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NEW ITERATIVE ALGORITHMS FOR ZEROS OF ACCRETIVE OPERATORS

YISHENG SONG

ABSTRACT. Two new iterative algorithms are provided to find zeros of accretive operators in a Banach space E with a uniformly Gâteaux differentiable norm. Strong convergence for two iterations is proved and as applications, the viscosity approximation results are obtained also.

1. Introduction

Throughout this paper, a Banach space E will always be over the real scalar field. We denote its norm by $\|\cdot\|$ and its dual space by E^*. The value of $x^* \in E^*$ at $y \in E$ is denoted by $\langle y, x\rangle$ and the *normalized duality mapping* J from E into 2^{E^*} is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|\|f\|, \|x\| = \|f\|\}, \quad \forall\, x \in E.$$

Let $F(T) = \{x \in E : Tx = x\}$ denote the set of all fixed point for a mapping T. It is well known (see, for example, [22]) that E is smooth if and only if J is single-valued.

A mapping $A : D(A) \subset E \to 2^E$ is called to be *accretive* if for all $x, y \in D(A)$ there exists $j(x-y) \in J(x-y)$ such that

$$\langle u - v, j(x-y)\rangle \geq 0 \quad \text{for } u \in Ax \text{ and } v \in Ay.$$

If E is a Hilbert space, accretive operators are also called *monotone*. An operator A is called *m-accretive* if it is accretive and $R(I + rA)$, range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy *the range condition* if $\overline{D(A)} \subset R(I + rA), \forall r > 0$, where I is an identity operator of E and $\overline{D(A)}$ denotes the closure of the domain of A.

Interest in accretive mappings stems mainly from their firm connection with equations of evolution. It is known (see, e.g., [25]) that many physically significant problems can be modeled by initial-value problems of the form

$$x'(t) + Ax(t) = 0, \; x(0) = x_0, \tag{1.1}$$

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where A is an accretive operator in an appropriate Banach space. Typical examples where such evolution equations occur can be found in the heat, wave, or Schrodinger equations. One of the fundamental results in the theory of accretive operators, due to Browder [2], states that if A is locally Lipschitzian and accretive then A is m-accretive. This result was subsequently generalized by Martin [8] to the continuous accretive operators. If in (1.1) $x(t)$ is independent of t , then (1.1) reduces to $Au = 0$ whose solutions correspond to the equilibrium points of system (1.1). Consequently, considerable research effects have been devoted, especially within the past 20 years or so, to iterative methods for approximating these equilibrium points. For example, Bruck [3] introduced an iteration process and proved the convergence of the process to a zero of a maximal monotone operator in a Hilbert space. In [12], Reich extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. In 2003, Benavides-Acedoand-Xu [1] used the proximal point algorithm of Rockafellar [14] and the iterative methods of Halpern [7] to find a zero of an m-accretive operator A in a uniformly smooth Banach space with a weakly continuous duality map J_φ with gauge φ in virtue of the resolvent J_r of A. Other investigation for a zero of an accretive operator can be found in [4, 5, 6, 10, 11, 13, 18, 20, 21, 23, 24].

In this paper, we shall introduce two new iterative schemes for approaching a zero of an accretive operator A:

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n) J_{r_n} x_n, \tag{1.2}$$

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) J_{r_n}(\alpha_n u + (1 - \alpha_n) x_n), \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying some appropriate conditions. The main purposes of this paper is to establish the strong convergence for the explicit iteration schemes (1.2) and (1.3) to a zero of an accretive operator A in a suitable framework of Banach spaces. Furthermore, as applications, we obtain the viscosity approximation results.

2. Preliminaries and basic results

Let $A : D(A) \subset E \to 2^E$ be an accretive operator and $A^{-1}0 = \{x \in D(A); 0 \in Ax\}$. We use J_r and A_r to denote the resolvent and Yosida's approximation of A, respectively. Namely,

$$J_r = (I + rA)^{-1} \text{ and } A_r = \frac{I - J_r}{r}, \ r > 0.$$

For J_r and A_r, the following is well known (see, [22, pp. 129–144]):

(i) $A_r x \in AJ_r x$ for all $x \in R(I + rA)$;
(ii) $\|A_r x\| \leq |Ax| = \inf\{\|y\|; y \in Ax\}$ for all $x \in D(A) \cap R(I + rA)$;
(iii) $J_r : R(I + rA) \to D(A)$ is nonexpansive (i.e., $\|J_r x - J_r y\| \leq \|x - y\|$ for all $x, y \in R(I + rA)$);
(iv) $A^{-1}0 = F(J_r) = \{x \in D(J_r); J_r x = x\}$;

(v) (The Resolvent Identity) For $r > 0$ and $t > 0$ and $x \in E$,

$$J_r x = J_t\left(\frac{t}{r}x + \left(1 - \frac{t}{r}\right)J_r x\right). \tag{2.1}$$

The norm of a Banach space E is said to be *Gâteaux differentiable* (or E is said to be *smooth*) if the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{$*$}$$

exists for each x, y on the unit sphere $S(E)$ of E. Moreover, if for each y in $S(E)$ the limit defined by $(*)$ is uniformly attained for x in $S(E)$, we say that the norm of E is *uniformly Gâteaux differentiable.* A Banach space E is said to (i) *uniformly smooth* if the limit $(*)$ is attained uniformly for $(x, y) \in S(E) \times S(E)$; (ii) *strictly convex* if $\frac{\|x+y\|}{2} < 1$ for all $\|x\| = \|y\| = 1$, $x \neq y$; (iii) *uniformly convex* if $\forall \varepsilon \in [0, 2]$, $\exists \delta_\varepsilon > 0$ such that $\frac{\|x+y\|}{2} < 1 - \delta_\varepsilon$ for all $\|x\| = \|y\| = 1$ with $\|x - y\| \geq \varepsilon$.

Lemma 2.1 (Suzuki [17, Lemma 2]). *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E and $\beta_n \in [0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n$ for all integers $n \geq 1$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let μ be a continuous linear functional on l^∞ satisfying $\|\mu\| = 1 = \mu(1)$. Then we know that μ is a mean on $\mathbb{N}$ if and only if

$$\inf\{a_n; n \in \mathbb{N}\} \leq \mu(a) \leq \sup\{a_n; n \in \mathbb{N}\}$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$. According to time and circumstances, we $\mu_n(a_n)$ instead of $\mu(a)$. A mean μ on $\mathbb{N}$ is called a *Banach limit* if

$$\mu_n(a_n) = \mu_n(a_{n+1})$$

for every $a = (a_1, a_2, \ldots) \in l^\infty$.

Lemma 2.2 ([23, Lemma 1]). *Let C be a nonempty closed convex subset of a Banach space E with a uniformly Gâteaux differentiable norm. Let $\{x_n\}$ be a bounded sequence of E and let μ_n be a Banach limit and $z \in C$. Then*

$$\mu_n \|x_n - z\|^2 = \min_{y \in C} \mu_n \|x_n - y\|^2$$

if and only if

$$\mu_n \langle y - z, J(x_n - z) \rangle \leq 0, \ \forall y \in C.$$

Lemma 2.3 ([15, Proposition 2]). *Let α is a real number and $(x_0, x_1, \ldots) \in l^\infty$ such that $\mu_n x_n \leq \alpha$ for all Banach limits. If $\limsup_{n\to\infty}(x_{n+1} - x_n) \leq 0$, then* $\limsup_{n\to\infty} x_n \leq \alpha$.

Lemma 2.4 ([1, Lemma 2.3]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the property:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\beta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\beta_n\} \subset \mathbb{R}$ such that (i) $\sum\limits_{n=0}^{\infty} \gamma_n = \infty$, (ii) $\limsup\limits_{n\to\infty} \beta_n \leq 0$. *Then $\{a_n\}$ converges to zero as $n \to \infty$.*

3. The strong convergence of iterations

In this section, we shall study the strong convergence of the Rockafellar type iteration (3.1) and Halpern type iteration (3.2):

$$x_{n+1} = \beta_n x_n + (1-\beta_n)J_{r_n}(\alpha_n u + (1-\alpha_n)x_n), \tag{3.1}$$

$$x_{n+1} = \alpha_n u + \beta_n x_n + (1-\alpha_n-\beta_n)J_{r_n}x_n, \tag{3.2}$$

where both $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the following conditions:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0$; (C2) $\sum\limits_{n=1}^{+\infty} \alpha_n = +\infty$; (C3) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;
(C4) $\liminf\limits_{n\to\infty} r_n > 0$ and $\lim\limits_{n\to\infty} \frac{r_n}{r_{n+1}} = 1$.

Theorem 3.1. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Assumed that $\overline{D(A)}$ is convex subset of E and every nonempty bounded closed convex subset of $\overline{D(A)}$ has the fixed point property for nonexpansive self-mappings. Suppose that $0 \in R(A)$ and for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by* (3.1). *If $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions* (C1), (C2), (C3), *and* (C4), *then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

Proof. The proof consists of the following steps.

Step 1. $\{x_n\}$ is bounded. Since $0 \in R(A)$, we can take $y \in A^{-1}0 = F(J_r)$. It follows from the nonexpansion of J_r for all $r > 0$ that

$$\begin{aligned}
\|x_{n+1} - y\| &\leq \|\beta_n x_n + (1-\beta_n)J_{r_n}(\alpha_n u + (1-\alpha_n)x_n) - y\| \\
&\leq \beta_n\|x_n - y\| + (1-\beta_n)\|J_{r_n}(\alpha_n u + (1-\alpha_n)x_n) - y\| \\
&\leq \beta_n\|x_n - y\| + (1-\beta_n)\alpha_n\|u - y\| + (1-\beta_n)(1-\alpha_n)\|x_n - y\| \\
&\leq [1 - \alpha_n(1-\beta_n)]\|x_n - y\| + (1-\beta_n)\alpha_n\|u - y\| \\
&\leq \max\{\|x_n - y\|, \|u - y\|\} \\
&\;\;\vdots \\
&\leq \max\{\|x_1 - y\|, \|u - y\|\}.
\end{aligned}$$

So, the set $\{x_n\}$ is bounded. Setting $z_n = \alpha_n u + (1-\alpha_n)x_n$, then so is $\{z_n\}$. This implies the boundness of $\{J_{r_n}z_n\}$ by $\|J_{r_n}z_n - y\| \le \|z_n - y\|$. Let $M = \sup_{n\in\mathbb{N}}\{\|u\|, \|z_n\|, \|x_n\|, \|J_{r_n}z_n\|\}$, where $\mathbb{N}$ denotes the set of all positive integers.

Step 2. $\lim_{n\to\infty}\|z_n - J_r z_n\| = 0$ for all $r > 0$ and $\lim_{n\to\infty}\|z_{n+1} - z_n\| = 0$.

From the control condition (C1), it follows that

$$\lim_{n\to\infty}\|z_n - x_n\| = \lim_{n\to\infty}\alpha_n\|u - x_n\| = 0. \tag{3.3}$$

From the resolvent identity (2.1), we have

$$y_n = J_{r_n}z_n = J_{r_{n-1}}\Big(\frac{r_{n-1}}{r_n}z_n + \big(1 - \frac{r_{n-1}}{r_n}\big)J_{r_n}z_n\Big).$$

Therefore,

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|J_{r_{n+1}}z_{n+1} - J_{r_n}z_n\| \\
&\le \Big\|\frac{r_n}{r_{n+1}}(z_{n+1} - z_n) + \big(1 - \frac{r_n}{r_{n+1}}\big)(J_{r_{n+1}}z_{n+1} - z_n)\Big\| \\
&\le \|z_{n+1} - z_n\| + \Big|1 - \frac{r_n}{r_{n+1}}\Big|(\|z_{n+1} - z_n\| + \|J_{r_{n+1}}z_{n+1} - z_n\|) \\
&\le \|x_{n+1} - x_n\| + \alpha_n\|x_n\| + \alpha_{n+1}\|x_{n+1}\| + |\alpha_n - \alpha_{n+1}|\|u\| + 4M\Big|1 - \frac{r_n}{r_{n+1}}\Big| \\
&\le \|x_{n+1} - x_n\| + 4M\Big(\alpha_n + \alpha_{n+1} + \Big|1 - \frac{r_n}{r_{n+1}}\Big|\Big).
\end{aligned}$$

Thus, by the conditions (C1) and (C4), as $n \to \infty$,

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le 4M\Big(\alpha_n + \alpha_{n-1} + \Big|1 - \frac{r_n}{r_{n+1}}\Big|\Big) \to 0.$$

Hence,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Applying Lemma 2.1 to (3.1) with the condition (C3), we obtain

$$\lim_{n\to\infty}\|x_n - J_{r_n}z_n\| = \lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.4}$$

Combing with (3.3), we have

$$\lim_{n\to\infty}\|z_n - J_{r_n}z_n\| = 0. \tag{3.5}$$

By the condition (C4), we have

$$\begin{aligned}
\|J_r J_{r_n}z_n - J_{r_n}z_n\| &= \|(I - J_r)J_{r_n}z_n\| = r\|A_r J_{r_n}z_n\| \le r|AJ_{r_n}z_n| \\
&\le r\|A_{r_n}z_n\| = r\frac{\|z_n - J_{r_n}z_n\|}{r_n} \to 0 \ (n \to \infty),
\end{aligned}$$

then

$$\lim_{n\to\infty}\|J_r J_{r_n}z_n - J_{r_n}z_n\| = 0. \tag{3.6}$$

For all $r > 0$, we also have

$$\begin{aligned}\|z_n - J_r z_n\| &\le \|z_n - J_{r_n} z_n\| + \|J_{r_n} z_n - J_r J_{r_n} z_n\| + \|J_r J_{r_n} z_n - J_r z_n\| \\ &\le 2\|z_n - J_{r_n} z_n\| + \|J_{r_n} z_n - J_r J_{r_n} z_n\|.\end{aligned}$$

Combining (3.5) and (3.6), we obtain that for all $r > 0$,

$$\lim_{n\to\infty} \|z_n - J_r z_n\| = 0. \tag{3.7}$$

Since $\lim_{n\to\infty} \|x_n - J_{r_n} z_n\| = 0$, it follows that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|J_{r_n} z_n - x_n\| = 0.$$

Since $\|z_{n+1} - z_n\| \le \|z_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| + \|x_n - z_n\|$, then, noting (3.3),

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0.$$

Step 3. $\limsup\limits_{n\to\infty}\langle u - p, J(z_n - p)\rangle \le 0$ for some $p \in A^{-1}0$. In fact, the following real valued function g can be defined on $\overline{D(A)}$ by

$$g(y) = \mu_n\|z_n - y\|^2 \quad \text{for all } y \in \overline{D(A)}.$$

Clearly, $g(y)$ is continuous and convex in $\overline{D(A)}$, and $\lim_{\|y\|\to\infty} g(y) = \infty$. Let

$$K = \{z \in \overline{D(A)};\ g(z) = \min\{g(y);\ y \in \overline{D(A)}\}\}.$$

Then, it follows that K is a nonempty bounded closed convex subset of $\overline{D(A)}$ [22, pp. 18–25]. Furthermore, $J_r(K) \subset K$ for all $r > 0$. In fact, for each $z \in K$, we have from (3.7) that

$$\begin{aligned}g(J_r z) &= \mu_n\|z_n - J_r z\|^2 \le \mu_n(\|z_n - J_r z_n\| + \|J_r z_n - J_r z\|)^2 \\ &\le \mu_n\|z_n - z\|^2 = g(z).\end{aligned}$$

So, $J_r z \in K$. Hence, by the hypothesis, there exists $p \in F(J_r) \cap K \subset A^{-1}0$. It follows from Lemma 2.2 that

$$\mu_n\langle u - p, J(z_n - p)\rangle \le 0.$$

Since $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$, then it follows from the norm-weak* uniform continuity of the duality mapping J that

$$\lim_{n\to\infty} (\langle u - p, J(z_{n+1} - p)\rangle - \langle u - p, J(z_n - p)\rangle) = 0.$$

Hence, an application of Lemma 2.3 for the sequence $\{\langle u - p, J(z_n - p)\rangle\}$ to yield

$$\limsup_{n\to\infty}\langle u - p, J(z_n - p)\rangle \le 0. \tag{3.8}$$

Step 4. $x_n \to p$. Using Eq.(3.1), we make the following estimates:

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&= \langle \beta_n(x_n - p) + (1 - \beta_n)(J_{r_n} z_n - p), J(x_{n+1} - p)\rangle \\
&\leq \beta_n \|x_n - p\| \|J(x_{n+1} - p)\| + (1 - \beta_n)\|J_{r_n} z_n - p\| \|J(x_{n+1} - p)\| \\
&\leq \beta_n \frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} + (1 - \beta_n)\frac{\|z_n - p\|^2 + \|x_{n+1} - z\|^2}{2}
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - p\|^2 &= \alpha_n \langle u - p, J(z_n - p)\rangle + (1 - \alpha_n)\langle x_n - p, J(z_n - p)\rangle \\
&\leq \alpha_n \langle u - p, J(z_n - p)\rangle + (1 - \alpha_n)\|x_n - p\| \|z_n - p\| \\
&\leq \alpha_n \langle u - p, J(z_n - p)\rangle + (1 - \alpha_n)\frac{\|x_n - p\|^2 + \|z_n - p\|^2}{2}.
\end{aligned}$$

Therefore,

$$\|x_{n+1} - p\|^2 \leq \beta_n \|x_n - p\|^2 + (1 - \beta_n)\|z_n - p\|^2$$

and

$$\|z_n - p\|^2 \leq 2\alpha_n \langle u - p, J(z_n - p)\rangle + (1 - \alpha_n)\|x_n - p\|^2.$$

Combing to yield

$$\|x_{n+1} - p\|^2 \leq [1 - \alpha_n(1 - \beta_n)]\|x_n - p\|^2 + 2\alpha_n(1 - \beta_n)\langle u - p, J(z_n - p)\rangle. \tag{3.9}$$

The assumptions (C2) and (C3) implies $\sum_{n=1}^{\infty} \alpha_n(1 - \beta_n) = \infty$. Hence, we apply Lemma 2.4 to (3.9) with (3.8) to obtain the desired result. □

Theorem 3.2. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Suppose that for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by (3.1) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $0 \in R(A)$ and $D(A)$ is convex, then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

Proof. We observe that E is assumed to be a strictly convex space instead of to have the fixed point property for nonexpansive self-mappings in Theorem 3.1. So, we only need show Step 3. Following the proof technique of Theorem 3.1, the function g can be defined on $\overline{D(A)}$ by

$$g(y) = \mu_n \|z_n - y\|^2 \text{ for all } y \in \overline{D(A)}$$

and

$$K = \{z \in \overline{D(A)}; g(z) = \min\{g(y); y \in \overline{D(A)}\}\}.$$

Then, it follows that K is a nonempty bounded closed convex subset of $\overline{D(A)}$ and $J_r(K) \subset K$ for all $r > 0$. Let $y \in F(J_r) = A^{-1}0$. It follows from Day-James Theorem ([9, Theorem 5.1.18, Corollary 5.1.19]) that there exists an unique $p \in K$ such that

$$\|p - y\| = \inf_{x \in K} \|p - x\|.$$

Since $y = J_r y$ and $J_r p \in K$,

$$\|y - J_r p\| = \|J_r y - J_r p\| \leq \|y - p\|.$$

Hence $p = J_r p$ by the uniqueness of $p \in K$. Thus $p \in K \cap F(J_r) \subset A^{-1}0$. The remainder proof is the same as Theorem 3.1, we omit it. □

Theorem 3.3. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Assumed that $D(A)$ is a convex subset of E and every nonempty bounded closed convex subset of $\overline{D(A)}$ has the fixed point property for nonexpansive self-mappings. Suppose that for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by (3.2) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $0 \in R(A)$, then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

Proof. As in the proof of Theorem 3.1, we proceed with the following steps.

Step 1. $\{x_n\}$ is bounded. Take $y \in A^{-1}0 = F(J_r)$. It follows that

$$\begin{aligned} \|x_{n+1} - y\| &= \|\alpha_n u + \beta_n x_n + (1 - \alpha_n - \beta_n) J_{r_n} x_n - y\| \\ &\leq \alpha_n \|u - y\| + \beta_n \|x_n - y\| + (1 - \alpha_n - \beta_n) \|J_{r_n} x_n - y\| \\ &\leq \alpha_n \|u - y\| + \beta_n \|x_n - y\| + (1 - \alpha_n - \beta_n) \|x_n - y\| \\ &\leq \alpha_n \|u - y\| + (1 - \alpha_n) \|x_n - y\| \\ &\leq \max\{\|x_n - y\|, \|u - y\|\} \\ &\ \ \vdots \\ &\leq \max\{\|x_1 - y\|, \|u - y\|\}. \end{aligned}$$

So, the sets $\{x_n\}$ is bounded. This implies the boundness of $\{J_{r_n} x_n\}$ by $\|J_{r_n} x_n - y\| \leq \|x_n - y\|$. Let $M = \sup_{n \in \mathbb{N}} \{\|u\|, \|x_n\|, \|J_{r_n} x_n\|\}$.

Step 2. $\lim_{n\to\infty} \|x_n - J_r x_n\| = 0$ for all $r > 0$ and also $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Let $y_n = \frac{\alpha_n}{1-\beta_n} u + (1 - \frac{\alpha_n}{1-\beta_n}) J_{r_n} x_n$. By the definition (3.2) of the sequence $\{x_n\}$, we have

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n. \tag{3.10}$$

Setting $\lambda_n = \frac{\alpha_n}{1-\beta_n}$, then by the conditions (C1) and (C3), we have $\lim_{n\to\infty} \lambda_n = 0$. Furthermore,

$$\lim_{n\to\infty} \|y_n - J_{r_n} x_n\| = \lim_{n\to\infty} \lambda_n \|u - J_{r_n} x_n\| = 0. \tag{3.11}$$

From the resolvent identity (2.1), we have

$$J_{r_n}x_n = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right).$$

Therefore,

$$\begin{aligned}
&\|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| \\
&\le \left\|\frac{r_n}{r_{n+1}}(x_{n+1} - x_n) + \left(1 - \frac{r_n}{r_{n+1}}\right)(J_{r_{n+1}}x_{n+1} - x_n)\right\| \\
&\le \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|(\|x_{n+1} - x_n\| + \|J_{r_{n+1}}x_{n+1} - x_n\|) \\
&\le \|x_{n+1} - x_n\| + 4M\left|1 - \frac{r_n}{r_{n+1}}\right|.
\end{aligned}$$

So, we also have the following estimates for y_n:

$$\begin{aligned}
&\|y_{n+1} - y_n\| \\
&= \|(\lambda_{n+1} - \lambda_n)u + (1 - \lambda_{n+1})J_{r_{n+1}}x_{n+1} - (1 - \lambda_n)J_{r_n}x_n\| \\
&\le |\lambda_{n+1} - \lambda_n|\|u\| + \|J_{r_{n+1}}x_{n+1} - J_{r_n}x_n\| + \lambda_{n+1}\|J_{r_{n+1}}x_{n+1}\| + \lambda_n\|J_{r_n}x_n\| \\
&\le \|x_{n+1} - x_n\| + 4M\left(\left|1 - \frac{r_n}{r_{n+1}}\right| + \lambda_{n+1} + \lambda_n\right).
\end{aligned}$$

Thus, by the conditions (C1) and (C4), as $n \to \infty$,

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le 4M\left(\left|1 - \frac{r_n}{r_{n+1}}\right| + \lambda_{n+1} + \lambda_n\right) \to 0.$$

Hence,

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Applying Lemma 2.1 to (3.10) with the condition (C3), we obtain

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Combing (3.11), we have

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{3.12}$$

Together with the condition (C4), it follows that

$$\begin{aligned}
\|J_rJ_{r_n}x_n - J_{r_n}x_n\| &= \|(I - J_r)J_{r_n}x_n\| = r\|A_rJ_{r_n}x_n\| \le r|AJ_{r_n}x_n| \\
&\le r\|A_{r_n}x_n\| = r\frac{\|x_n - J_{r_n}x_n\|}{r_n} \to 0\ (n \to \infty).
\end{aligned}$$

Then

$$\lim_{n\to\infty}\|J_rJ_{r_n}x_n - J_{r_n}x_n\| = 0. \tag{3.13}$$

For all $r > 0$, we also have

$$\begin{aligned}
\|x_n - J_rx_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rJ_{r_n}x_n\| + \|J_rJ_{r_n}x_n - J_rx_n\| \\
&\le 2\|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_rJ_{r_n}x_n\|.
\end{aligned}$$

Combining (3.12) and (3.13), we get that for all $r > 0$,

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0. \tag{3.14}$$

Moreover,

$$\|x_{n+1} - x_n\| \le \alpha_n \|u - x_n\| + (1 - \alpha_n - \beta_n)\|J_{r_n} x_n - x_n\|.$$

So, noting (3.12) and the condition (C1), we also obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Step 3. $\limsup_{n\to\infty}\langle u - p, J(x_n - p)\rangle \le 0$ for some $p \in A^{-1}0$. We observe that in Step 3 of Theorem 3.1, the result still holds if z_n is replaced by x_n. Thus, using the same argumentation as Theorem 3.1 we obtain that for some $p \in A^{-1}0 = F(J_r)$,

$$\limsup_{n\to\infty}\langle u - p, J(x_{n+1} - p)\rangle \le 0. \tag{3.15}$$

Step 4. $\|x_n - p\| \to 0$ as $n \to \infty$. In fact, we can make the following estimates:

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \alpha_n\langle u - p, J(x_{n+1} - p)\rangle + \beta_n\langle x_n - p, J(x_{n+1} - p)\rangle \\
&\quad + (1 - \alpha_n - \beta_n)\langle J_{r_n} x_n - p, J(x_{n+1} - p)\rangle \\
&\le \alpha_n\langle u - p, J(x_{n+1} - p)\rangle + \beta_n \frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} \\
&\quad + (1 - \alpha_n - \beta_n)\frac{\|J_{r_n} x_n - p\|^2 + \|x_{n+1} - p\|^2}{2} \\
&\le \alpha_n\langle u - p, J(x_{n+1} - p)\rangle + (1 - \alpha_n)\frac{\|x_n - p\|^2 + \|x_{n+1} - p\|^2}{2}.
\end{aligned}$$

And thus,

$$\|x_{n+1} - p\|^2 \le (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u - p, J(x_{n+1} - p)\rangle. \tag{3.16}$$

Applying Lemma 2.4 to (3.16) with (3.15) and the assumption (C2) to yield the desired result. □

Using the same argumentation as Theorem 3.2, the following result is reached easily.

Theorem 3.4. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Suppose that for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by (3.2) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $D(A)$ is convex and $0 \in R(A)$, then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

Remark 1. (i) Theorem 3.1 (3.3) appears to be independent of Theorem 3.2 (3.4). On the one hand, it is easy to find examples of spaces which satisfies the fixed point property for nonexpansive self-mappings, which are not strictly convex. On the other hand, it appears to be unknown whether a reflexive and strictly convex Banach space satisfies the fixed point property for nonexpansive self-mappings.

(ii) There are many spaces which satisfy the fixed point property for nonexpansive self-mappings in the known results. For example, uniformly convex Banach space, uniformly smooth Banach space, reflexive Banach space with normal structure, Banach space with Opial's condition and so on.

Corollary 3.5. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be m-accretive. Assumed that $\overline{D(A)}$ is a convex subset of E and every nonempty bounded closed convex subset of $\overline{D(A)}$ has the fixed point property for nonexpansive self-mappings. Suppose that for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by (3.1) or (3.2) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $0 \in R(A)$, then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

Proof. Since A is m-accretive, A is accretive and satisfies the range condition $\overline{D(A)} \subset E = R(I+rA)$ for all $r > 0$. Following Theorem 3.1 or 3.2, the desired result is reached. □

Corollary 3.6. *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be m-accretive. Suppose that for an anchor point $u \in \overline{D(A)}$ and an initial value $x_1 \in \overline{D(A)}$, $\{x_n\}$ is defined by (3.1) or (3.2) and $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $\overline{D(A)}$ is convex and $0 \in R(A)$, then as $n \to \infty$, $\{x_n\}$ converges strongly to an element p of $A^{-1}0$.*

4. Some applications

In this section, as applications, we present several viscosity approximation results using similar proof technique to Song [19] and Suzuki [16].

Lemma 4.1. *Let $E, A, D(A), \{x_n\}, \{\alpha_n\}, \{\beta_n\}, \{r_n\}$ be as Theorem 3.1 or Theorem 3.3. For each $u \in \overline{D(A)}$, put $Pu = \lim_{n\to\infty} x_n$. Then P is a nonexpansive mapping on $\overline{D(A)}$.*

Proof. Let $\{x_n\}$ be defined by (3.1). Fix $u \in \overline{D(A)}$. Define sequences $\{x_n\}$ and $\{y_n\}$ by $x_1 = u$ and $y_1 \in \overline{D(A)}$,

$$x_{n+1} = \beta_n x_n + (1-\beta_n) J_{r_n}(\alpha_n u + (1-\alpha_n)x_n)$$

and

$$y_{n+1} = \beta_n y_n + (1-\beta_n) J_{r_n}(\alpha_n u + (1-\alpha_n)y_n).$$

Then we have

$$\begin{aligned}
&\|x_{n+1} - y_{n+1}\| \\
&\leq \beta_n \|x_n - y_n\| + (1 - \beta_n)\|(\alpha_n u + (1 - \alpha_n)x_n) - (\alpha_n u + (1 - \alpha_n)y_n)\| \\
&\leq (1 - \alpha_n(1 - \beta_n))\|x_n - y_n\|,
\end{aligned}$$

and hence $\lim_{n\to\infty} \|x_n - y_n\| = 0$ by Lemma 2.4. This means that Pu does not depend on the initial value y_1. That is, $Pu = \lim_{n\to\infty} x_n$ is well defined.

We fix $v \in \overline{D(A)}$ and define a sequences $\{y_n\}$ in $\overline{D(A)}$ by $y_1 = v$,

$$y_{n+1} = \beta_n y_n + (1 - \beta_n)J_{r_n}(\alpha_n v + (1 - \alpha_n)y_n).$$

Then $Pv = \lim_{n\to\infty} y_n$. We also have

$$\begin{aligned}
\|x_{n+1} - y_{n+1}\| \leq &(1 - \beta_n)\|(\alpha_n u + (1 - \alpha_n)x_n) - (\alpha_n v + (1 - \alpha_n)y_n)\| \\
&+ \beta_n \|x_n - y_n\| \\
\leq &(1 - \alpha_n(1 - \beta_n))\|x_n - y_n\| + \alpha_n(1 - \beta_n)\|u - v\|.
\end{aligned}$$

Therefore by induction, $\|x_{n+1} - y_{n+1}\| \leq \|u - v\|$ for all $n \in \mathbb{N}$. This implies $\|Pu - Pv\| \leq \|u - v\|$.

Similarly, we also obtain the same conclusion when $\{x_n\}$ be defined by (3.2). This completes the proof. □

Theorem 4.2. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Assumed that $\overline{D(A)}$ is a convex subset of E and every nonempty bounded closed convex subset of $\overline{D(A)}$ has the fixed point property for nonexpansive self-mappings. Suppose that for an initial value $y_1 \in \overline{D(A)}$, $\{y_n\}$ is defined by*

$$y_{n+1} = \beta_n y_n + (1 - \beta_n)J_{r_n}(\alpha_n f(y_n) + (1 - \alpha_n)y_n), \tag{4.1}$$

where f is a contractive self-mapping on $\overline{D(A)}$ with the contractive coefficient $\beta \in (0, 1)$. Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ and $\{r_n\} \subset (0, +\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $0 \in R(A)$, then as $n \to \infty$, $\{y_n\}$ converges strongly to an element x^* of $A^{-1}0$.

Proof. Let P be as Lemma 4.1. Then P is a nonexpansive mapping from $\overline{D(A)}$ to $A^{-1}0$ and also Pf is a contractive mapping of $\overline{D(A)}$ into itself since

$$\|Pf(x) - Pf(y)\| \leq \|f(x) - f(y)\| \leq \beta\|x - y\| \quad \text{for all } x, y \in \overline{D(A)}.$$

Thus, Banach Contraction Principle guarantees that there exists a unique element $x^* \in \overline{D(A)}$ such that $x^* = Pf(x^*)$. Define a sequence $\{x_n\}$ in $\overline{D(A)}$ by $x_1 \in \overline{D(A)}$ and

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)J_{r_n}(\alpha_n f(x^*) + (1 - \alpha_n)x_n), \quad \forall n \geq 1.$$

Then, by Theorem 3.1 and Lemma 4.1, $\lim_{n\to\infty} x_n = Pf(x^*) = x^* \in A^{-1}0$. Next, we show $y_n \to x^*$ as $n \to \infty$. Indeed, it follows that

$$\begin{aligned}
&\|y_{n+1} - x_{n+1}\| \\
&\le \beta_n \|y_n - x_n\| \\
&\quad + (1-\beta_n)\|J_{r_n}(\alpha_n f(y_n) + (1-\alpha_n)y_n) - J_{r_n}(\alpha_n f(x^*) + (1-\alpha_n)x_n)\| \\
&\le \beta_n\|y_n - x_n\| + (1-\beta_n)\alpha_n\beta\|y_n - x^*\| + (1-\alpha_n)(1-\beta_n)\|y_n - x_n\| \\
&\le [1-(1-\beta)(1-\beta_n)\alpha_n]\|x_n - y_n\| + \beta\alpha_n(1-\beta_n)\|x_n - x^*\|.
\end{aligned}$$

Applying Lemma 2.4 to obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore, $\lim_{n\to\infty}\|y_n - x^*\| = 0$. This completes the proof. □

Theorem 4.3. *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and $A : D(A) \subset E \to 2^E$ be an accretive operator that satisfies the range condition. Assumed that $\overline{D(A)}$ is a convex subset of E and every nonempty bounded closed convex subset of $\overline{D(A)}$ has the fixed point property for nonexpansive self-mappings. Suppose that for an initial value $y_1 \in \overline{D(A)}$, $\{y_n\}$ is defined by*

$$y_{n+1} = \alpha_n f(y_n) + \beta_n y_n + (1-\alpha_n-\beta_n)J_{r_n} y_n, \tag{4.2}$$

where f is a contractive self-mapping on $\overline{D(A)}$ with the contractive coefficient $\beta \in (0,1)$. Assumed that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\{r_n\} \subset (0,+\infty)$ satisfying the conditions (C1), (C2), (C3), and (C4). If $0 \in R(A)$, then as $n \to \infty$, $\{y_n\}$ converges strongly to an element x^* of $A^{-1}0$.

Proof. Using the same argumentation as Theorem 4.2, we can obtain that there exists a unique element $x^* \in \overline{D(A)}$ such that $x^* = Pf(x^*)$. Define a sequence $\{x_n\}$ in $\overline{D(A)}$ by $x_1 \in \overline{D(A)}$ and

$$x_{n+1} = \alpha_n f(x^*) + \beta_n x_n + (1-\alpha_n-\beta_n)J_{r_n}x_n, \quad \forall n \ge 1.$$

Then, by Theorem 3.3 and Lemma 4.1, $\lim_{n\to\infty} x_n = Pf(x^*) = x^* \in A^{-1}0$. Next, we show $y_n \to x^*$ as $n \to \infty$. Indeed, it follows that

$$\begin{aligned}
&\|y_{n+1} - x_{n+1}\| \\
&\le \alpha_n\|f(y_n) - f(x^*)\| + \beta_n\|x_n - y_n\| + (1-\alpha_n-\beta_n)\|J_{r_n}y_n - J_{r_n}x_n\| \\
&\le \alpha_n\beta\|y_n - x^*\| + (1-\alpha_n)\|x_n - y_n\| \\
&\le [1-(1-\beta)\alpha_n]\|x_n - y_n\| + \beta\alpha_n\|x_n - x^*\|.
\end{aligned}$$

Applying Lemma 2.4 to obtain $\lim_{n\to\infty}\|x_n - y_n\| = 0$. Therefore, $\lim_{n\to\infty}\|y_n - x^*\| = 0$. This completes the proof. □

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College of Mathematics and Information Science
Henan Normal University
Xinxiang 453007, P. R. China
E-mail address: songyisheng123@yahoo.com.cn or songyisheng123@163.com