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THE CLASS OF MODULES WITH PROJECTIVE COVER

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ABSTRACT. Let R be a ring. A right R-module M is called perfect if M possesses a projective cover. In this paper, we consider the relationship between the class of perfect modules and other classes of modules. Some known rings are characterized by these relationships.

1. Introduction

Throughout this paper, R is an associative ring with identity and all modules are unitary. Let U and M be two right R-modules. U is said to be M-projective in case for each R-epimorphism $g : M \to N$ and each R-homomorphism $f : U \to N$ there is an R-homomorphism $h : U \to M$ such that $f = gh$. If U is M-projective for every right R-module M, then U is said to be projective. Projective right R-modules are just direct summands of free right R-modules. $N \leq M$ means N is a submodule of M. A submodule N of a module M is said to be *superfluous or small* ($N \ll M$) if for any submodule K of M, $N + K = M$ implies that $K = M$.

Dualizing the concept of an injective hull of a module, Bass [3] defined a projective cover of a module M to be an epimorphism $p : P \to M$ such that P is a projective module and $\mathrm{Ker}(p) \ll P$. Thus modules having projective covers are, up to isomorphism, of the form P/K, where P is a projective module and K its small submodule. Unlike injective hulls, projective covers of modules seldom exist. For instance, over a semiprimitive ring R (i.e., $J(R) = 0$, where $J(R)$ denotes the Jacobson radical of R) a module has a projective cover if and only if it is already projective (see [11, 24.11(5)]). Recall that a right R-module M is called perfect if M possesses a projective cover [18]. A ring R is said to be right perfect if every right R-module is perfect [3].

In Section 2 of this paper, we investigate some properties of the class of perfect modules. We prove that the class of perfect modules is closed under extensions. New characterizations of semisimple rings and perfect rings are given using the relationship between the class of perfect modules and other

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classes of modules. For example, it is shown that a ring R is right perfect if and only if every singular right R-module has a projective cover if and only if every perfect right R-module is cotorsion.

In Section 3, we study $\mathcal{C}_P$-hereditary rings (rings whose small right ideals are projective). It is proven that a ring R is right $\mathcal{C}_P$-hereditary if and only if every perfect right R-module has projective dimension at most 1.

In what follows, $\mathcal{M}_R$ is the category of right R-modules. The projective (resp. injective) dimension of right R-module M is denoted by $pd(M)$ (resp. $id(M)$). Let $\mathcal{C}_P = \{M \in \mathcal{M}_R \,|\, M \text{ is perfect}\}$, $\mathcal{P}_n = \{M \in \mathcal{M}_R \,|\, pd(M) \leq n\}$, where n is a fixed nonnegative integer and $\mathcal{I}_n = \{M \in \mathcal{M}_R \,|\, id(M) \leq n\}$, where n is a fixed nonnegative integer.

2. Some properties of $\mathcal{C}_P$ and perfect rings

We begin with the following

Lemma 2.1. *Let R be a ring and M a right R-module. If $A \ll M$ and M is a direct summand of B, then $A \ll B$.*

Proof. It is a special case of [1, Proposition 5.20]. □

Lemma 2.2. *Consider the following commutative diagram of right R-modules with exact rows and columns:*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \xrightarrow{\alpha_1} & K_2 & \xrightarrow{\alpha_2} & K_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \\
0 & \longrightarrow & A & \xrightarrow{\beta_1} & B & \xrightarrow{\beta_2} & C & \longrightarrow & 0,
\end{array}$$

where $t_1(K_1) \ll A$, $t_3(K_3) \ll C$. If C is a projective module, then $t_2(K_2) \ll B$.

Proof. We may assume, without loss of generality, that $K_1 \leq K_2$, $K_1 \leq A$, $K_2 \leq B$, $K_3 \leq C$ and $A \leq B$; t_1, t_2, t_3, α_1, β_1 are inclusion homomorphisms. Since $K_1 \ll K_2 \leq B$, we have $K_1 \ll B$ by [1, Lemma 5.18]. Note that $K_1 \subseteq A = \mathrm{Ker}(\beta_2)$, so there exists a unique epimorphism $h : B/K_1 \to C$ such that $\beta_2 = h\pi$, where π is the natural epimorphism $B \to B/K_1$. Thus $B/K_1 \xrightarrow{h} C \longrightarrow 0$ is split since C is a projective module, and hence C is a direct summand of B/K_1. Note that $K_2/K_1 \cong K_3 \ll C$, we have $K_2/K_1 \ll B/K_1$ by Lemma 2.1. So $K_2 \ll B$ by [1, Proposition 5.17]. □

Proposition 2.3. $\mathcal{C}_P = \{M \in \mathcal{M}_R \,|\, M$ *is perfect*$\}$ *is closed under extensions.*

Proof. Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be a short exact sequence with $A, C \in \mathcal{C}_P$. It suffices to prove that $B \in \mathcal{C}_P$. We have the following commutative diagram by [10, Exercise 9F, P82] or the proof of [14, Lemma 6.20]:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & K_2 & \longrightarrow & K_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle t_2} & & \downarrow{\scriptstyle t_3} & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & ,
\end{array}$$

where all rows and columns are exact, P_1, P_2, P_3 are projective modules, and t_1, t_3 are projective covers of A, B respectively. Then $K_2 \ll P_2$ by Lemma 2.2, and so $t_2 : P_2 \to B$ is the projective cover of B, as desired. □

Corollary 2.4. *If M is a perfect right R-module and P is a projective right R-module, then $P \oplus M$ is a perfect right R-module.*

Remark. For a short exact sequence $0 \to A \to B \to C \to 0$, in general, C needn't have a projective cover when A and B have projective covers. For example, $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$ is exact, and $\mathbb{Z}_2$ does not have a projective cover. But we have the following:

Proposition 2.5. *Let R be a ring and M a right R-module, the following statements are equivalent:*

(1) $M \in \mathcal{C}_P$;
(2) *There is a small submodule A of M such that $M/A \in \mathcal{C}_P$;*
(3) $M/A \in \mathcal{C}_P$ *for every small submodule A of M.*

Proof. It's easy to verify by [1, Lemma 27.5]. □

Theorem 2.6. *For a ring R, the following statements are equivalent:*

(1) *R is semisimple;*
(2) $\mathcal{C}_P = \{M \mid M$ *is a semisimple right R-module*$\}$;
(3) $\mathcal{C}_P = \{M \mid M$ *is a flat right R-module*$\}$;
(4) $\mathcal{C}_P = \mathcal{I}_0$;
(5) $\mathcal{C}_P \subseteq \mathcal{I}_0$.

Proof. (1) ⇒ (2), (3), (4), and (4) ⇒ (5) are clear.

(2) ⇒ (1) Since every semisimple right R-module has a projective cover, R is right perfect by [16, Theorem 5]. Then every right R-module has a projective cover, and hence every right R-module is semisimple. Thus R is semisimple.

(3) ⇒ (1) Since every flat right R-module has a projective cover, R is right perfect by [6, Theorem 2.1]. Therefore every right R-module has a projective cover, and so every right R-module is flat. Thus R is a von Neumann regular ring, and hence R is semisimple.

(5) ⇒ (1) Since every projective right R-module is injective, R is a QF ring by [1, Theorem 31.9]. Hence R is right perfect. Thus every right R-module is injective, as desired. □

Theorem 2.7. *For a ring R, the following statements are equivalent:*

(1) *R is right perfect;*
(2) *$\mathcal{C}_P \supseteq \{M \mid M$ is a singular right R-module$\}$;*
(3) *$\mathcal{C}_P = \{C \mid C$ is a cotorsion right R-module$\}$;*
(4) *$\mathcal{C}_P \subseteq \{C \mid C$ is a cotorsion right R-module$\}$.*

Proof. (1) ⇒ (2), and (3) ⇒ (4) are clear.

(2) ⇒ (1) Let M be any semisimple right R-module. By [8, Proposition 1.24], any simple right R-module is either singular or projective, but not both. Let $M = (\oplus P_i) \oplus (\oplus S_j)$, where every P_i is a projective simple right R-module and every S_j is a singular simple right R-module. Hence $\oplus P_i$ is a projective module and $\oplus S_j$ is a singular module by [8, Proposition 1.22(b)]. Consequently M is perfect by Corollary 2.4 and R is right perfect by [16, Theorem 5].

(1) ⇒ (3) follows from [19, Proposition 3.3.1].

(4) ⇒ (1) Since the free right R-module $R^{(\mathbb{N})}$ is cotorsion, R is right perfect by [9, Corollary 20]. □

Remark. Cotorsion modules have turned out to be a very useful in characterizing rings. For a more detailed discussion of these aspects, we refer the reader to [13] and [19]. We just give a result which improves [13, Theorem 2.18(6)]. A ring R is a right perfect ring if and only if there is a right T-nilpotent ideal I such that R/I is a right perfect ring: Since R/I is right perfect, every right R/I-module is cotorsion by [19, Proposition 3.3.1]. Thus every right R/I-module is cotorsion as a right R-module by [19, Proposition 3.3.3]. Let M be a flat right R-module. Then M/MI is a projective R/I-module by [15, Lemma 5.1]. By [15, Theorem 5.2], M is a projective right R-module, hence R is right perfect by [1, Theorem 28.4].

Corollary 2.8. *For a ring R, if $\mathcal{C}_P \subseteq \{C \mid C$ is a pure-injective right R-module$\}$, then R is a right artinian ring.*

Proof. If $\mathcal{C}_P \subseteq \{C \mid C$ is a pure-injective right R-module$\}$, then R is right perfect by Theorem 2.7 and so every right R-module is pure-injective. Thus R is a right artinian ring by [2, Proposition 5]. □

3. $\mathcal{C}_P$-hereditary rings

Proposition 3.1. *A ring R is semiprimitive if and only if $\mathcal{C}_P = \mathcal{P}_0$.*

Proof. If $J(R) = 0$, then $\mathcal{C}_P = \mathcal{P}_0$ by [11, 24.11(5)]. Conversely, if $J(R) \neq 0$, then $R/J(R) \in \mathcal{C}_P = \mathcal{P}_0$. Thus the exact sequence $0 \to J(R) \to R \to R/J(R) \to 0$ is split, and hence $J(R)$ is the direct summand of R. So $J(R) = 0$, as desired. □

Corollary 3.2. *For a semiprimitive ring R, the following statements are equivalent:*

(1) $\mathcal{C}_P = \mathcal{P}_1$;
(2) $\mathcal{C}_P = \mathcal{P}_n$ *for any integer n with $1 \leq n < \infty$.*

Lemma 3.3. *If A is a small submodule of $M = M_1 \oplus M_2$, then $A \cap M_1 \ll M_1$.*

Proof. Let $\pi_1 : M_1 \oplus M_2 \to M_1$ be the 1st coordinate map. By [1, Lemma 5.18], $\pi_1(A) \ll M_1$. Hence $A \cap M_1 \leq \pi_1(A)$ is a small submodule of M_1. □

Let R be a ring and $\mathscr{L}$ a class of right R-modules. Fay and Joubert [7] introduced $\mathscr{L}$-injective modules which generalize the notion of injective modules. A right R-module M is called $\mathscr{L}$-injective if $\mathrm{Ext}^1_R(L, M) = 0$ for each $L \in \mathscr{L}$. Following [7], a ring R is called right $\mathscr{L}$-hereditary if homomorphic images of $\mathscr{L}$-injective right R-modules are $\mathscr{L}$-injective. Let $\mathscr{L} = \mathcal{C}_P$, then a right R-module M is $\mathcal{C}_P$-injective if and only if every homomorphism from a small submodule of a projective right R-module P to M can be extended to an R-homomorphism from P to M. A $\mathcal{C}_P$-injective right R-module must be a small injective right R-module [17]. A ring R is called right $\mathcal{C}_P$-hereditary if homomorphic images of $\mathcal{C}_P$-injective right R-modules are $\mathcal{C}_P$-injective. Recall that a ring R is right hereditary if every right ideal of R is projective, or equivalently, if every submodule of a projective right R-module is projective. A right $\mathcal{C}_P$-hereditary ring has the similar property as shown by the following theorem.

Theorem 3.4. *For a ring R, the following statements are equivalent:*

(1) $\mathcal{C}_P \subseteq \mathcal{P}_1$;
(2) *R is right $\mathcal{C}_P$-hereditary;*
(3) *every small right ideal is projective;*
(4) *every small submodule of a projective right R-module is projective;*
(5) *every factor module of an injective right R-module is $\mathcal{C}_P$-injective.*

Proof. (1) $\Leftrightarrow$ (4), and (4) $\Rightarrow$ (3) are clear. (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) by [20, Theorem 3.3].

(3) $\Rightarrow$ (4) The proof is modeled on that of [4, Theorem I.5.3] or [14, Theorem 4.17]. Let A be a small submodule of a projective right R-module P. Suppose P is a direct summand of F, where F is a free module with a well ordered base $\{x_\alpha\}$. Then A is a small submodule of F by [5, I.2.2(6)]. We denote by F_α (or G_α) the submodule of F consisting of elements which can be expressed by means of generators x_β with $\beta < \alpha$ (or $\beta \leq \alpha$), i.e.,

$$F_\alpha = \bigoplus_{\beta<\alpha} x_\beta R \text{ (or } G_\alpha = \bigoplus_{\beta\leq\alpha} x_\beta R).$$

Then each element $y \in A \cap G_\alpha$ has a unique decomposition $y = x + x_\alpha r$ with $x \in F_\alpha$ and $r \in R$. The mapping $\varphi_\alpha : y \mapsto r$ maps $A \cap G_\alpha$ onto a right ideal I_α with kernel $A \cap F_\alpha$. Since $A \cap G_\alpha$ is a small submodule of G_α by Lemma 3.3, I_α is a small right ideal by [1, Lemma 5.18]. Thus I_α is projective by hypothesis, and so the following exact sequence is split:

$$0 \longrightarrow A \cap F_\alpha \longrightarrow A \cap G_\alpha \longrightarrow I_\alpha \longrightarrow 0,$$

where $I_\alpha = \mathrm{Im}(\varphi_\alpha)$. Hence $A \cap G_\alpha = (A \cap F_\alpha) \oplus C_\alpha$, where $C_\alpha \cong I_\alpha$. We shall show that A is a direct sum of the modules C_α.

Firstly, the relation $c_1 + \cdots + c_n = 0$ with $c_i \in C_{\alpha_i}$, $\alpha_1 < \cdots < \alpha_n$, implies that $c_i = 0$; indeed, the sum of $A \cap F_{\alpha_n}$ and C_{α_n} being direct, we have $c_1 + \cdots + c_{n-1} = 0$, $c_n = 0$; the assertion then follows by recursion on n. Secondly, A is the sum of the modules C_α; assume to the contrary $A \neq \sum_\alpha C_\alpha$. Then there is a least index β such that there is an element $a \in A \cap G_\beta$ which is not in $\sum_\alpha C_\alpha$. Since $a = b + c$ with $b \in A \cap F_\beta$, $c \in C_\beta$, it follows that b is not in $\sum_\alpha C_\alpha$. However $b \in A \cap G_\gamma$ for some $\gamma < \beta$, thus contradicting the minimality of β. □

Examples:

(1) Semiprimitive rings are $\mathcal{C}_P$-hereditary rings. In particular, von Neumann regular rings are $\mathcal{C}_P$-hereditary rings.
(2) Right hereditary rings are right $\mathcal{C}_P$-hereditary rings. But the converse is not true in general. For example, the global dimension of the ring $\prod_{i=1}^{\infty} \mathbb{C}$ is ≥ 2, where $\mathbb{C}$ is the complex field, but it is a von Neumann regular ring.
(3) **Small's Example.** A right $\mathcal{C}_P$-hereditary need not be left $\mathcal{C}_P$-hereditary. Let $R = \left(\begin{smallmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$. Then R is right hereditary left semihereditary, but R is not left $\mathcal{C}_P$-hereditary since $I = \left(\begin{smallmatrix} 0 & \mathbb{Q} \\ 0 & 0 \end{smallmatrix}\right)$ is a small left ideal which is not projective by [12, Example 2.33]. This example also shows that a left semihereditary ring need not be a left $\mathcal{C}_P$-hereditary ring.

Proposition 3.5. *For a right perfect ring R, the following statements are equivalent:*

(1) *R is right $\mathcal{C}_P$-hereditary;*

(2) *R is right hereditary.*

Proof. Since $\mathcal{M}_R = \mathcal{C}_P \subseteq \mathcal{P}_1$, every right R-module M, $pd(M) \leq 1$, as desired. $\square$

There exists an artinian ring which is not $\mathcal{C}_P$-hereditary. For example, the ring $\mathbb{Z}_4$ is artinian but $J(\mathbb{Z}_4) = 2\ \mathbb{Z}_4$ is not a projective $\mathbb{Z}_4$-module, hence $\mathbb{Z}_4$ is not a $\mathcal{C}_P$-hereditary ring.

Corollary 3.6. *For a right artinian ring R, the following statements are equivalent:*

(1) *R is right $\mathcal{C}_P$-hereditary;*
(2) *R is right hereditary;*
(3) *$J(R)$ is projective as a right R-module;*
(4) *every maximal right ideal of R is projective as a right R-module.*

Proof. It follows from Proposition 3.5 and [12, Theorem 2.35]. $\square$

Recall that a right R-module M is called a lifting module if for any submodule N of M, there exists a direct summand K of M such that $K \leq N$ and $N/K \ll M/K$. For more details of lifting modules we refer the reader to [5].

Proposition 3.7. *For a right $\mathcal{C}_P$-hereditary ring R, if every projective right R-module is lifting, then R is right hereditary.*

Proof. Let N be a submodule of projective right R-module P. It is enough to prove N is projective. Since P is lifting, there exists a direct summand K of P such that $K \leq N$ and $N/K \ll P/K$. Since N/K is projective by Theorem 3.4, N is a projective module. $\square$

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