SOME CONSTRUCTIONS OF
IMPLICATIVE/COMMUTATIVE $d$-ALGEBRAS

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Abstract. In this paper, we give some constructions of implicative/commutative
$d$-algebras which are not $BCK$-algebras. This demonstrates that the notion of
implicative/commutative $d$-algebras are indeed generalizations of the same in $BCK$-algebras.

1. Preliminaries

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([5, 6]). $BCK$-algebras have some connections with
other areas: D. Mundici [10] proved that $MV$-algebras are categorically equivalent to bounded commutative $BCK$-algebras, and J. Meng [8] proved that
implicative commutative semigroups are equivalent to a class of $BCK$-algebras.
Z. Riečanová [14] showed that extendable commutative $BCK$-algebras directed
upwards are equivalent to generalized $MV$-effect algebras. G. Georgescu and
A. Iorgulescu [2] introduced the notion of pseudo-$BCK$ algebras as an extension
and weak pseudo-$BL$ algebras (pseudo-$MTL$ algebras). J. Neggers and H. S.
Kim introduced the notion of $d$-algebras which is another useful generalization
of $BCK$-algebras, and then investigated several relations between $d$-algebras
and $BCK$-algebras as well as several other relations between $d$-algebras and
oriented digraphs [12]. After that some further aspects were studied ([7, 11,
13]). J. S. Han et al. [3] defined a variety of special $d$-algebras, such as strong
$d$-algebras, (weakly) selective $d$-algebras and others. The main assertion is that
the squared algebra $(X; \boxtimes, 0)$ of a $d$-algebra is a $d$-algebra if and only if the
root $(X; *, 0)$ of the squared algebra $(X; \boxtimes, 0)$ is a strong $d$-algebra.

In this paper, we give some constructions of implicative/commutative $d$-
algebras which are not $BCK$-algebras. This demonstrates that the notion of
implicative/commutative $d$-algebras are indeed generalizations of the same in $BCK$-algebras.

2. Introduction

An \textit{(ordinary) $d$-algebra} ([12, 13]) is a non-empty set $X$ with a constant $0$ and a binary operation "$*$" satisfying the following axioms:

(A) $x * x = 0$,
(B) $0 * x = 0$,
(C) $x * y = 0$ and $y * x = 0$ imply $x = y$ for all $x, y \in X$.

A $BCK$-algebra is a $d$-algebra $X$ satisfying the following additional axioms:

(D) $(x * y) * (x * z) = (z * y)$,
(E) $x * (x * y) * y = 0$ for all $x, y, z \in X$.

\textbf{Example 2.1} ([3]). Consider the real numbers $\mathbb{R}$, and suppose that $(\mathbb{R}; *, e)$ has the multiplication

$$x * y = (x - y)(x - e) + e.$$ 

Then $x * x = e; e * x = e; x * y = y * x = e$ yields $(x - y)(x - e) = 0, (y - x)(y - e) = e$ and $x = y$ or $x = e = y$, i.e., $x = y$, i.e., $(\mathbb{R}; *, e)$ is a $d$-algebra.

\textbf{Theorem 2.2} ([4, p. 162]). Let $X$ be a set with $0 \in X$. If we define a binary operation $*$ on $X$ by

$$x * y := \begin{cases} 
0 & \text{if } x = y, \\
x & \text{if } x \neq y,
\end{cases}$$

then $(X, *, 0)$ is an implicative $BCK$-algebra.

3. Commutative $d$-algebra

\textbf{Definition 3.1}. A field $(X, +, \cdot)$ is called $\sqrt{3}$-\textit{exponential} if there is a function $\varphi : X \rightarrow X$ such that

(E1) $\varphi(\varphi(x)) = x^3$,
(E2) $\varphi(xy) = \varphi(x)\varphi(y)$,
(E3) if $x \neq 0$, then $\varphi(x) \neq 0$,
(E4) $\varphi(0) = 0$

for any $x, y \in X$.

\textbf{Example 3.2}. Let $X := \mathbb{R}$ be the set of all real numbers. If we define a map $\varphi : X \rightarrow X$ by

$$\varphi(x) := \begin{cases} x\sqrt{3} & \text{if } x > 0, \\
0 & \text{if } x = 0, \\
-y\sqrt{3} & \text{if } x = -y < 0,
\end{cases}$$

then $(\mathbb{R}, +, \cdot)$ is $\sqrt{3}$-exponential.
**Proposition 3.3.** Let \((X, +, \cdot)\) be a \(\sqrt{3}\)-exponential field. If we define a new binary operation \(\ast\) on \(X\) by \(x \ast y := x^2\varphi(y)y\) for any \(x, y \in X\), then \(x \ast (x \ast y) = y \ast (y \ast x)\) for any \(x, y \in X\).

**Proof.** Given \(x, y \in X\), we have
\[
x \ast (x \ast y) = x^2\varphi(x \ast y)(x \ast y)
= x^2\varphi(x^2\varphi(y)y)(x^2\varphi(y)y)
= x^4y^4\varphi(x^2\varphi(y)y)^2.
\]
Similarly, we obtain \(y \ast (y \ast x) = y^4x^4\varphi(y)^2\varphi(x)^2\), proving the proposition. 

Using the notion of \(\sqrt{3}\)-exponential field, we construct a commutative \(d\)-algebra which is not a \(BCK\)-algebra.

**Theorem 3.4.** Let \((X, +, \cdot)\) be a \(\sqrt{3}\)-exponential field and let \(x \ast y := x^2\varphi(y)y\) for any \(x, y \in X\). If we define a binary operation \(\ast\) on \(X\) by
\[
x \ast y := \begin{cases} 
0 & \text{if } x = 0 \text{ or } x = y, \\
x & \text{if } y = 0, \\
x \ast y & \text{otherwise}, 
\end{cases}
\]
then \((X, \ast, 0)\) is a commutative \(d\)-algebra.

**Proof.** Let \(x \ast y = y \ast x = 0\). If \(x = 0\) or \(y = 0\), then it is easy to see that \(x = y\). If we assume that \(xy \neq 0\) and \(x \neq y\), then \(x^2\varphi(y)y = y^2\varphi(x)x = 0\), which leads to \(\varphi(x) = \varphi(y) = 0\). By (E3) we obtain \(x = y\), a contradiction. Hence \((X, \ast, 0)\) is a \(d\)-algebra.

We claim that \((X, \ast)\) is commutative. If \(xy \neq 0\) and \(x \neq y\), then \(x \ast (x \ast y) = x \ast (x \ast y) = y \ast (y \ast x) = y \ast (y \ast x)\) by Proposition 3.3. The other cases are trivial. This proves the theorem. 

Note that the commutative \(d\)-algebra \((X, \ast, 0)\) described in Theorem 3.4 need not be a \(BCK\)-algebra, as in Example 3.2, where \((2 \ast (2 \ast 1)) \ast 1 = 2^8(2\sqrt{3})^4 = 2^{8+4}\sqrt{3} \neq 0\). Moreover, it is not implicative, since \(x \ast (y \ast x) = x^{6+2\sqrt{3}}y^{2\sqrt{3}} \neq x\).

We give another method for finding commutative \(d\)-algebras which are not \(BCK\)-algebras.

### 4. (Positive-)Implicative \(d\)-algebras

**Proposition 4.1.** Let \(X\) be a field and let \(x, y \in X\). If we define
\[
x \ast y := x(x - y)\varphi(x, y)
\]
where \(\varphi : X \times X \to X\) is a function with \(\varphi(x, y) \neq 0\) for any \(x, y \in X\). Then \((X, \ast, 0)\) is a \(d\)-algebra.
Proof. If we assume that \( x * y = y * x = 0 \), then \( x(x - y)\varphi(x, y) = 0 \) and \( y(y - x)\varphi(y, x) = 0 \) and hence \( x(x - y) = 0 = y(y - x) \). This leads to \( x = y \), since \( x - y \neq 0 \) implies \( x = 0, y = 0 \), i.e., \( x = y \), a contradiction. Hence \((X, *, 0)\) is a \(d\)-algebra. \(\Box\)

The \(d\)-algebra \((X, *, 0)\) described in Proposition 4.1 is called a \(\varphi\)-function \(d\)-algebra.

A \(d/BCK\)-algebra \((X, *, 0)\) is said to be implicative ([2, 10]) if \(x = x*(y*x)\) for any \(x, y \in X\).

**Proposition 4.2.** If \((X, *, 0)\) is an implicative \(d\)-algebra, then \(x * 0 = x\) for any \(x \in X\).

**Proof.** If \(X\) is implicative, then \(x = x*(y*x)\) for any \(x, y \in X\). If we let \(y := x\), then \(x = x*(x*x) = x * 0\), proving the proposition. \(\Box\)

**Proposition 4.3.** Let \((X, *, 0)\) be a \(\varphi\)-function \(d\)-algebra. Then \((X, *, 0)\) is implicative if and only if \(\varphi\) satisfies the condition:

\[
\varphi(x, y * x) := \begin{cases} 
\frac{1}{x-y*z} & \text{if } x \neq 0, \\
 a & \text{otherwise},
\end{cases}
\]

where \(a\) is an arbitrary element of \(X\).

**Proof.** Straightforward. \(\Box\)

Note that if \(x \neq 0\), then \(x \neq y * x\) in Proposition 4.3. A \(d/BCK\)-algebra \((X, *, 0)\) is said to be positive implicative ([2, 10]) if \((x * y) * z = (x * z) * (y * z)\) for any \(x, y, z \in X\).

**Proposition 4.4.** There are no positive implicative \(\varphi\)-function \(d\)-algebras which are not \(BCK\)-algebras.

**Proof.** Assume that the implicative \(\varphi\)-function \(d\)-algebra \((X, *, 0)\) which is not a \(BCK\)-algebra is positive implicative. Then \((x * y) * z = (x * z) * (y * z)\) for any \(x, y, z \in X\). If we let \(z := x\), then \((x * y) * x = (x * x) * (y * x) = 0 * (y * x) = 0\), i.e., \((x * y) * x = 0\). Since \((X, *, 0)\) is a \(\varphi\)-function \(d\)-algebra, we have \(0 = (x * y)(x * y - x)\varphi(x * y, x)\). Since \(\varphi(x, y) \neq 0, \forall x, y \in X\), we obtain \(0 = (x * y)(x * y - x)\). Therefore, either \(x * y = 0\) or \(x * y = x\), i.e., \(x * y \in \{0, x\}, \forall x, y \in X\). Assume that there are \(x, y \in X\) such that \(x \neq 0, x \neq y\) and \(x * y = 0\). Then \(0 = x * y = x(x - y)\varphi(x, y) \neq 0\), a contradiction. Hence we have \(x * y = 0\) if \(x = y\) and \(x * y = x\) if \(x \neq y\), i.e., \((X, *, 0)\) is an implicative \(BCK\)-algebra by Theorem 2.2, a contradiction. \(\Box\)

**Theorem 4.5 ([9]).** A \(BCK\)-algebra \(X\) is positive implicative if and only if \((x * y) * y = x * y\) for any \(x, y \in X\).

**Theorem 4.6.** If the \(\varphi\)-function \(d\)-algebra \((X, *, 0)\) is implicative, then \((x * y) * y = x * y\) for any \(x, y \in X\).
Proof. Let the $\varphi$-function $d$-algebra $(X,*,0)$ be implicative. Then we have $x = x(x - y*x)\varphi(x,y*x)$ for any $x, y \in X$. Assume that $x \neq 0$, since $x = 0$ implies $(0*y)*z = 0 = (0*z)*(y*z)$. Then we have $1 = (x - y*x)\varphi(x,y*x)$.

Hence, $\varphi(x,y*x) = \frac{1}{x - y*x}$. Also, $y*x \neq 0$ and $y*x \neq x*(y*x)$. Then $\varphi(y*x,x) = \varphi(y*x,x*(y*x)) = \frac{1}{y*x - x*(y*x)} = \frac{1}{x - y*x} = -\varphi(x,y*x)$, i.e., we obtain

$$\varphi(y*x,x) = -\varphi(x,y*x).$$

Given $x, y \in X$, we have

$$
(y*x)*x = (y*x)(y*x - x)\varphi(y*x,x)
= (y*x)(y*x - x)[-\varphi(x,y*x)]
= (y*x)(x - y*x)\varphi(x,y*x).
$$

Since $x = x*(y*x) = x(x - y*x)\varphi(x,y*x)$, we have

$$x - (y*x)*x = (y*x - x)^2\varphi(x,y*x)
= (y*x - x)^2\frac{1}{x - y*x}
= x - y*x,$$

proving the theorem. \qed

Note that in $BCK$-algebras, the condition $(x*y)*(x*z) = (x*y)*x$ is equivalent to the condition $(x*y)*y = x*y$, but it is not equivalent in $d$-algebras in general. This can be demonstrated by Theorem 4.6 and Example 4.8.

Example 4.7. If we define a map $\varphi : X \to X$ by

$$\varphi(x,y) := \begin{cases} 
\frac{1}{x - y} & \text{if } x(x - y) \neq 0, \\
a & \text{if } x = y, \\
b & \text{if } x = 0,
\end{cases}$$

then the function $\varphi$ satisfies the conditions of Proposition 3.2, and so it defines a $\varphi$-function $d$-algebra $(X,*,0)$ where

$$x*y := \begin{cases} 
x & \text{if } x \neq y, \\
0 & \text{if } x = y,
\end{cases}$$

which is an implicative $BCK$-algebra as described in Theorem 2.2.

We need to find an implicative $d$-algebra which is not a $BCK$-algebra. Consider the following example.

Example 4.8. If we define a map $\varphi$ on $X$ by

$$\varphi(x,y) := \begin{cases} 
\frac{-y}{x(y-x)} & \text{if } y(y-x) \neq 0, \\
a & \text{otherwise}
\end{cases}$$
for an arbitrary element \(a\) in \(X\), then

\[
x * y := \begin{cases} 
-y & \text{if } y(y - x) \neq 0, \\
0 & \text{if } x = 0 \text{ or } x = y, \\
x & \text{if } y = 0
\end{cases}
\]

leads to a \(d\)-algebra. If \(y(y - x) \neq 0\), then \(x * (y * x) = x * (-x) = x\) for any \(x, y \in X\), showing that \((X, *, 0)\) is an implicat\(ive d\)-algebra. Indeed, it is not a \(BCK\)-algebra, since \(((3 * 4) * (3 * 5)) * (5 * 4) = 4 \neq 0\).

**Example 4.9.** If we apply Example 4.8 to the finite field \(\mathbb{Z}_5\), then we obtain the following table:

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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>4</td>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then it is an implicat\(ive d\)-algebra, which is not a \(BCK\)-algebra, since \(((3 * 4) * (3 * 2)) * (2 * 4) = 4 \neq 0\). Moreover, it is not positive implicat\(ive\), since \((3 * 4) * 5 = -4 * 5 = -5\) and \((3 * 5) * (4 * 5) = -5 * -5 = 5\).

**Remark.** In \(BCK\)-algebras, \(X\) is an implicat\(ive \(BCK\)-algebra if and only if it is both a positive implicat\(ive\) and a commutat\(ive \(BCK\)-algebra. But this does not hold in \(d\)-algebras. See Example 4.9.

**References**


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