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WHEN IS AN ENDOMORPHISM RING P-COHERENT?

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ABSTRACT. A ring is called left P-coherent if every principal left ideal is finitely presented. Let M be a right R-module with the endomorphism ring S. We mainly study the P-coherence of S. It is shown that S is a left P-coherent ring if and only if the left annihilator $\mathrm{ann}_S(X)$ is a finitely generated left ideal of S for any M-cyclic submodule X of M if and only if every cyclically M-presented right R-module has an M-torsionfree preenvelope. As applications, we investigate when the endomorphism ring S is left PP or von Neumann regular.

1. Introduction

As a new generalization of coherent rings, the concept of P-coherent rings has been recently introduced and studied in [6]. R is called a *left P-coherent ring* if every principal left ideal of R is finitely presented, or equivalently, if the left annihilator of a in R is a finitely generated left ideal of R for any $a \in R$. The examples of left P-coherent rings include left coherent rings and domains. Recall that R is called a *left PP ring* (resp., *PF ring*) if every principal left ideal of R is projective (resp., flat). It is obvious that R is a left PP ring if and only if R is a left P-coherent and PF ring. Another interesting fact is that R is a left coherent ring if and only if every $n \times n$ matrix ring $M_n(R)$ is a left P-coherent ring for every $n \geq 1$ (see [6, Proposition 2.4]). P-coherent rings are also closely related to two classes of classical modules, i.e., torsionfree and divisible modules. In fact, it has been shown that R is a left P-coherent ring if and only if any direct product of torsionfree right R-modules is torsionfree, if and only if any direct limit of divisible left R-modules is divisible (see [6, Theorem 2.7]).

In this article, we will further consider the P-coherence of the endomorphism ring S of a right R-module M. To this aim, we first introduce the concept of M-torsionfree modules in Section 2. Some elementary properties of M-torsionfree

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modules are given. For example, if N is an M-torsionfree right R-module, then $\mathrm{Hom}_R(M, N)$ is a torsionfree right S-module.

In Section 3, we study when the endomorphism ring S of a right R-module M_R is left P-coherent. It is shown that S is a left P-coherent ring if and only if every cyclically M-presented right R-module has an M-torsionfree preenvelope if and only if the left annihilator $\mathrm{ann}_S(X)$ is a finitely generated left ideal of S for any M-cyclic submodule X of M_R.

Section 4 is devoted to some applications. We show that every cyclically M-presented right R-module has an M-torsionfree preenvelope which is a monomorphism if and only if the endomorphism ring S is left P-coherent and $_SM$ is divisible. It is also proven that S is a left P-coherent ring and submodules of M-torsionfree right R-modules are M-torsionfree if and only if every cyclically M-presented right R-module has an $\mathrm{add}M_R$-envelope which is an epimorphism. In addition, we further investigate when the endomorphism ring of M_R is left PP or von Neumann regular in case M_R is quasi-projective.

Throughout this paper, all rings are associative with identity and all modules are unitary. As usual, $E(M)$ denotes the injective envelope of M, M^I stands for the direct product of copies of M indexed by a set I. M_R ($_RM$) denotes a right (left) R-module. For a module M_R, we denote by $S = \mathrm{End}(M_R)$ the endomorphism ring of M_R and by $\mathrm{add}M_R$ the category consisting of all modules isomorphic to direct summands of finite direct sums of copies of M_R. The reader may consult [1], [2] and [7] for background materials in ring and module theory.

2. M-torsionfree modules

Let M_R be a right R-module. Recall that a right R-module is called *M-cyclic* if it is isomorphic to M/X for some submodule X of M. A right R-module N is called *cyclically M-presented* if there is an exact sequence $0 \to K \to M \to N \to 0$ with K M-cyclic. A right R-module is called *cyclically presented* if it is cyclically R_R-presented.

Definition 2.1. Let M_R be a right R-module. A right R-module N is called *M-torsionfree* if every R-homomorphism $K \to N$ with K cyclically M-presented factors through a right R-module in $\mathrm{add}M_R$, or equivalently, factors through M^n for some integer $n \geq 0$.

Remark 2.2. (1) By definitions, the class of M-torsionfree right R-modules is closed under direct summands and finite direct sums.

(2) If $N \in \mathrm{add}M_R$, then N is clearly M-torsionfree. The converse holds if N is cyclically M-presented.

(3) Recall that a right R-module N is called *torsionfree* [3] if $\mathrm{Tor}_1^R(N, R/Ra) = 0$ for all $a \in R$. By [10, Proposition 2], it is easy to verify that a right R-module N is torsionfree if and only if every R-homomorphism $K \to N$ with K cyclically presented factors through a finitely generated projective right R-module. So R_R-torsionfree modules are exactly torsionfree right R-modules.

Let M_R be a right R-module with $S = \text{End}(M_R)$. For a right R-module A, there is a natural R-homomorphism $\theta_A : \text{Hom}_R(M, A) \otimes_S M \to A$ defined via $\theta_A(f \otimes x) = f(x)$ for $f \in \text{Hom}_R(M, A)$ and $x \in M$.

On the other hand, for a right S-module B, there is a natural S-homomorphism $\xi_B : B \to \text{Hom}_R(M, B \otimes_S M)$ defined via $\xi_B(b)(x) = b \otimes x$ for $b \in B$ and $x \in M$.

Recall that a right R-module N is called *M-projective* if, for every quotient module Q of M, the sequence $\text{Hom}_R(N, M) \to \text{Hom}_R(N, Q) \to 0$ is exact. M is called *quasi-projective* if M is M-projective.

Proposition 2.3. *Let M_R be a right R-module with $S = \text{End}(M_R)$. If N is an M-torsionfree right R-module, then $\text{Hom}_R(M, N)$ is a torsionfree right S-module. The converse holds if M_R is a quasi-projective right R-module.*

Proof. Suppose that N is an M-torsionfree right R-module. For any cyclically presented right S-module G, there is a right R-module exact sequence

$$S \otimes_S M \to S \otimes_S M \to G \otimes_S M \to 0.$$

So $G \otimes_S M$ is a cyclically M-presented right R-module. Let $\alpha : G \to \text{Hom}_R(M, N)$ be any S-homomorphism. Since N is M-torsionfree, the right R-homomorphism $G \otimes_S M \overset{\alpha\otimes 1}{\to} \text{Hom}_R(M, N) \otimes_S M \overset{\theta_N}{\to} N$ factors through M^n for some $n \geq 0$. Thus, applying $\text{Hom}_R(M, -)$, we get the right S-homomorphism

$$\text{Hom}_R(M, G \otimes_S M) \overset{(\alpha\otimes 1)_*}{\to} \text{Hom}_R(M, \text{Hom}_R(M, N) \otimes_S M) \overset{(\theta_N)_*}{\to} \text{Hom}_R(M, N),$$

which factors through $\text{Hom}_R(M, M^n) \cong S^n$. Therefore the right S-homomorphism $G \overset{\xi_G}{\to} \text{Hom}_R(M, G \otimes_S M) \overset{(\alpha\otimes 1)_*}{\to} \text{Hom}_R(M, \text{Hom}_R(M, N) \otimes_S M) \overset{(\theta_N)_*}{\to} \text{Hom}_R(M, N)$ factors through S^n. It is easy to verify that

$$\alpha = (\theta_N)_*(\alpha \otimes 1)_*\xi_G.$$

So $\text{Hom}_R(M, N)$ is a torsionfree right S-module by [10, Proposition 2].

Conversely, assume that M is a quasi-projective right R-module and $\text{Hom}_R(M, N)$ is a torsionfree right S-module. Let H be a cyclically M-presented right R-module and $f : H \to N$ any R-homomorphism. Then $\text{Hom}_R(M, H)$ is a cyclically presented right S-module since M is quasi-projective. So the right S-homomorphism $\text{Hom}_R(M, H) \overset{f_*}{\to} \text{Hom}_R(M, N)$ factors through S^m for some integer $m \geq 0$. Thus the induced R-homomorphism $\text{Hom}_R(M, H) \otimes_S M \overset{f_*\otimes 1}{\to} \text{Hom}_R(M, N) \otimes_S M$ factors through $S^m \otimes_S M \cong M^m$. Since H is cyclically M-presented, we have $\text{Hom}_R(M, H) \otimes_S M \overset{\theta_H}{\to} H$ is an isomorphism. Note that

$$f\theta_H = \theta_N(f_* \otimes 1).$$

So f factors through M^m, and hence N is an M-torsionfree right R-module. □

Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. For right R-modules A, B, there is a natural homomorphism

$$\sigma_{A,B} : \mathrm{Hom}_R(M, A) \otimes_S \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(B, A)$$

defined via $\sigma_{A,B}(f \otimes g)(b) = f(g(b))$ for $f \in \mathrm{Hom}_R(M, A)$, $g \in \mathrm{Hom}_R(B, M)$, $b \in B$.

It is easy to check that $\sigma_{A,B}$ is an isomorphism if $A \in \mathrm{add}M_R$ or $B \in \mathrm{add}M_R$.

Proposition 2.4. *Let M and A be right R-modules. Then the following conditions are equivalent:*

(1) *A is M-torsionfree;*
(2) *For any cyclically M-presented right R-module B, $\sigma_{A,B}$ is an epimorphism.*

Proof. (1) $\Rightarrow$ (2). Let $f \in \mathrm{Hom}_R(B, A)$. By (1), f factors through a right R-module M^n for some integer $n \geq 0$, i.e., there exist $g : B \to M^n$ and $h : M^n \to A$ such that $f = hg$. Let $\pi_i : M^n \to M$ be the ith projection and $\lambda_i : M \to M^n$ the ith injection, $i = 1, 2, \ldots, n$. Put $f_i = h\lambda_i$ and $g_i = \pi_i g$. It is easy to check that

$$f = \sigma_{A,B}(\sum_{i=1}^{n} f_i \otimes g_i).$$

So $\sigma_{A,B}$ is an epimorphism.

(2) $\Rightarrow$ (1). Let B be a cyclically M-presented right R-module and $f \in \mathrm{Hom}_R(B, A)$. By (2), there are $f_i \in \mathrm{Hom}_R(M, A)$ and $g_i \in \mathrm{Hom}_R(B, M), i = 1, 2, \ldots, n$, such that

$$f = \sigma_{A,B}(\sum_{i=1}^{n} f_i \otimes g_i).$$

Define $g : B \to M^n$ via

$$g(b) = (g_1(b), g_2(b), \ldots, g_n(b)), \quad b \in B$$

and define $h : M^n \to A$ via

$$h(m_1, m_2, \ldots, m_n) = \sum_{i=1}^{n} f_i(m_i), \quad m_i \in M.$$

Then $f = hg$ and hence (1) follows. □

Proposition 2.5. *Let M_R be a finitely presented right R-module. Then every pure submodule of an M-torsionfree right R-module is M-torsionfree.*

Proof. Let N be a pure submodule of an M-torsionfree right R-module L and $j : N \to L$ the inclusion. Since L is M-torsionfree, for any cyclically M-presented right R-module P and any homomorphism $f : P \to N$, there are $Q \in \mathrm{add}M_R$ and $g : P \to Q$ and $h : Q \to L$ such that $jf = hg$. Note that there is a pure epimorphism $\phi : H \to L$ with H pure-projective by [10, Proposition 1], so we have the pullback diagram of j and ϕ:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{\lambda} & H & \xrightarrow{\pi\phi} & L/N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\phi} & & \| & & \\ 0 & \longrightarrow & N & \xrightarrow{j} & L & \xrightarrow{\pi} & L/N & \longrightarrow & 0. \end{array}$$

Since M_R is finitely presented, Q is finitely presented. Thus there exists $l : Q \to H$ such that $h = \phi l$ since ϕ is pure. Therefore we have $\pi\phi lg = \pi hg = \pi jf = 0$, which implies that $lg(P) \subseteq K$ (here λ is regarded as the inclusion). Since P is finitely generated, so is $lg(P)$. Note that j and ϕ are pure, it is easily seen that λ is also pure. On the other hand, since H is pure-projective, by [11, Proposition 1.4(3)], we get a homomorphism $k : H \to K$ such that $klg(p) = lg(p)$ for all $p \in P$. Put $\beta = \alpha kl$, then $\beta \in \mathrm{Hom}_R(Q, N)$.

For all $p \in P$, we have

$$\beta g(p) = j\alpha klg(p) = \phi\lambda klg(p) = \phi\lambda lg(p) = \phi lg(p) = hg(p) = jf(p) = f(p).$$

So $f = \beta g$. Thus N is M-torsionfree. □

3. When is the endomorphism ring of a right R-module P-coherent?

Let $\mathcal{C}$ be a class of right R-modules and N a right R-module. A homomorphism $\phi : N \to F$ with $F \in \mathcal{C}$ is called a *$\mathcal{C}$-preenvelope of N* [2] if for any homomorphism f: $N \to F'$ with $F' \in \mathcal{C}$, there is a homomorphism $g : F \to F'$ such that $g\phi = f$. Moreover, if every endomorphism $g : F \to F$ such that $g\phi = \phi$ is an isomorphism, the $\mathcal{C}$-preenvelope ϕ is called a *$\mathcal{C}$-envelope of N*.

For $\mathcal{C}$ the class of M-torsionfree right R-modules, $\mathcal{C}$-(pre)envelopes will simply be called *M-torsionfree (pre)envelopes.*

Theorem 3.1. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*

(1) *S is a left P-coherent ring;*
(2) *$\mathrm{Hom}_R(N, M)$ is a finitely generated left S-module for any cyclically M-presented right R-module N;*
(3) *Every cyclically M-presented right R-module has an M-torsionfree preenvelope;*
(4) *The left annihilator $ann_S(X)$ is a finitely generated left ideal of S for any M-cyclic submodule X of M_R;*
(5) *All direct products of copies of M_R are M-torsionfree;*
(6) *All direct products of M-torsionfree right R-modules are M-torsionfree.*

Proof. (1) ⇒ (2). Let N be a cyclically M-presented right R-module, i.e., there is a right R-module exact sequence $M \xrightarrow{f} M \to N \to 0$. Then we have an induced left S-module exact sequence

$$0 \to \mathrm{Hom}_R(N, M) \to S \xrightarrow{f^*} S.$$

Note that $\mathrm{im}(f^*)$ is a principal left ideal of S. Thus $\mathrm{im}(f^*)$ is finitely presented by (1), and so $\mathrm{Hom}_R(N, M)$ is a finitely generated left S-module.

(2) $\Rightarrow$ (1). Let $a \in S$. Then there is a left S-module exact sequence $0 \to K \xrightarrow{g} S \xrightarrow{f} Sa \to 0$, which induces a right R-module exact sequence

$$0 \to \mathrm{Hom}_S(Sa, M) \xrightarrow{f^*} \mathrm{Hom}_S(S, M) \xrightarrow{g^*} \mathrm{Hom}_S(K, M).$$

Note that the following diagram is commutative:

$$\begin{array}{ccccc} S & \xrightarrow{f} & Sa & \longrightarrow & 0 \\ \downarrow{\scriptstyle \delta_S} & & \downarrow{\scriptstyle \delta_{Sa}} & & \\ \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), M) & \xrightarrow{f^{**}} & \mathrm{Hom}_R(\mathrm{Hom}_S(Sa, M), M), & & \end{array}$$

where δ_{Sa} and δ_S are the canonical maps.

Let $i : Sa \to S$ be the inclusion. We get the following commutative diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & Sa & \xrightarrow{i} & S \\ & & \downarrow{\scriptstyle \delta_{Sa}} & & \downarrow{\scriptstyle \delta_S} \\ & & \mathrm{Hom}_R(\mathrm{Hom}_S(Sa, M), M) & \xrightarrow{i^{**}} & \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), M) \end{array}$$

So $i^{**}\delta_{Sa} = \delta_S i$ is a monomorphism.

On the other hand, there exists a right R-module exact sequence

$$\mathrm{Hom}_S(S, M) \xrightarrow{f^*i^*} \mathrm{Hom}_S(S, M) \to G \to 0.$$

Since G is cyclically M-presented, we have that $\mathrm{Hom}_R(G, M)$ is a finitely generated left S-module by (2). Therefore there exists $\psi : K \to \mathrm{Hom}_R(G, M)$ such that the following diagram with exact rows is commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{g} & S & \xrightarrow{f} & Sa & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \delta_S} & & \downarrow{\scriptstyle i^{**}\delta_{Sa}} & & \\ 0 & \longrightarrow & \mathrm{Hom}_R(G, M) & \longrightarrow & \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), M) & \xrightarrow{i^{**}f^{**}} & \mathrm{Hom}_R(\mathrm{Hom}_S(S, M), M) & & \end{array}$$

By the Five Lemma, ψ is an isomorphism. Thus K is finitely generated, and so Sa is finitely presented. Hence S is a left P-coherent ring.

(2) $\Rightarrow$ (3). Let N be a cyclically M-presented right R-module. By (2), $\mathrm{Hom}_R(N, M)$ is a finitely generated left S-module, so there is a generating set $\{f_j \in \mathrm{Hom}_R(N, M) : 1 \leq j \leq n\}$ of $\mathrm{Hom}_R(N, M)$. Define $f : N \to M^n$ via

$$x \mapsto (f_1(x), f_2(x), \ldots, f_n(x)), \quad x \in N.$$

We will show that f is an M-torsionfree preenvelope of N. In fact, let G be an M-torsionfree right R-module and $\varphi : N \to G$ any R-homomorphism.

Then there are $g : N \to M^m$ and $\psi : M^m \to G$ such that $\varphi = \psi g$ for some integer $m \geq 0$. Let $\pi_i : M^m \to M$ be the ith projection, $1 \leq i \leq m$. Note that $\pi_i g \in \mathrm{Hom}_R(N, M)$, so there exist $s_{ij} \in S$ $(1 \leq j \leq n)$ such that $\pi_i g = \sum_{j=1}^n s_{ij} f_j$. Define $h_i : M^n \to M$ via

$$(a_1, a_2, \ldots, a_n) \mapsto \sum_{j=1}^{n} s_{ij} a_j, \qquad a_j \in M.$$

Then there exists $h : M^n \to M^m$ such that $h_i = \pi_i h$. So $\pi_i h f = h_i f = \pi_i g$ and hence $g = hf$. Thus $\varphi = \psi g = (\psi h) f$. Consequently f is an M-torsionfree preenvelope.

(3) $\Rightarrow$ (4). Let X be an M-cyclic submodule of the right R-module M. Consider the right R-module exact sequence $0 \to X \xrightarrow{i} M \xrightarrow{\pi} M/X \to 0$, where i is the inclusion and π is the canonical map. Since M/X is cyclically M-presented, M/X has an M-torsionfree preenvelope $\theta : M/X \to Y$ by (3). So there exist $\alpha : M/X \to M^m$ and $\eta : M^m \to Y$ such that $\theta = \eta\alpha$ for some integer $m \geq 0$. Let $p_k : M^m \to M$ (resp., $\lambda_k : M \to M^m$) be the kth canonical projection (resp., injection) and $\beta_k = p_k \alpha \pi \in S$, $k = 1, 2, \ldots, m$. It is clear that $\beta_k \in \mathrm{ann}_S(X)$.

On the other hand, for any $f \in \mathrm{ann}_S(X)$, there is a right R-homomorphism $\gamma : M/X \to M$ such that $\gamma\pi = f$. Since θ is an M-torsionfree preenvelope, there exists $\phi : Y \to M$ such that $\phi\theta = \gamma$. Thus

$$f = \phi\theta\pi = \sum_{k=1}^{m} \phi\eta\lambda_k p_k \alpha\pi = \sum_{k=1}^{m} (\phi\eta\lambda_k)\beta_k \in \sum_{k=1}^{m} S\beta_k,$$

which implies that $\mathrm{ann}_S(X) = \sum\limits_{k=1}^{m} S\beta_k$ is a finitely generated left ideal of S.

(4) $\Rightarrow$ (3). Let N be a cyclically M-presented right R-module. Then there is a right R-module exact sequence $0 \to K \to M \xrightarrow{g} N \to 0$, where K is M-cyclic. Thus $\mathrm{ann}_S(K)$ is a finitely generated left ideal of S by (4). Suppose that $f_1, f_2, \ldots, f_m$ is a generating set of $\mathrm{ann}_S(K)$. Then K is contained in the kernel of the product map $f : M \to M^m$ induced by the f_i (we set $\pi_i f = f_i$, where $\pi_i : M^m \to M$ is the ith canonical projection, $i = 1, 2, \ldots, m$), and hence there is a map $h : N \to M^m$ such that $f = hg$. We claim that h is an M-torsionfree preenvelope of N. In fact, for any homomorphism $\psi : N \to G$ with G M-torsionfree, there exist $\alpha : N \to M^n$ and $\beta : M^n \to G$ such that $\gamma = \beta\alpha$ for some integer $n \geq 0$. Let $\pi_j : M^n \to M$ be the jth canonical projection, $j = 1, 2, \ldots, n$. It is obvious that $\pi_j \alpha g \in \mathrm{ann}_S(K)$. Thus there exist $t_{ij} \in S$, $i = 1, 2, \ldots, m$ such that

$$\pi_j \alpha g = \sum_{i=1}^{m} t_{ij} f_i = (\sum_{i=1}^{m} t_{ij} \pi_i) f = (\sum_{i=1}^{m} t_{ij} \pi_i) h g.$$

Since g is epic, we have $\pi_j\alpha = (\sum_{i=1}^m t_{ij}\pi_i)h$. In addition, there exists $\varphi : M^m \to M^n$ such that $\pi_j\varphi = \sum_{i=1}^m t_{ij}\pi_i$ for any $j = 1, 2, \ldots, n$. Hence $\pi_j\varphi h = (\sum_{i=1}^m t_{ij}\pi_i)h = \pi_j\alpha$, and so $\varphi h = \alpha$. Thus $\gamma = \beta\alpha = (\beta\varphi)h$. It follows that h is an M-torsionfree preenvelope.

(3) $\Rightarrow$ (6). Let $\{M_i\}_{i\in I}$ be a family of M-torsionfree right R-modules and N any cyclically M-presented right R-module. By (3), N has an M-torsionfree preenvelope $f : N \to F$. It follows that the sequence

$$\mathrm{Hom}_R(F, M_i) \to \mathrm{Hom}_R(N, M_i) \to 0$$

is exact. Thus we get the exact sequence

$$(\mathrm{Hom}_R(F, M_i))^I \to (\mathrm{Hom}_R(N, M_i))^I \to 0.$$

Note that

$$(\mathrm{Hom}_R(F, M_i))^I \cong \mathrm{Hom}_R(F, M_i^I) \text{ and } (\mathrm{Hom}_R(N, M_i))^I \cong \mathrm{Hom}_R(N, M_i^I).$$

So every homomorphism from N to M_i^I factors through F. In addition, since F is M-torsionfree, every homomorphism from N to F factors through a right R-module in addM_R. Thus every homomorphism from N to M_i^I factors through a right R-module in addM_R. So M_i^I is M-torsionfree.

(6) $\Rightarrow$ (5) is trivial.

(5) $\Rightarrow$ (2). Let A be a cyclically M-presented right R-module. For any index set I, we have the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{Hom}_R(A, M))^I & \xrightarrow{\quad\theta\quad} & \mathrm{Hom}_R(A, M^I) \\ \uparrow{\scriptstyle\varphi} & & \uparrow{\scriptstyle\sigma_{M^I,A}} \\ S^I \otimes_S \mathrm{Hom}_R(A, M) & \xrightarrow{\ \psi\ } & \mathrm{Hom}_R(M, M^I) \otimes_S \mathrm{Hom}_R(A, M), \end{array}$$

where φ is a canonical homomorphism, θ and ψ are isomorphisms. By Proposition 2.4, $\sigma_{M^I,A}$ is epic since M^I is M-torsionfree. Thus φ is epic, and hence $\mathrm{Hom}_R(A, M)$ is a finitely generated left S-module by [9, Lemma 13.1, p. 41]. □

Corollary 3.2. *Let M_R be a finitely presented right R-module with $S = \mathrm{End}(M_R)$. Then the following conditions are equivalent:*

(1) *S is a left P-coherent ring;*

(2) *Every right R-module has an M-torsionfree preenvelope.*

Proof. (1) $\Rightarrow$ (2). Let N be any right R-module. By [2, Lemma 5.3.12], there is a cardinal number $\aleph_\alpha$ such that for any R-homomorphism $f : N \to L$ with L M-torsionfree, there is a pure submodule Q of L such that $\mathrm{Card}(Q) \leq \aleph_\alpha$ and $f(N) \subseteq Q$. Thus f has a factorization $N \to Q \to L$, where Q is M-torsionfree by Proposition 2.5. Now let $(\varphi_i)_{i\in I}$ give all such homomorphisms $\varphi_i : N \to Q_i$ with $\mathrm{Card}(Q_i) \leq \aleph_\alpha$ and Q_i M-torsionfree. Then any homomorphism $N \to H$ with H M-torsionfree has a factorization $N \to Q_j \to H$ for some $j \in I$. Thus

$N \to Q_i^I$ is an M-torsionfree preenvelope since Q_i^I is M-torsionfree by (1) and Theorem 3.1.

(2) $\Rightarrow$ (1) is clear by Theorem 3.1. □

Theorem 3.3. *Let M_R be a quasi-projective right R-module with $S=\mathrm{End}(M_R)$. Then the following conditions are equivalent:*

(1) *S is a right P-coherent ring;*
(2) *For any $s \in S$, there exist $n \geq 0$ and a right R-homomorphism $\alpha : M^n \to \ker(s)$ such that $\mathrm{Hom}_R(M, \mathrm{coker}(\alpha)) = 0$.*

Proof. (1) $\Rightarrow$ (2). Let $s \in S$. Then there is a right R-module exact sequence

$$0 \to \ker(s) \to M \xrightarrow{s} M,$$

which gives rise to the right S-module exact sequence

$$0 \to \mathrm{Hom}_R(M, \ker(s)) \to S \xrightarrow{s_*} S.$$

So $\mathrm{Hom}_R(M, \ker(s))$ is a finitely generated right S-module since $\mathrm{im}(s_*)$ is finitely presented by (1). Thus there is a right S-module exact sequence

$$S^n \to \mathrm{Hom}_R(M, \ker(s)) \to 0,$$

which induces a right R-module exact sequence

$$M^n \xrightarrow{\beta} \mathrm{Hom}_R(M, \ker(s)) \otimes_S M \to 0.$$

Put

$$\alpha = \theta_{\ker(s)}\beta : M^n \xrightarrow{\beta} \mathrm{Hom}_R(M, \ker(s)) \otimes_S M \xrightarrow{\theta_{\ker(s)}} \ker(s).$$

Next we will show $\mathrm{Hom}_R(M, \mathrm{coker}(\alpha)) = 0$. Let $f \in \mathrm{Hom}_R(M, \mathrm{coker}(\alpha))$ and $\gamma : \ker(s) \to \mathrm{coker}(\alpha)$ be the canonical map. Since M is a quasi-projective right R-module, M is $\ker(s)$-projective by [1, Proposition 16.12]. So there exists $g : M \to \ker(s)$ such that $f = \gamma g$. For any $x \in M$, we have $g(x) = \theta_{\ker(s)}(g \otimes x) \in \mathrm{im}(\alpha)$. Thus $f(x) = \gamma g(x) = 0$, and hence $f = 0$. So $\mathrm{Hom}_R(M, \mathrm{coker}(\alpha)) = 0$.

(2) $\Rightarrow$ (1). Let $0 \to K \to S \xrightarrow{\varphi} S$ be an exact sequence of right S-modules. It is enough to show that K is finitely generated. By tensoring with ${}_SM$, $S \xrightarrow{\varphi} S$ induces a right R-homomorphism $\psi : M \to M$. By Five Lemma, we have $K \cong \mathrm{Hom}_R(M, \ker(\psi))$. In addition, there exists $\alpha : M^n \to \ker(\psi)$ such that $\mathrm{Hom}_R(M, \mathrm{coker}(\alpha)) = 0$ by (2). Since M is a quasi-projective right R-module, we get the right S-module exact sequence

$$S^n \to \mathrm{Hom}_R(M, \ker(\psi)) \to 0.$$

So K is finitely generated. It follows that S is a right P-coherent ring. □

4. Applications

Recall that a left R-module N is said to be *divisible* [3] if $\mathrm{Ext}_R^1(R/Ra, N) = 0$ for all $a \in R$.

Let ${}_SM_R$ be a bimodule. A right R-module L is called M_R-*torsionless* [1] if the canonical map $L \to \mathrm{Hom}_S(\mathrm{Hom}_R(L, M), M)$ is a monomorphism.

Lemma 4.1. *Let M_R be a right R-module with $S = \mathrm{End}(M_R)$. Then ${}_SM$ is divisible if and only if every cyclically M-presented right R-module is M_R-torsionless.*

Proof. " $\Rightarrow$ ". Let N be a cyclically M-presented right R-module, i.e., there is a right R-module exact sequence $M \xrightarrow{f} M \to N \to 0$. Then by [5, Lemma 4.1], there exists an exact sequence

$$0 \to \mathrm{Ext}_S^1(L, M) \to N \to \mathrm{Hom}_S(\mathrm{Hom}_R(N, M), M),$$

where $L = \mathrm{coker}(f^*)$ is a cyclically presented left S-module. Thus $\mathrm{Ext}_S^1(L, M) = 0$, which implies that N is M_R-torsionless.

" $\Leftarrow$ ". Let G be a cyclically presented left S-module, i.e., there is a left S-module exact sequence $S \xrightarrow{g} S \to G \to 0$. Thus we obtain an induced exact sequence

$$0 \to \mathrm{Hom}_S(G, M) \to \mathrm{Hom}_S(S, M) \xrightarrow{g^*} \mathrm{Hom}_S(S, M) \to H \to 0,$$

where $H = \mathrm{coker}(g^*)$ is a cyclically M-presented right R-module, and so is M_R-torsionless by hypothesis. Now applying [5, Lemma 4.1] to the right half of the above sequence, we get the exact sequence

$$0 \to \mathrm{Ext}_S^1(Q, M) \to H \to \mathrm{Hom}_S(\mathrm{Hom}_R(H, M), M),$$

where $Q = \mathrm{coker}(g^{**}) \cong G$. Thus $\mathrm{Ext}_S^1(G, M) = 0$, and so ${}_SM$ is divisible. $\square$

Theorem 4.2. *The following conditions are equivalent for a right R-module M_R with $S = \mathrm{End}(M_R)$:*

1. *(1) S is a left P-coherent ring and ${}_SM$ is divisible;*
2. *(2) S is a left P-coherent ring, and every cyclically M-presented right R-module embeds in L with $L \in addM_R$;*
3. *(3) S is a left P-coherent ring, and every injective right R-module is M-torsionfree;*
4. *(4) S is a left P-coherent ring, and the injective envelope of every simple right R-module is M-torsionfree;*
5. *(5) Every cyclically M-presented right R-module has an M-torsionfree preenvelope which is a monomorphism.*

Proof. (1) $\Rightarrow$ (2) holds by Theorem 3.1 and Lemma 4.1.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) are straightforward.

(4) $\Rightarrow$ (5). Let N be a cyclically M-presented right R-module. We will show that N is M_R-torsionless. It is enough to show that, for any $0 \neq m \in N$,

there exists $f : N \to M$ such that $f(m) \neq 0$. In fact, there is a maximal submodule K of mR, and so mR/K is simple. By the injectivity of $E(mR/K)$, there exists $j : N \to E(mR/K)$ such that $j\iota = i\pi$, where $\iota : mR \to N$ and $i : mR/K \to E(mR/K)$ are the inclusions, and $\pi : mR \to mR/K$ is the canonical map. Note that $j(m) = j\iota(m) = i\pi(m) \neq 0$. On the other hand, since $E(mR/K)$ is M-torsionfree by (4), there exist $n \in \mathbb{N}$, $g : N \to M^n$ and $h : M^n \to E(mR/K)$ such that $j = hg$. Therefore $g(m) = (x_1, x_2, \ldots, x_n) \neq 0$. So there exists some $x_i \neq 0$. Let $p_i : M^n \to M$ be the ith projection. Then $p_i g(m) \neq 0$. Hence N is M_R-torsionless. Thus N embeds in an M-torsionfree right R-module by Theorem 3.1, and so N has an M-torsionfree preenvelope which is a monomorphism by Theorem 3.1 again.

(5) $\Rightarrow$ (1) follows from Theorem 3.1 and Lemma 4.1. □

Corollary 4.3. *The following conditions are equivalent for an injective right R-module M_R with $S = \mathrm{End}(M_R)$:*

1. *S is a left P-coherent ring and ${}_SM$ is divisible;*
2. *Every cyclically M-presented right R-module has an M-torsionfree envelope which is a monomorphism.*

Proof. (1) $\Rightarrow$ (2). Let N be a cyclically M-presented right R-module. By Theorem 4.2, $E(N)$ is M-torsionfree. We claim that the inclusion $i : N \to E(N)$ is an M-torsionfree envelope of N. In fact, for any M-torsionfree right R-module F and any homomorphism $f : N \to F$, f factors through a module L in addM_R, i.e., there exist $g : N \to L$ and $h : L \to F$ such that $f = hg$. Since M_R is injective, we have L is injective. Therefore there is $j : E(N) \to L$ such that $g = ji$. Thus $f = h(ji) = (hj)i$, which means that i is an M-torsionfree preenvelope, and hence i is an M-torsionfree envelope of N since i is an injective envelope.

(2) $\Rightarrow$ (1) is clear Theorem 4.2. □

Theorem 4.4. *The following conditions are equivalent for a right R-module M_R with $S = \mathrm{End}(M_R)$:*

1. *S is a left P-coherent ring, and every submodule of an M-torsionfree right R-module is M-torsionfree;*
2. *Every cyclically M-presented right R-module has an M-torsionfree envelope which is an epimorphism;*
3. *Every cyclically M-presented right R-module has an addM_R-envelope which is an epimorphism.*

   If M_R is quasi-projective, then the conditions above are also equivalent to

4. *S is a left PP ring.*

Proof. (1) $\Rightarrow$ (2). Let N be a cyclically M-presented right R-module. Then N has an M-torsionfree preenvelope $f : N \to F$ by Theorem 3.1 since S is a left P-coherent ring. However im(f) is M-torsionfree by (1), it follows that $N \to \mathrm{im}(f)$ is an epic M-torsionfree envelope.

(2) $\Rightarrow$ (3). Let N be a cyclically M-presented right R-module. Then N has an epic M-torsionfree envelope $f : N \to F$ by (2). So f factors through a module L in addM_R, i.e., there exist $g : N \to L$ and $h : L \to F$ such that $f = hg$. On the other hand, since L is M-torsionfree, there exists $\alpha : F \to L$ such that $g = \alpha f$. Thus $f = h\alpha f$, and so $h\alpha = 1$ since f is epic. Hence F is isomorphic to a direct summand of L. Therefore $F \in$ addM_R and (3) follows.

(3) $\Rightarrow$ (1). It is clear that S is a left P-coherent ring by Theorem 3.1. Now suppose that N is a submodule of L with L an M-torsionfree right R-module, and $\iota : N \to L$ is the inclusion. For any cyclically M-presented right R-module K and $\alpha \in \mathrm{Hom}_R(K, N)$, $\iota\alpha$ factors through a module H in addM_R, i.e., there exist $g : K \to H$ and $h : H \to L$ such that $\iota\alpha = hg$. By (3), K has an epic addM_R-envelope $\beta : K \to Q$. Thus there exists $\gamma : Q \to H$ such that $g = \gamma\beta$, which implies that $\ker(\beta) \subseteq \ker(\alpha)$ and so there exists $\varphi : Q \to N$ such that $\alpha = \varphi\beta$. Therefore N is M-torsionfree.

(1) $\Rightarrow$ (4). Let $\alpha \in S$. Then there is a right R-module exact sequence $M \xrightarrow{\alpha} M \to N \to 0$. Since M is quasi-projective, we get a right S-module exact sequence

$$\mathrm{Hom}_R(M, M) \xrightarrow{\alpha_*} \mathrm{Hom}_R(M, M) \to \mathrm{Hom}_R(M, N) \to 0.$$

Let $\lambda : \alpha(M) \to M$ be the inclusion. By (1), $\alpha(M)$ is M-torsionfree, so $\mathrm{Hom}_R(M, \alpha(M))$ is a torsionfree right S-module by Proposition 2.3. Note that

$$\alpha S = \alpha_*(S) = \lambda_*(\mathrm{Hom}_R(M, \alpha(M))) \cong \mathrm{Hom}_R(M, \alpha(M)).$$

Thus αS is torsionfree, and so is flat by [8, p. 2047, 5(a)]. Hence S is a right PF ring. But the property that S is a PF ring is left-right symmetric (see [4]), therefore S is a left PP ring.

(4) $\Rightarrow$ (1). Let N be a submodule of H with H an M-torsionfree right R-module. Then there is a right R-module exact sequence $0 \to N \to H \to L \to 0$, which induces a right S-module exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, H) \to G \to 0.$$

Note that $\mathrm{Hom}_R(M, H)$ is a torsionfree right S-module by Proposition 2.3.

Let $a \in S$. Then Sa is projective by (4). So the exact sequence $0 \to Sa \to S \to S/Sa \to 0$ induces the exactness of the sequence

$$0 = \mathrm{Tor}_2^S(G, S) \to \mathrm{Tor}_2^S(G, S/Sa) \to \mathrm{Tor}_1^S(G, Sa) = 0.$$

Thus $\mathrm{Tor}_2^S(G, S/Sa) = 0$. Therefore we have the exact sequence

$$0 = \mathrm{Tor}_2^S(G, S/Sa) \to \mathrm{Tor}_1^S(\mathrm{Hom}_R(M, N), S/Sa)$$
$$\to \mathrm{Tor}_1^S(\mathrm{Hom}_R(M, H), S/Sa) = 0.$$

Hence $\mathrm{Tor}_1^S(\mathrm{Hom}_R(M, N), S/Sa) = 0$, which implies that $\mathrm{Hom}_R(M, N)$ is a torsionfree right S-module, and so N is an M-torsionfree right R-module by Proposition 2.3. $\square$

Corollary 4.5. *Let M_R be a quasi-projective right R-module with S=End(M_R). Then the following conditions are equivalent:*

(1) *S is a von Neumann regular ring;*
(2) *S is a left PP ring and $_SM$ is divisible;*
(3) *Every cyclically M-presented right R-module is M-torsionfree;*
(4) *Every right R-module is M-torsionfree.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (3) follows from Theorems 4.2 and 4.4.

(3) $\Rightarrow$ (4) is clear by Remark 2.2 (2).

(4) $\Rightarrow$ (1). Let $\alpha \in S$. Then there is a right R-module exact sequence $M \overset{\alpha}{\to} M \overset{\beta}{\to} N \to 0$. Note that $N \in \mathrm{add}M_R$ by (4). Therefore N is M-projective by [1, Proposition 16.10] since M is quasi-projective. Thus β is a split epimorphism. So we get a right S-module exact sequence

$$\mathrm{Hom}_R(M,M) \overset{\alpha_*}{\to} \mathrm{Hom}_R(M,M) \overset{\beta_*}{\to} \mathrm{Hom}_R(M,N) \to 0,$$

where β_* is a split epimorphism. Thus $\alpha S = \alpha_*(S)$ is a direct summand of S. Hence S is a von Neumann regular ring. □

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References

[1] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13. Springer-Verlag, New York-Heidelberg, 1974.
[2] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, de Gruyter Expositions in Mathematics, 30. Walter de Gruyter & Co., Berlin, 2000.
[3] A. Hattori, *A foundation of torsion theory for modules over general rings*, Nagoya Math. J. **17** (1960), 147–158.
[4] S. Jøndrup, p.p. *rings and finitely generated flat ideals*, Proc. Amer. Math. Soc. **28** (1971), 431–435.
[5] L. X. Mao, *Relatively endocoherent modules*, Algebra Colloq. **14** (2007), no. 2, 327–340.
[6] L. X. Mao and N. Q. Ding, *On divisible and torsionfree modules*, Comm. Algebra **36** (2008), no. 2, 708–731.
[7] J. J. Rotman, *An Introduction to Homological Algebra*, Academic Press, New York, 1979.
[8] A. Shamsuddin, *n-injective and n-flat modules*, Comm. Algebra **29** (2001), no. 5, 2039–2050.
[9] B. Stenström, *Rings of Quotients*, Springer-Verlag, Berlin-Heidelberg-New York, 1975.
[10] R. B. Warfield, Jr., *Purity and algebraic compactness for modules*, Pacific J. Math. **28** (1969), 699–719.
[11] W. Zimmermann, *On locally pure-injective modules*, J. Pure Appl. Algebra **166** (2002), no. 3, 337–357.

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