|f_0(x) - H(x)\|_B \leq \frac{1}{2}\widetilde{\phi}(x) + \phi(x, 0, 0),
\]
\[
\|f_1(x) - H(x)\|_B \leq \frac{1}{2}\widetilde{\phi}(x) + \phi(0, x, 0),
\]
\[
\|f_2(x) - H(x)\|_B \leq \frac{1}{2}\widetilde{\phi}(x) + \phi(0, 0, x)
\]
for all \(x \in A\).

**Proof.** Similar to Theorem 4.1, we get
\[
\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \leq \Psi(x, y, z),
\]
(4.20)
where
\[
\Psi(x, y, z) := \phi(x, y, z) + \phi(x, 0, 0) + \phi(0, y, 0) + \phi(0, 0, z)
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x, y, z \in A\). Letting \(y = x, z = 0\) and \(\mu = 1\) in (4.20), we get
\[
\|f(2x) - 2f(x)\|_B \leq \Psi(x, x, 0)
\]
(4.21)
for all \(x \in A\). Replacing \(x\) by \(\frac{x}{2^{m+1}}\) in (4.21) and multiplying both sides of (4.21) to \(2^n\), we get
\[
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|_B \leq 2^n \Psi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, 0\right)
\]
(4.22)
for all \(x \in A\) and all non-negative integers \(n\). By (4.22), we get
\[
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B \leq \sum_{i=m}^{n} 2^i \Psi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}, 0\right)
\]
(4.23)
for all \(x \in A\) and all non-negative integers \(n\) and \(m\) with \(n \geq m\). Thus we conclude from (4.16) and (4.23) that the sequence \(\{2^n f\left(\frac{x}{2^n}\right)\}\) is a Cauchy sequence in \(B\) for all \(x \in A\). Since \(B\) is complete, the sequence \(\{2^n f\left(\frac{x}{2^n}\right)\}\) converges in \(B\) for all \(x \in A\). So one can define the mapping \(H : A \to B\) by
\[
H(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right) = \lim_{n \to \infty} 2^n f_i\left(\frac{x}{2^n}\right)\quad (i = 0, 1, 2)
\]
for all $x \in A$. Letting $m = 0$ and passing the limit when $n \to \infty$ in (4.23), we get (4.19).

The rest of proof is similar to the proof of Theorem 4.1.

Corollary 4.3. Let $\theta, r_i$ ($0 \leq i \leq 2$) be non-negative real numbers such that $0 < r_i < 1$ (respectively, $r_i > 3$) for all $0 \leq i \leq 2$. Suppose that $f, f_i : A \to B$ $(0 \leq i \leq 2)$ are mappings with $f(0) = f_i(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$ and all $0 \leq i \leq 2$. Let

$$
\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \\
\leq \theta (\|x\|_A^{r_0} + \|y\|_A^{r_1} + \|z\|_A^{r_2}), \\
\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_B \\
\leq \theta (\|w_0\|_A^{r_0} + \|w_1\|_A^{r_1} + \|w_2\|_A^{r_2}), \\
\|f(x) - f(y)\|_B \\
\leq \theta (\|x\|_A^{r_0} + \|y\|_A^{r_1})
$$

for all $\mu \in \mathbb{T}^3$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$. Then there exists a unique proper $JCQ^*$-triple homomorphism $H : A \to B$ such that

$$
\|f(x) - H(x)\|_B \leq 2\theta \left[ \frac{\|x\|_A^{r_0}}{|2r_0 - 2r_1|} + \frac{\|y\|_A^{r_1}}{|2r_0 - 2r_1|} \right], \\
\|f_i(x) - H(x)\|_B \leq 2\theta \left[ \frac{\|x\|_A^{r_0}}{|2r_0 - 2r_1|} + \frac{\|y\|_A^{r_1}}{|2r_0 - 2r_1|} \right] + \theta \|x\|_A^{r_1}
$$

for all $x \in A$ and all $0 \leq i \leq 2$.

Theorem 4.4. Let $\theta, r_i$ ($0 \leq i \leq 2$) be non-negative real numbers such that $r_0 + r_1 + r_2 < 3$ and $0 < r_i < 1$ for some $0 \leq i \leq 2$. Suppose that $f, f_i : A \to B$ $(0 \leq i \leq 2)$ are mappings with $f(0) = 0$ and $f(w), f_i(w) \in B_0$ for all $w \in A_0$. Let

$$
\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \leq \theta \|x\|_A^{r_0} \|y\|_A^{r_1} \|z\|_A^{r_2}; \\
\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_B \leq \theta \|w_0\|_A^{r_0} \|w_1\|_A^{r_1} \|w_2\|_A^{r_2}
$$

for all $\mu \in \mathbb{T}^3$, all $w_0, w_1, w_2 \in A_0$ and all $x, y, z \in A$ (by putting $\|\cdot\|_A^0 = 1$). Then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism. Moreover, if $r_i, r_j > 0$ for some $0 \leq i < j \leq 2$, then

$$
f(x) = f_0(x) - f_0(0) = f_1(x) - f_1(0) = f_2(x) - f_2(0)
$$

for all $x \in A$.

Proof. Without loss of generality, we may assume that $0 < r_2 < 1$. It is clear that $f_0(0) + f_1(0) + f_2(0) = 0$. By letting $y = z = 0$ and $\mu = 1$ in (4.24), we get

$$
f(x) = f_0(x) - f_0(0)
$$

for all $x \in A$. Similarly, we have

$$
f(x) = f_1(x) - f_1(0)
$$
for all \( x \in A \). We have two cases:

**Case I.** \( r_0 = r_1 = 0 \). We infer from (4.24) that

\[
\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \\
\leq \|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \\
+ \|f(z) - f_0(0) - f_1(0) - f_2(z)\|_B \\
\leq 2\theta \|z\|_A^2
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \). By letting \( z = 0 \) in (4.28), we get

\[ f(\mu x + \mu y) = \mu f(x) + \mu f(y) \]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y \in A \). By the same reasoning as in the proof of Theorem 2.1 of [11], the mapping \( f : A \to B \) is \( C \)-linear. It follows from (4.24), (4.26), and (4.27) that

\[ f(x) = \lim_{n \to \infty} \frac{1}{2^n} f_0(2^n x) = \lim_{n \to \infty} \frac{1}{2^n} f_1(2^n x) = \lim_{n \to \infty} \frac{1}{2^n} f_2(2^n x) \]

for all \( x \in A \). So (4.25) implies that

\[
\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\|_B \\
= \lim_{n \to \infty} \frac{1}{8^n} \|f(\{2^n w_0, 2^n x, 2^n w_2\}) - \{f_0(2^n w_0), f_1(2^n x), f_2(2^n w_2)\}\|_B \\
\leq \theta \lim_{n \to \infty} \frac{2^{nr_2}}{8^n} \|w_2\|_A^n = 0
\]

for all \( w_0, w_2 \in A_0 \) and all \( x \in A \). Therefore,

\[ f(\{w_0, x, w_1\}) = \{f(w_0), f(x), f(w_2)\} \]

for all \( w_0, w_2 \in A_0 \) and \( x \in A \). So the mapping \( f : A \to B \) is a proper \( JCQ^* \)-triple homomorphism.

**Case II.** \( r_0 > 0 \) or \( r_1 > 0 \). Without loss of generality, we may assume that \( r_1 > 0 \). Letting \( x = y = 0 \) and \( \mu = 1 \) in (4.24), we get that \( f(z) = f_2(z) - f_2(0) \) for all \( z \in A \). It follows from (4.24) that

\[
\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \\
= \|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \\
\leq \theta \|x\|_A^n \|y\|_A^n \|z\|_A^n
\]

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \). By putting \( z = 0 \) in the last inequality, we infer that the mapping \( f \) is \( C \)-linear. The rest of the proof is similar to the proof of Case I.

\[ \square \]

The following theorem is an alternative result of Theorem 4.4 and its proof is similar to the proof of Theorem 4.4.

**Theorem 4.5.** Let \( \theta, r_i \) \((0 \leq i \leq 2)\) be non-negative real numbers such that \( r_i > 3 \) for some \( 0 \leq i \leq 2 \). Suppose that \( f, f_i : A \to B \) \((0 \leq i \leq 2)\) are mappings satisfying (4.24) and (4.25) (by putting \( \|\cdot\|_A = 1 \)) with \( f(0) = f_i(0) = 0 \) and \( f(w), f_0(w), f_2(w) \in B_0 \) for all \( w \in A_0 \). Then the mapping \( f : A \to B \)
is a proper JCQ*-triple homomorphism. Moreover, if \( r_i, r_j > 0 \) for some \( 0 \leq i < j \leq 2 \), then
\[
f(x) = f_i(x)
\]
for all \( x \in A \) and all \( 0 \leq i \leq 2 \).

For \( r_0 = r_1 = r_2 = 0 \), we have the following theorem.

**Theorem 4.6.** Let \( \theta \) be non-negative real number and let \( f, f_i : A \to B (0 \leq i \leq 2) \) be mappings such that \( f(w), f_i(w) \in B_0 \) for all \( w \in A_0 \) and
\[
\|f(\mu x + \mu y + \mu z) - \mu f_0(x) - \mu f_1(y) - \mu f_2(z)\|_B \leq \theta, \\
\|f(w_0 + w_1 + w_2) - f_0(w_0) - f_1(w_1) - f_2(w_2)\|_{B_0} \leq \theta, \\
\|f(\{w_0, x, w_1\}) - \{f_0(w_0), f_i(x), f_2(w_2)\}\|_B \leq \theta
\]
for all \( \mu \in \mathbb{T}^1 \), all \( w_0, w_1, w_2 \in A_0 \) and all \( x, y, z \in A \). Then there exists a unique proper JCQ*-triple homomorphism \( H : A \to B \) such that
\[
\|f(x) + f(0) - H(x)\|_B \leq 4\theta + 2M, \\
\|f_i(x) - f_i(0) - H(x)\|_B \leq 6\theta + 4M \quad (i = 0, 1, 2)
\]
for all \( x \in A \), where \( M = \|f_0(0) + f_1(0) + f_2(0)\|_B \).

**Proof.** Similar to the proof of Theorem 4.1, we have
\[
\|f(\mu x + \mu y + \mu z) - \mu f(x) - \mu f(y) - \mu f(z)\|_B \leq 4\theta + 2M
\]
for all \( x, y, z \in A \) and all \( \mu \in \mathbb{T}^1 \), where \( M = \|f_0(0) + f_1(0) + f_2(0)\|_B \). Using the same proof as in Theorem 4.1, we infer that
\[
\|\frac{1}{2^{n+1}} f(2^{n+1}x) - \frac{1}{2^n} f(2^nx) - \sum_{i=m}^{n} \frac{1}{2^{i+1}} f(0)\|_B \leq (2\theta + M) \sum_{i=m}^{n} \frac{1}{2^i}
\]
for all \( x \in A \) and all non-negative integers \( n \) and \( m \) with \( n \geq m \). Thus we conclude from (4.29) that the sequence \( \{\frac{1}{2^n} f(2^n x)\} \) is a Cauchy sequence in \( B \) for all \( x \in A \). Since \( B \) is complete, the sequence \( \{\frac{1}{2^n} f(2^n x)\} \) converges in \( B \) for all \( x \in A \). So one can define the mapping \( H : A \to B \) by
\[
H(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x) = \lim_{n \to \infty} \frac{1}{2^n} f_i(2^n x) \quad (i = 0, 1, 2)
\]
for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 4.1. \( \square \)

**Theorem 4.7.** Let \( \theta \geq 0, r_0, r_1, r_2 \) be real numbers such that \( r_0 + r_1 > 0 \) and \( r_2 < 0 \). Assume that \( f, f_i : A \to B (0 \leq i \leq 2) \) are mappings with \( f_0(0) = f_i(0) = 0 \) and \( f(w), f_0(w), f_2(w) \in B_0 \) for all \( w \in A_0 \) and satisfying
\[
\|f(x + y + z) - f_0(x) - f_1(y) - f_2(z)\|_B \leq \theta\|x\|_A \|y\|_A \|z\|_A, \\
\|f(\{w_0, x, w_2\}) - \{f_0(w_0), f_i(x), f_2(w_2)\}\|_B \leq \theta\|w_0\|_A \|x\|_A \|w_2\|_A
\]
for all \( w_2 \in A_0 \setminus \{0\} \), all \( w_0 \in A_0 \setminus \{0\} \) if \( r_0 < 0 \) and all \( x, y, z \) in \( A \) (\( x \in A \setminus \{0\} \) if \( r_0 < 0 \) and \( y \in A \setminus \{0\} \) if \( r_1 < 0 \), \( z \in A \setminus \{0\} \). If the
mappings \( t \rightarrow f(tx) \) and \( t \rightarrow f_i(tx) \) \((0 \leq i \leq 2)\) are continuous in \(0 \in \mathbb{R}\) for each fixed \(x \in A\), then

(i) \( f = f_0 = f_1 = f_2 \),

(ii) the mapping \( f : A \rightarrow B \) is a proper \( JCQ^* \)-triple homomorphism.

**Proof.** Without loss of generality, we may assume that \( r_1 > 0 \). Letting \( y = 0 \) in (4.30) and \( x = 0 \) in (4.31), we get

\[
(4.32) \quad f(0) = 0, \quad f(x + z) = f_0(x) + f_2(z), \quad (x, z \neq 0).
\]

Replacing \( x \) and \( z \) by \( \frac{x}{n} \) and \( \frac{z}{n} \), respectively, in (4.32) and letting \( n \to \infty \), we get that \( f_2(0) = 0 \). Letting \( y = -x \) in (4.30) and using (4.32), we get

\[
(4.33) \quad \| f(z) - f(x + z) - f_1(-x) \|_B \leq \theta \| x \|^p + r_1 \| z \|^p, \quad (x, z \neq 0).
\]

Therefore

\[
(4.34) \quad \lim_{n \to \infty} f\left(\frac{x}{n} + z\right) = f(z), \quad (x, z \neq 0).
\]

Since \( f(0) = 0 \), (4.34) holds for all \( x, z \in A \). It follows from (4.32) and (4.34) that \( f = f_2 \). So by replacing \( z \) by \( \frac{z}{n} \) in (4.32) and letting \( n \to \infty \) and using (4.34), we get \( f = f_0 \). Hence (4.32) implies that the mapping \( f \) is additive. Thus (4.30) means that

\[
\| f(y) - f_1(y) \|_B \leq \theta \| x \|^p \| y \|^q \| z \|^q, \quad (y \in A, x, z \neq 0).
\]

So \( f = f_1 \), and this proves (i).

To prove (ii), since the mapping \( f \) is additive, by the same reasoning as in the proof of the main result of [13], the mapping \( f : A \rightarrow B \) is \( C \)-linear. Now, (4.31) implies that

\[
\left\| f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\right\|_B = \lim_{n \to \infty} \frac{1}{n} \left\| f(\{w_0, x, nw_2\}) - \{f(w_0), f(x), f(nw_2)\}\right\|_B \leq \theta \lim_{n \to \infty} n^{r_2-1} \left\| w_0 \right\|^p \left\| x \right\|^q \left\| w_2 \right\|^q = 0
\]

for all \( x \in A \) and all \( w_0, w_2 \in A_0 \setminus \{0\} \). Since \( f(0) = 0 \), therefore

\[
f(\{w_0, x, w_2\}) = \{f(w_0), f(x), f(w_2)\}
\]

for all \( x \in A \) and all \( w_0, w_2 \in A_0 \). So the mapping \( f : A \rightarrow B \) is a proper \( JCQ^* \)-triple homomorphism. \( \square \)

**Theorem 4.8.** Let \( \theta \geq 0 \) and \( r_0, r_1 < 0 \) be real numbers and let \( s_0, s_1, s_2 \) be real numbers such that \( s_j \neq 1 \) for some \( 0 \leq j \leq 2 \). Assume that \( f : A \rightarrow B \) is a mapping with \( f(0) = 0 \) and \( f(w) \in B_0 \) for all \( w \in A_0 \) and

\[
(4.35) \quad \| f(x + y) - f(x) - f(y) \|_B \leq \theta \| x \|^p \| y \|^q,
\]

\[
(4.36) \quad \| f(\{w_0, x, w_1\}) - \{f(w_0), f(x), f(w_2)\}\|_B \leq \theta \| w_0 \|^p \| x \|^q \| w_2 \|^q
\]
for all $w_0, w_2 \in A_0 \setminus \{0\}$ and all $x, y \in A \setminus \{0\}$. If the mapping $t \mapsto f(tx)$ is continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism.

Proof. Let $y \in A \setminus \{0\}$. Replacing $x$ and $y$ in (4.35) by $\frac{y}{2} + ny$ and $\frac{y}{2} - ny$, respectively, we get

(4.37) $\quad f(y) = \lim_{n \to \infty} [f(\frac{y}{2} + ny) + f(\frac{y}{2} - ny)]$

for all $y \in A \setminus \{0\}$. Since $f(0) = 0$, (4.37) holds for all $y \in A$. Let $x, y \in A \setminus \{0\}$. It follows from (4.35) and (4.37) that

$$
\|f(x + y) - f(x) - f(y)\|_B \\
= \lim_{n \to \infty} \|f(\frac{x + y}{2} + n(x + y)) + f(\frac{x + y}{2} - n(x + y)) \\
- f(\frac{x}{2} + nx) - f(\frac{x}{2} - nx) - f(\frac{y}{2} + ny) - f(\frac{y}{2} - ny)\|_B \\
\leq \limsup_{n \to \infty} \|f(\frac{x + y}{2} + n(x + y)) - f(\frac{x}{2} + nx) - f(\frac{y}{2} + ny)\|_B \\
+ \limsup_{n \to \infty} \|f(\frac{x + y}{2} - n(x + y)) - f(\frac{x}{2} - nx) - f(\frac{y}{2} - ny)\|_B \\
\leq \theta \left( \lim_{n \to \infty} \left( \frac{1}{2} + n \right)^{r_0 + r_1} + \lim_{n \to \infty} \left( n - \frac{1}{2} \right)^{r_0 + r_1} \right) \|x\|_A^{s_0} \|y\|_A^{s_1} = 0.
$$

So we have $f(x + y) = f(x) + f(y)$ for all $x, y \in A \setminus \{0\}$. Since $f(0) = 0$, we get that the mapping $f$ is additive. By the same reasoning as in the proof of the main result of [13], the mapping $f : A \to B$ is $C$-linear. Without loss of generality we may assume that $s_0 \neq 1$. Let $s_0 < 1$ (we have similar proof when $s_0 > 1$). It follows from (4.36) that

$$
\|f(\{w_0, x, w_2\}) - \{f(w_0), f(x), f(w_2)\}\|_B \\
= \lim_{n \to \infty} \frac{1}{n} \|f(\{n w_0, x, w_2\}) - \{f(n w_0), f(x), f(w_2)\}\|_B \\
\leq \theta \lim_{n \to \infty} n^{s_0 - 1} \|w_0\|_A^{s_0} \|x\|_A^{s_1} \|w_2\|_A^{s_2} = 0
$$

for all $x \in A \setminus \{0\}$ and all $w_0, w_2 \in A_0 \setminus \{0\}$. Since $f(0) = 0$, we get that

$$
f(\{w_0, x, w_2\}) = \{f(w_0), f(x), f(w_2)\}
$$

for all $x \in A$ and all $w_0, w_2 \in A_0$. So the mapping $f : A \to B$ is a proper $JCQ^*$-triple homomorphism.\[\square\]

References


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