BLASCHKE PRODUCTS AND RATIONAL FUNCTIONS
WITH SIEGEL DISKS

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ABSTRACT. Let $m$ be a positive integer. We show that for any given real number $\alpha \in [0,1]$ and complex number $\mu$ with $|\mu| \leq 1$ which satisfy $e^{2\pi i \alpha} \mu^m \neq 1$, there exists a Blaschke product $B$ of degree $2m + 1$ which has a fixed point of multiplier $\mu^m$ at the point at infinity such that the restriction of the Blaschke product $B$ on the unit circle is a critical circle map with rotation number $\alpha$. Moreover if the given real number $\alpha$ is irrational of bounded type, then a modified Blaschke product of $B$ is quasiconformally conjugate to some rational function of degree $m + 1$ which has a fixed point of multiplier $\mu^m$ at the point at infinity and a Siegel disk whose boundary is a quasicircle containing its critical point.

1. Introduction

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. In the theory of the complex dynamics, there are two important sets called the Fatou set and the Julia set. The Fatou set $F(f)$ is the set of normality in the sense of Montel for the family $\{f^n\}_{n=1}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is $n$ iterates of $f$. The Julia set $J(f)$ is the complement $\hat{\mathbb{C}} \setminus F(f)$. A solution $z_0$ of the equation $f(z) = z$ is called a fixed point of $f$ and $\lambda = f'(z_0)$ is called the multiplier of $z_0$ if $z_0 \in \mathbb{C}$. The multiplier of $z_0 = \infty$ is defined as the multiplier of the origin for $\psi \circ f \circ \psi^{-1}$, where $\psi(z) = 1/z$. The fixed point $z_0$ is attracting, repelling or indifferent if its multiplier $\lambda$ satisfies that $|\lambda| < 1$, $|\lambda| > 1$ or $|\lambda| = 1$ respectively. Attracting fixed points belong to the Fatou set and repelling fixed points belong to the Julia set. In the case that $z_0$ is indifferent, the classification is more complicated. The fixed point $z_0$ is parabolic, a Siegel point or a Cremer point if its multiplier is a root of unity, $z_0 \in F(f)$ or $z_0 \in J(f)$ respectively. Parabolic fixed points belong to the Julia set. The Fatou component containing a Siegel point is called a Siegel disk centered at $z_0$. Non-repelling fixed points “capture” at least one critical point of $f$, which is a solution of the equation $f'(z) = 0$.

In this paper, we investigate rational functions with Siegel disks. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$ with a fixed point of multiplier
$e^{2\pi i \alpha}$ at the origin, where $\alpha \in [0, 1]$ is irrational. If the origin is a Siegel point, then there exists a local holomorphic change of coordinate $\Phi : \mathbb{D} \to \mathbb{C}$ with $0 = \Phi(0)$ such that $\Phi^{-1} \circ f \circ \Phi(z) = e^{2\pi i \alpha} z$, where $\mathbb{D}$ is the unit disk. The Siegel disk $\Delta$ centered at the origin contains $\Phi(\mathbb{D})$.

For the irrational number $\alpha$, we consider the continued fraction expansion

$$\alpha = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots}}}$$

of $\alpha$ and then a sequence of rational numbers

$$\frac{p_n}{q_n} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}$$

converges to $\alpha$, where $a_n$ is a positive integer uniquely determined by $\alpha$ for all $n \in \mathbb{N}$. The irrational number $\alpha$ is a Diophantine number of order $\kappa \geq 2$ if there exists $\varepsilon > 0$ such that

$$\left| \alpha - \frac{p}{q} \right| > \frac{\varepsilon}{q^{\kappa}}$$

for all rational numbers $p/q$. The class of Diophantine numbers of order $\kappa$ is denoted by $\mathcal{D}_\kappa$. The irrational number $\alpha$ belongs to $\mathcal{D}_\kappa$ if and only if the sequence

$$\left\{ \frac{q_{n+1}}{q_n} \right\}_{n=1}^{\infty}$$

is bounded. In the case that $\kappa = 2$, the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded if and only if $\{q_{n+1}/q_n\}_{n=1}^{\infty}$ is bounded. Therefore Diophantine numbers of order 2 are said to be of bounded type. The irrational number $\alpha$ is a Bryuno number if the sum

$$\sum_{n=1}^{\infty} \frac{\log q_{n+1}}{q_n}$$

converges. The class of Bryuno numbers is denoted by $\mathcal{B}$. Note that for $\kappa > 2$, $\mathcal{D}_2 \subset \mathcal{D}_\kappa \subset \mathcal{B}$ and $\mathcal{D}_\kappa$ has full measure on $\mathbb{R}/\mathbb{Z}$ (see [7] or [11]). Bryuno showed that if $\alpha$ is a Bryuno number, then $f$ is linearizable at the origin. Yoccoz showed that if a quadratic polynomial $P_\alpha(z) = z^2 + e^{2\pi i \alpha} z$ is linearizable at the origin, then $\alpha$ is a Bryuno number, that is, $P_\alpha$ is linearizable at the origin if and only if $\alpha$ is a Bryuno number. Moreover the following theorem holds if $\alpha$ is of bounded type (see [10] or [11]).

**Theorem 1.1** (Ghys-Douady-Herman-Shishikura-Świątek). If an irrational number $\alpha \in [0, 1]$ is of bounded type, then the boundary of the Siegel disk $\Delta$ of $P_\alpha$ centered at the origin is a quasicircle containing its critical point $-e^{2\pi i \alpha}/2$. 
Moreover if the irrational number $\alpha$ is of bounded type, then the following holds:

(a) (Petersen). The Julia set $J(P_\alpha)$ of $P_\alpha$ is locally connected and has measure zero.

(b) (McMullen). The Hausdorff dimension of $J(P_\alpha)$ is less than 2.

(c) (Graczyk-Jones). The Hausdorff dimension of $\partial \Delta$ is greater than 1.

Conversely, Petersen showed that if $\partial \Delta$ is a quasicircle containing the finite critical point $-e^{2\pi i \alpha}/2$ of $P_\alpha$, then $\alpha \in [0, 1]$ is of bounded type. Zakeri extended Theorem 1.1 to the case of cubic polynomials.

**Theorem 1.2** ([12]). *Let $P$ be a cubic polynomial with fixed point of multiplier $e^{2\pi i \alpha}$ at the origin. If an irrational number $\alpha \in [0, 1]$ is of bounded type, then the boundary of the Siegel disk $\Delta$ of $P$ centered at the origin is a quasicircle containing one or both critical points.*

Geyer showed the following theorem which is extended to some polynomials. Let $Q_m(z) = e^{2\pi i \alpha} z (1 + z/m)^m$. Note that $P_\alpha$ is conformally conjugate to $Q_1$.

**Theorem 1.3** ([4]). *Let $m \geq 1$ be a positive integer. If an irrational number $\alpha \in [0, 1]$ is of bounded type, then the boundary of the Siegel disk $\Delta$ of $Q_m$ centered at the origin is a quasicircle containing its critical point $-m/(m + 1)$.*

Let $F_{\lambda, \mu}(z) = z(z + \lambda)/(\mu z + 1)$ with $\lambda, \mu \neq 1$. The origin and the point at infinity are fixed points of $F_{\lambda, \mu}$ of multiplier $\lambda$ and $\mu$ respectively. In the case that $\mu = 0$, $F_{\lambda, 0}(z) = \lambda z + z^2$. Therefore the quadratic rational function $F_{\lambda, \mu}$ is considered as a perturbation of the quadratic polynomial $z \mapsto \lambda z + z^2$. In the case that $\lambda = e^{2\pi i \alpha}$ and $\alpha$ is irrational of bounded type, the author showed the following theorem which is a generalization of Theorem 1.1.

**Theorem 1.4** ([5]). *If an irrational number $\alpha \in [0, 1]$ is of bounded type, $\lambda = e^{2\pi i \alpha}$ and $\mu \in \mathbb{D}$ with $\lambda \mu \neq 1$, then the boundary of the Siegel disk $\Delta$ of $F_{\lambda, \mu}$ centered at the origin is a quasicircle containing its critical point.*

For complex numbers $\lambda$ and $\mu$ with $\lambda \mu \neq 1$ and a positive integer $m$, let

$$F_{\lambda, \mu, m}(z) = z \left( \frac{z + \lambda}{\mu z + 1} \right)^m.$$

Note that $F_{\lambda, \mu, 1} = F_{\lambda, \mu}$. The origin and the point at infinity are fixed points of $F_{\lambda, \mu, m}$ of multiplier $\lambda^m$ and $\mu^m$ respectively. In the case that $\mu = 0$,

$$F_{\lambda, 0, m}(z) = z (z + \lambda)^m.$$

Therefore the rational function $F_{\lambda, \mu, m}$ of degree $m + 1$ is considered as a perturbation of the polynomial $F_{\lambda, 0, m}$ of degree $m + 1$. Note that $F_{\lambda, 0, m}$ is conformally conjugate to $Q_m$ if $\lambda^m = e^{2\pi i \alpha}$. We show the following theorem which contains Theorem 1.4.
Theorem 1.5. Let \( m \geq 1 \) be a positive integer and let \( \mu \in \overline{D} \). If an irrational number \( \alpha \in [0,1] \) is of bounded type and \( e^{2\pi i \alpha} \mu^m \neq 1 \), then there exist suitable pairs \( \{ (\lambda_j, \mu_j) \}_{j=1}^{m} \) with

(i) \( \lambda_j^m = e^{2\pi i \alpha} \), \( \mu_j^m = \mu^m \) and \( \lambda_j \neq \mu_j \) for \( j \in \{1, \ldots, m\} \)

(ii) \( \lambda_j \neq \lambda_k \) if \( j \neq k \)

such that for each \( j \in \{1, \ldots, m\} \), the boundary of the Siegel disk \( \Delta \) of \( F_{\lambda_j, \mu_j, m} \) centered at the origin is a quasicircle containing its critical point.

<table>
<thead>
<tr>
<th>Theorem 1.5</th>
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<tbody>
<tr>
<td>( m = 1, \mu = 0 )</td>
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<td>( m = 1 )</td>
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<td>( \mu = 0 )</td>
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Table 1. Special cases of Theorem 1.5

Theorem 1.5 contains Theorems 1.1, 1.3, and 1.4. Moreover we obtain the following corollary.

Corollary 1.6. Let \( m \geq 1 \) be a positive integer, \( \alpha \in [0,1] \) be an irrational number of bounded type, \( \mu^m = e^{2\pi i \phi} \) with \( e^{2\pi i \alpha} \mu^m \neq 1 \) and \( \{ (\lambda_j, \mu_j) \}_{j=1}^{m} \) be as in Theorem 1.5. If \( \beta \in [0,1] \) is an irrational number of bounded type, then the boundaries of Siegel disks \( \Delta \) and \( \Delta_\infty \) of \( F_{\lambda_j, \mu_j, m} \) centered at the origin and the point at infinity respectively are quasicircles containing its critical point.

2. Blaschke products with a critical point on the unit circle

2.1. Existence of Blaschke products

Let \( m \geq 1 \) be a positive integer. We consider a Blaschke product

\[
B(z) = e^{2\pi im \theta} \left( \frac{z-a}{1-\bar{a}z} \right)^m \left( \frac{z-b}{1-\bar{b}z} \right)^m
\]

of degree \( 2m + 1 \) with \( ab \neq 1 \) and \( 0 < |a| \leq |b| < \infty \). Let \( \lambda = abe^{2\pi i \theta} \) and let \( \mu = \bar{a}e^{-2\pi i \theta} \). The derivative \( B' \) of \( B \) is

\[
B'(z) = \frac{e^{2\pi im \theta}}{(1-\bar{a}z)^2(1-\bar{b}z)^2} \left( \frac{z-a}{1-\bar{a}z} \right)^{m-1} \left( \frac{z-b}{1-\bar{b}z} \right)^{m-1} g(z),
\]

where

\[
g(z) = \bar{a}b \bar{z}^4 + \left\{ -(m+1)(\bar{a}+\bar{b}) + (m-1)\bar{a}b(a+b) \right\} z^3
\]

\[
+ \left\{ 2m+1 - (2m-1)|ab|^2 + |a+b|^2 \right\} z^2
\]

\[
+ \left\{ -(m+1)(a+b) + (m-1)ab(\bar{a}+\bar{b}) \right\} z + ab.
\]
So multipliers of fixed points $z = 0$ and $z = \infty$ are $\lambda^m$ and $\mu^m$ respectively. Let $c_1, c_2, c_3 = 1/\bar{c}_2$ and $c_4 = 1/\bar{c}_1$ be solutions of the equation $g(z) = 0$. Therefore critical points of $B$ are $a, 1/\bar{a}$, $b, 1/\bar{b}$ $c_1$, $c_2$, $c_3$ and $c_4$ and multiplicities of critical points $a, 1/\bar{a}$, $b$ and $1/\bar{b}$ are $m - 1$. Since $c_1$, $c_2$, $c_3$ and $c_4$ are solutions of $g(z) = 0$, we obtain that

$$g(z) = \bar{a}\bar{b}(z - c_1)(z - c_2)(z - c_3)(z - c_4) = \bar{a}\bar{b}\left\{z^4 - C_3z^3 + C_2z^2 - C_1z + C_0\right\},$$

where

$$C_3 = c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2},$$

$$C_2 = \frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right),$$

$$C_1 = \frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right),$$

$$C_0 = \frac{c_1c_2}{\bar{c}_1\bar{c}_2}.$$

Comparing coefficients of two representations of $g(z)$ implies that

$$c_1 + \frac{1}{\bar{c}_1} + c_2 + \frac{1}{\bar{c}_2} = \frac{(m + 1)(\bar{a} + \bar{b}) - (m - 1)(a + b)\bar{a}\bar{b}}{\bar{a}\bar{b}}, \tag{1}$$

$$\frac{c_1}{\bar{c}_1} + \frac{c_2}{\bar{c}_2} + \left(c_1 + \frac{1}{\bar{c}_1}\right)\left(c_2 + \frac{1}{\bar{c}_2}\right) = \frac{2m + 1 - (2m - 1)|ab|^2 + |a + b|^2}{\bar{a}\bar{b}}, \tag{2}$$

$$\frac{c_1}{\bar{c}_1}\left(c_2 + \frac{1}{\bar{c}_2}\right) + \frac{c_2}{\bar{c}_2}\left(c_1 + \frac{1}{\bar{c}_1}\right) = \frac{(m + 1)(a + b) - (m - 1)(\bar{a} + \bar{b})ab}{\bar{a}\bar{b}}, \tag{3}$$

$$\frac{c_1c_2}{\bar{c}_1\bar{c}_2} = \frac{ab}{\bar{a}\bar{b}}. \tag{4}$$

Eliminating $c_1$ and $c_2$ from equations (1), (2), and (4) gives that

$$|a + b|^2 - (m + 1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(\bar{a} + \bar{b}) - \left(\frac{\bar{c}_2}{c_2}\right)ab$$

$$+ \left\{\left(c_2 + \frac{1}{\bar{c}_2}\right)^2 - \frac{c_2}{\bar{c}_2}\right\}\bar{a}\bar{b} + (m - 1)\left(c_2 + \frac{1}{\bar{c}_2}\right)(a + b)\bar{a}\bar{b}$$

$$+ 2m + 1 - (2m - 1)|ab|^2 = 0.$$
and eliminating \( c_1 \) and \( \bar{c}_1 \) from equations (1), (3), and (4) gives that

\[
\frac{c_2}{\bar{c}_2} \left( c_2 + \frac{1}{\bar{c}_2} \right) ab + (m + 1) \left( \frac{c_2}{\bar{c}_2} \right) (\bar{a} + \bar{b}) - (m - 1) \left( \frac{c_2}{\bar{c}_2} \right) (a + b) \bar{a} \bar{b}
\]

(6)

\[
= \frac{c_2}{\bar{c}_2} \left( c_2 + \frac{1}{\bar{c}_2} \right) \bar{a} \bar{b} + (m + 1) (a + b) - (m - 1)(\bar{a} + \bar{b}) ab.
\]

We obtain that

\[
|a + b|^2 - 2(m + 1)e^{2\pi i \varphi} (\bar{a} + \bar{b}) - e^{2\pi i (-2\varphi)} ab + 3e^{2\pi i:2\varphi} \bar{a} \bar{b}
\]

+ 2(m - 1)e^{2\pi i \varphi} (a + b) \bar{a} \bar{b} + 2m + 1 - (2m - 1)|ab|^2 = 0

(7)

and

\[
e^{2\pi i (-2\varphi)} ab + \frac{m + 1}{2} e^{2\pi i \varphi} (\bar{a} + \bar{b}) - \frac{m - 1}{2} e^{2\pi i (-\varphi)} (a + b) \bar{a} \bar{b}
\]

\[
= e^{2\pi i (-\varphi)} \bar{a} \bar{b} + \frac{m + 1}{2} e^{2\pi i (-\varphi)} (a + b) - \frac{m - 1}{2} e^{2\pi i (-\varphi)} (\bar{a} + \bar{b}) ab
\]

(8)

by substituting \( c_2 = e^{2\pi i \varphi} \) into equations (5) and (6). Eliminating \( ab \) from equations (7) and (8) gives that

\[
|a + b|^2 - \frac{3}{2} (m + 1)e^{2\pi i \varphi} (\bar{a} + \bar{b})
\]

\[- \frac{m + 1}{2} e^{2\pi i (-\varphi)} (a + b) + 2e^{2\pi i:2\varphi} \bar{a} \bar{b} + \frac{m - 1}{2} e^{2\pi i (-\varphi)} (\bar{a} + \bar{b}) ab
\]

\[+ \frac{3}{2} (m - 1)e^{2\pi i \varphi} (a + b) \bar{a} \bar{b} + 2m + 1 - (2m - 1)|ab|^2 = 0.
\]

Let \( \zeta = a + b \). Then

\[
|\zeta|^2 - \frac{3}{2} (m + 1)e^{2\pi i \varphi} \zeta - \frac{m + 1}{2} e^{2\pi i (-\varphi)} \zeta + 2e^{2\pi i:2\varphi} \bar{a} \bar{b} + \frac{m - 1}{2} e^{2\pi i (-\varphi)} ab \zeta
\]

\[+ \frac{3}{2} (m - 1)e^{2\pi i \varphi} \bar{a} \bar{b} \zeta + 2m + 1 - (2m - 1)|ab|^2 = 0.
\]

The real part of the left side of the equation (10) is

\[
x^2 + y^2 - 2x \left( (m + 1) \cos 2\pi \varphi - (m - 1)r \cos 2\pi (\varphi + \theta + \omega) \right)
\]

\[- 2y \left( (m + 1) \sin 2\pi \varphi + (m - 1)r \sin 2\pi (\varphi + \theta + \omega) \right)
\]

\[+ 2r \cos 2\pi (2\varphi + \theta + \omega) + 2m + 1 - (2m - 1)r^2 = 0
\]

and the imaginary part of the left side of the equation (10) is

\[
y \left( (m + 1) \cos 2\pi \varphi + (m - 1)r \cos 2\pi (\varphi + \theta + \omega) \right)
\]

\[- x \left( (m + 1) \sin 2\pi \varphi - (m - 1)r \sin 2\pi (\varphi + \theta + \omega) \right)
\]

\[+ 2r \sin 2\pi (2\varphi + \theta + \omega) = 0,
\]
where $\zeta = x + iy$ and $\mu = a\bar{b}e^{-2\pi i \theta} = re^{2\pi i \omega}$. The solutions of simultaneous equations (11) and (12) are

$$x = \left\{ (m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1) r \cos 2\pi (2\varphi + \theta + \omega) \right\}^{-1} \times \left\{ C_4 \cos 2\pi \varphi + C_5 \cos 2\pi (\varphi + \theta + \omega) + C_6 \cos 2\pi (3\varphi + \theta + \omega) + C_7 \cos 2\pi (3\varphi + 2\theta + 2\omega) \right\},$$

and

$$y = \left\{ (m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1) r \cos 2\pi (2\varphi + \theta + \omega) \right\}^{-1} \times \left\{ C_4 \sin 2\pi \varphi - C_5 \sin 2\pi (\varphi + \theta + \omega) + C_6 \sin 2\pi (3\varphi + \theta + \omega) - C_7 \sin 2\pi (3\varphi + 2\theta + 2\omega) \right\},$$

where

$$C_4 = (m + 1)^2(2m + 1) - 2m(m^2 - 1)r^2,$$
$$C_5 = 2m(m^2 - 1)r - (m - 1)^2(2m - 1)r^3,$$
$$C_6 = -(m + 1)^2r,$$
$$C_7 = -(m - 1)^2r^2.$$ 

Hence $\zeta = x + iy$ satisfies the equation (10). Conversely, we show the following theorem.

**Theorem 2.1.** Let $\mu = re^{2\pi i \omega} \in \overline{D}$ and let $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ with $|a| \leq |b|$ be complex numbers satisfying relations $a + b = x + iy$ and $ab = re^{-2\pi i (\theta + \omega)}$, that is, $a$ and $b$ are the solutions of the equation

$$Z^2 - (x + iy)Z + re^{-2\pi i(\theta + \omega)} = 0,$$

where $x$ and $y$ are as above and $(\theta, \varphi) \in [0, 1]^2$. Then the following holds:

(a) In the case that $r = 0$, solutions of the equation (†) are $a = 0$ and $b = (2m + 1)e^{2\pi i \varphi}$.  

(b) In the case that $0 < r < 1$, the equation (†) does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.  

(c) In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod{1}$, the equation (†) has double roots and $a = b = e^{2\pi i \varphi}$.  

(d) In the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod{1}$, the equation (†) does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.  

(e) In the case (a), (b) or (d),

$$B(z) = B_{\theta, \varphi, m}(z) = e^{2\pi im\theta} \left( \frac{z - a}{1 - \bar{a}z} \right)^m \left( \frac{z - b}{1 - \bar{b}z} \right)^m.$$
is a Blaschke product of degree $2m + 1$ and the point at infinity is a fixed point of $B$ with multiplier $\mu^m$. Moreover $z = e^{2\pi i \varphi}$ is a critical point of $B$ and $B|_T : T \to T$ is a homeomorphism, where $T$ is the unit circle.

Proof. First, we show the following lemma.

**Lemma 2.2.** $|x + iy| \geq 2$. Moreover the equality holds if and only if $r = 1$ and $2\varphi + \theta = \omega \equiv 0 \pmod{1}$ hold.

Proof of Lemma 2.2. We calculate that

$$
|x + iy|^2 = \left\{ (m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1) r \cos 2\pi (2\varphi + \theta + \omega) \right\}^{-2}
\times \left\{ C_4 e^{2\pi i \varphi} + C_5 e^{-2\pi i (\varphi + \theta + \omega)} + C_6 e^{2\pi i (3\varphi + \theta + \omega)} + C_7 e^{-2\pi i (3\varphi + 2\theta + \omega)} \right\}^2
\times \left\{ C_4 e^{2\pi i \varphi} + C_5 e^{-2\pi i (\varphi + \theta + \omega)} + C_6 e^{2\pi i (3\varphi + \theta + \omega)} + C_7 e^{-2\pi i (3\varphi + 2\theta + \omega)} \right\}
= \left\{ (m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1) r \cos 2\pi (2\varphi + \theta + \omega) \right\}^{-2}
\times \left\{ C_4^2 + C_5^2 + C_6^2 + C_7^2 + (2C_4 C_5 + 2C_4 C_6 + 2C_5 C_7) \cos 2\pi (2\varphi + \theta + \omega)
+ \left( 2C_4 C_7 + 2C_5 C_6 \right) \cos 2\pi \cdot 2(2\varphi + \theta + \omega)
+ 2C_6 C_7 \cos 2\pi \cdot 2(2\varphi + \theta + \omega)
+ \left( C_4 C_7 + C_5 C_6 \right) \cos 2\pi (2\varphi + \theta + \omega) + 8C_6 C_7 \cos 3\pi (2\varphi + \theta + \omega) \right\}
$$

where

$$
\cos 2\pi \cdot 2(2\varphi + \theta + \omega) = 2 \cos^2 2\pi (2\varphi + \theta + \omega) - 1
$$

and

$$
\cos 2\pi \cdot 3(2\varphi + \theta + \omega) = 4 \cos^3 2\pi (2\varphi + \theta + \omega) - 3 \cos 2\pi (2\varphi + \theta + \omega).
$$
Therefore

\[ |x + iy|^2 \]
\[ = \left\{(m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1)r \cos 2\pi(2\varphi + \theta + \omega)\right\}^{-2} \]
\[ \times \left[4m^6 + 20m^5 + 41m^4 + 44m^3 + 8m^2 + 8m + 1 \right. \]
\[ + (4m^6 - 12m^5 - 5m^4 + 12m^3 + 14m^2 + 8m + 3) r^2 \]
\[ + (4m^6 - 12m^5 - 5m^4 - 12m^3 + 14m^2 - 8m + 3) r^4 \]
\[ + (4m^6 - 20m^5 + 41m^4 - 44m^3 + 26m^2 - 8m + 1) r^6 \]
\[ + \left\{(8m^6 + 16m^5 - 10m^4 - 48m^3 - 44m^2 - 16m - 2) r \right. \]
\[ + (-16m^6 + 16m^5 - 10m^4 + 48m^3 - 44m^2 + 16m - 2) r^3 \]
\[ + \left. (16m^5 - 20m^4 + 41m^3 + 24m^2 - 4) r^2 \right\} \cos 2\pi(2\varphi + \theta + \omega) \]
\[ + \left. (8m^4 - 16m^3 + 8) r^3 \cos^3 2\pi(2\varphi + \theta + \omega) \right] \]
\[ = \left\{(m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1)r \cos 2\pi(2\varphi + \theta + \omega)\right\}^{-1} \]
\[ \times \left[\left((m + 1)^2 + (m - 1)^2 r^2 + 2(m^2 - 1)r \cos 2\pi(2\varphi + \theta + \omega)\right) \right. \]
\[ \times \left[\left((m + 1)^2(2m + 1)^2 - 2(4m^4 - 5m^2 - 1)r^2 + (m - 1)^2(2m - 1)^2 r^4 \right. \right. \]
\[ + \left(4m(m + 1)(2m + 1)r + 4m(m - 1)(2m - 1) r^3 \right) \cos 2\pi(2\varphi + \theta + \omega) \]
\[ + 4(m^2 - 1) r^2 \cos^2 2\pi(2\varphi + \theta + \omega) \right] \]
Let \(X = \cos 2\pi(2\phi + \theta + \omega)\) and we consider the function
\[
f(X) = \left\{(m + 1)^2 + (m - 1)^2r^2 + 2(m^2 - 1)rX\right\}^{-1}
\times \left[(m + 1)^2(2m + 1)^2 - 2(4m^4 - 5m^2 - 1)r^2 + (m - 1)^2(2m - 1)^2r^4
+ 4mr\left\{(m + 1)(2m + 1)+(m - 1)(2m - 1)r^2\right\}X + 4(m^2 - 1)r^2X^2\right].
\]
Then the function \(f\) is monotone decreasing on \([-1, 1]\) and
\[
f(1) = \left\{2m + 1 - (2m - 1)r\right\}^2.
\]
In the case that \(0 \leq r < 1\), we obtain that
\[
|x + iy| \geq \sqrt{f(1)} = 2m + 1 - (2m - 1)r > 2.
\]
In the case that \(r = 1\) and \(2\phi + \theta + \omega \equiv 0 \pmod{1}\), we obtain that
\[
|x + iy| > \sqrt{f(1)} = 2m + 1 - (2m - 1)\cdot 1 = 2.
\]
Moreover in the case that \(r = 1\) and \(2\phi + \theta + \omega \equiv 0 \pmod{1}\), we obtain that
\[
|x + iy| = \sqrt{f(1)} = 2m + 1 - (2m - 1)\cdot 1 = 2.
\]

**Proof of (a).** It is clear.

**Proof of (b).** By Lemma 2.2, \(|a + b| = |x + iy| > 2\). In the case that \(0 < r < 1\),
either \(0 < |a| < 1 \leq |b| < \infty\) or \(0 < |a| \leq |b| \leq 1\) hold since \(|a||b| = r\). If
\(0 < |a| \leq |b| \leq 1\), then
\[
2 < |a + b| \leq |a| + |b| \leq 2.
\]
This is a contradiction and hence the situation \(0 < |a| < 1 \leq |b| < \infty\) happens.
If \(|b| = 1\), then
\[
2 < |a + b| \leq |a| + |b| = |a| + 1 < 2.
\]
This is a contradiction. Therefore the equation \((\dagger)\) does not have double roots
and \(0 < |a| < 1 < |b| < \infty\).

**Proof of (c).** By assumptions, we obtain that \(x + iy = 2e^{2\pi i\phi}\) and \(re^{-2\pi i(\phi + \omega)} = e^{2\pi i - 2\phi}\). Therefore the equation \((\dagger)\) is
\[
Z^2 - 2e^{2\pi i\phi}Z + e^{2\pi i - 2\phi} = 0
\]
and hence \(a = b = e^{2\pi i\phi}\).

**Proof of (d).** By Lemma 2.2, \(|a + b| = |x + iy| > 2\). In the case that \(r = 1\), either
\(0 < |a| < 1 < |b| < \infty\) or \(|a| = |b| = 1\) hold since \(|a||b| = 1\). If \(|a| = |b| = 1\), then
\[
2 < |a + b| \leq |a| + |b| = 2.
\]
This is a contradiction. Therefore the equation \((\dagger)\) does not have double roots
and \(0 < |a| < 1 < |b| < \infty\).
Proof of (e). Let
\[ u(z) = \left( \frac{z - a}{1 - \bar{a}z} \right) \left( \frac{z - b}{1 - \bar{b}z} \right) = \frac{z^2 - (a + b)z + ab}{\bar{a}b z^2 - (\bar{a} + \bar{b})z + 1}. \]

The necessary and sufficient condition that the degree of the Blaschke product \( B \) be \( 2m + 1 \) is that the function \( u \) be not constant. So the necessary and sufficient condition that the degree of the Blaschke product \( B \) be 1 is that the function \( u \) be constant. In the case that \( r = 0 \), the function \( u \) is not constant since
\[ u(z) = \frac{z^2 - (2m + 1)e^{2\pi i \varphi} z}{-(2m + 1)e^{-2\pi i \varphi} z + 1}. \]

If \( r \neq 0 \), then
\[ u(z) = \frac{1}{\bar{a}b} \cdot \frac{\bar{a}\bar{b} z^2 - \bar{a}\bar{b}(a + b)z + |ab|^2}{\bar{a}b z^2 - (\bar{a} + \bar{b})z + 1}. \]

In the case that \( 0 < r < 1 \), the degree of the Blaschke product \( B \) is \( 2m + 1 \) since \( |ab| = r < 1 \). In the case that \( r = 1 \), we obtain that
\[ \bar{a}\bar{b}(a + b) - (\bar{a} + \bar{b}) = \frac{-2me^{-2\pi i (3\varphi + \theta + \omega)} \left\{ e^{2\pi i (2\varphi + \theta + \omega)} - 1 \right\}^3}{m^2 + 1 + (m^2 - 1) \cos 2\pi (2\varphi + \theta + \omega)}. \]

Therefore in the case that \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \pmod{1} \), the degree of the Blaschke product \( B \) is \( 2m + 1 \). On the other hand, if \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \pmod{1} \), then
\[ u(z) = \frac{1}{\bar{a}b} = e^{2\pi i \cdot 2\varphi} \]
and the degree of the Blaschke product \( B \) is 1. It is clear that the point at infinity is a fixed point of \( B \) with multiplier \( \mu^m \). Moreover it is clear that \( g(e^{2\pi i \varphi}) = 0 \) and hence \( z = e^{2\pi i \varphi} \) is a critical point of \( B \), where
\[ B'(z) = \frac{e^{2\pi im \theta}}{(1 - \bar{a}z)^2(1 - \bar{b}z)^2} \left( \frac{z - a}{1 - \bar{a}z} \right)^{m-1} \left( \frac{z - b}{1 - \bar{b}z} \right)^{m-1} g(z) \]
and
\[ g(z) = \bar{a}\bar{b} z^4 + \left\{ -(m + 1)(\bar{a} + \bar{b}) + (m - 1)\bar{a}\bar{b}(a + b) \right\} z^3 \]
\[ + \left\{ 2m + 1 - (2m - 1)|ab|^2 + |a + b|^2 \right\} z^2 \]
\[ + \left\{ -(m + 1)(a + b) + (m - 1)ab(\bar{a} + \bar{b}) \right\} z + ab. \]

Finally we show that two critical points of \( B \) other than \( a, 1/\bar{a}, b, 1/\bar{b} \) (if \( m \geq 2 \)) and \( e^{2\pi i \varphi} \) are in \( \mathbb{C} \setminus \mathbb{T} \). In the case that \( r = 0 \), we obtain that
\[ g(z) = -(m + 1)(2m + 1)e^{-2\pi i \varphi} z \left( z - e^{2\pi i \varphi} \right)^2. \]
Therefore critical points of $B$ are $b, 1/b$ (if $m \geq 2$), $0$, $\infty$ and $e^{2\pi i \varphi}$. In the case that $r \neq 0$, let

$$h(z) = z^2 + \frac{e^{2\pi i \varphi}}{C_{10}} \left\{ C_9 e^{-2\pi i (2(\varphi + \theta + \omega))} + C_8 e^{-2\pi i (2\varphi + \theta + \omega)} + C_9 \right\} z
+ e^{-2\pi i (\varphi + \theta + \omega)},$$

where

$$C_8 = -(m + 1)^3(2m + 1) + 2(2m^4 - m^2 - 1)r^2 - (m - 1)^3(2m - 1)r^4,$$

$$C_9 = (m + 1)^3r - (m - 1)^3r^3,$$

$$C_{10} = (m + 1)^2r + (m - 1)^2r^3 + 2(m^2 - 1)r \cos 2\pi(2\varphi + \theta + \omega).$$

Then we can factor $r^{-1}e^{-2\pi i (\theta + \omega)}g(z)$ as

$$\frac{1}{r} \cdot e^{-2\pi i (\theta + \omega)} \cdot g(z) = \left( z - e^{2\pi i \varphi} \right)^2 \cdot h(z).$$

Let

$$h_1(z) = \frac{e^{2\pi i \varphi}}{C_{10}} \left\{ C_9 e^{-2\pi i (2(\varphi + \theta + \omega))} + C_8 e^{-2\pi i (2\varphi + \theta + \omega)} + C_9 \right\} z$$

and

$$h_2(z) = z^2 + e^{-2\pi i (\varphi + \theta + \omega)}.$$

For $z \in \mathbb{T}$, $|h_2(z)| \leq 2$.

**Lemma 2.3.** $|h_1(z)| > 2$ on $\mathbb{T}$.

**Proof of Lemma 2.3.** In the case that $0 < r < 1$, we obtain that

$$|h_1(z)| = \frac{1}{|C_{10}|} \left| C_9 e^{-2\pi i (2(\varphi + \theta + \omega))} + C_8 e^{-2\pi i (2\varphi + \theta + \omega)} + C_9 \right|$$

$$\geq \frac{|C_8| - 2|C_9|}{|C_{10}|}$$

$$= \frac{-C_8 - 2C_9}{|C_{10}|}$$

$$\geq v(m, r)$$

on $\mathbb{T}$, where

$$v(m, r) = \left\{ (3m - 1)(m + 1)r + (m - 1)^2r^3 \right\}^{-1}$$

$$\times \left\{ (m + 1)^3(2m + 1) - 2(m + 1)^3r - 2(2m^4 - m^2 - 1)r^2$$

$$+ 2(m - 1)^3r^3 + (m - 1)^3(2m - 1)r^4 \right\}.$$ 

Since the function $r \mapsto v(m, r)$ is monotone decreasing on $(0, 1]$ and $v(m, 1) = 2$, we obtain that $|h_1(z)| > 2$ on $\mathbb{T}$. In the case that $r = 1$ and $2\varphi + \theta + \omega \neq 0$
(mod 1), we obtain that
\[
|h_1(z)| = \left| \frac{C_9}{C_{10}} \right| \left| e^{2\pi i(2\varphi + \theta + \omega)} + \frac{C_8}{C_9} e^{-2\pi i(2\varphi + \theta + \omega)} + 1 \right|
\]
\[
= \frac{C_9}{C_{10}} \left| e^{2\pi i(2\varphi + \theta + \omega)} + 1 \right|^2 - \left| \frac{C_8}{C_9} - 2 \right|
\]
\[
\geq \frac{C_9}{C_{10}} \left| e^{2\pi i(2\varphi + \theta + \omega)} + 1 \right|^2 - \frac{4(4m^2 + 1)}{3m^2 + 1}
\]
\[
> \frac{3m^2 + 1}{2m^2} \left\{ \frac{4(4m^2 + 1)}{3m^2 + 1} - \left| e^{2\pi i(2\varphi + \theta + \omega)} + 1 \right|^2 \right\}
\]
\[
= 2
\]
on \mathbb{T}.

By the Rouché's theorem, the number of roots of \( h(z) = h_1(z) + h_2(z) \) on \( \mathbb{D} \) is one since \( |h_1(z)| > 2 \geq |h_2(z)| \) on \( \mathbb{T} \) and the number of roots of \( h_1(z) \) on \( \mathbb{D} \) is one. So one of critical points of \( B \) other than \( a, 1/\bar{a}, b, 1/\bar{b} \) (if \( m \geq 2 \)) and \( e^{2\pi i \varphi} \) is in \( \mathbb{D} \). Since critical points of a Blaschke product are symmetric with respect to the unit circle, the other one critical point of \( B \) is in \( \mathbb{C} \setminus \mathbb{D} \). In this case, the inverse image \( B^{-1}(\mathbb{T}) \) of the unit circle \( \mathbb{T} \) is the union of \( \mathbb{T} \) and a figure eight \( 8 \) which crosses at \( z = e^{2\pi i \varphi} \). Refer to Figure 1. Then \( B|_S : \mathbb{T} \to \mathbb{T} \) is a 2\( m \)-to-1 map and therefore \( B|_T : \mathbb{T} \to \mathbb{T} \) is a homeomorphism.

\[\]

\[\]

**Figure 1.** The inverse image \( B_{\theta, \varphi, m}^{-1}(\mathbb{T}) \) of the unit circle \( \mathbb{T} \).
Remark 2.4. Two complex numbers $a = a(\theta, \varphi)$ and $b = b(\theta, \varphi)$ satisfy that
\[ a(\theta + 1, \varphi) = a(\theta, \varphi) = a(\theta, \varphi + 1) \]
and
\[ b(\theta + 1, \varphi) = b(\theta, \varphi) = b(\theta, \varphi + 1). \]

2.2. Rotation numbers of Blaschke products

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism and let $\tilde{f} : \mathbb{R} \to \mathbb{R}$ be a lift of $f$ via $x \mapsto e^{2\pi i x}$ which satisfies $\tilde{f}(x + 1) = \tilde{f}(x) + 1$ for all $x \in \mathbb{R}$. The lift $\tilde{f}$ of $f$ is unique up to addition of an integer constant. The rotation number $\rho(\tilde{f})$ of $\tilde{f}$ is defined as
\[ \rho(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(x)}{n}, \]
which is independent of $x \in \mathbb{R}$. The rotation number $\rho(f)$ is defined as the residue class of $\rho(\tilde{f})$ modulo $\mathbb{Z}$. Poincaré showed that the rotation number is rational with denominator $q$ if and only if $f$ has a periodic point with period $q$. The following theorem is important (see [6]).

**Theorem 2.5.** Let $\mathcal{F}$ be the set of all orientation preserving homeomorphisms from the unit circle onto itself with the topology of uniform convergence. Then the rotation number function $\rho : \mathcal{F} \to \mathbb{R}/\mathbb{Z}$ defined as $f \mapsto \rho(f)$ is continuous.

Let $a(\theta, \varphi)$ and $b(\theta, \varphi)$ be as in Theorem 2.1. We define a map $\Gamma_m : [0, 1]^3 \to \mathbb{T}$ as
\[ \Gamma_m(x, \theta, \varphi) = \left( \frac{e^{2\pi i x} - a(\theta, \varphi)}{1 - a(\theta, \varphi) e^{2\pi i x}} \right)^m \left( \frac{e^{2\pi i x} - b(\theta, \varphi)}{1 - b(\theta, \varphi) e^{2\pi i x}} \right)^m, \]
and a map $H_m : [0, 1]^4 \to \mathbb{T}$ as
\[ H_m(x, \theta, \varphi, t) = \left( \frac{e^{2\pi i x} - a(\theta, \varphi, t)}{1 - a(\theta, \varphi, t) e^{2\pi i x}} \right)^m \left( \frac{e^{2\pi i x} - b(\theta, \varphi, t)}{1 - b(\theta, \varphi, t) e^{2\pi i x}} \right)^m, \]
where
\[ a(\theta, \varphi, t) = (1 - t) a(\theta, \varphi) + t e^{2\pi i \varphi} \]
and
\[ b(\theta, \varphi, t) = (1 - t) b(\theta, \varphi) + t e^{2\pi i \varphi}. \]

Note that $\Gamma_m(x, \theta, \varphi) = e^{2\pi i \cdot 2m\varphi} \mod$ $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \mod 1$. The following three lemmas play important roles in the proof of Theorem 2.9.

**Lemma 2.6.** A map $H_m(\cdot, \theta, \varphi, \cdot) : [0, 1]^2 \to \mathbb{T}$ is a homotopy between a loop $x \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $x \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(\theta, \varphi) \in [0, 1]^2$.

**Proof.** It is clear since $H_m(\cdot, \theta, \varphi, 0) = \Gamma_m(\cdot, \theta, \varphi)$ and $H_m(\cdot, \theta, \varphi, 1) = e^{2\pi i \cdot 2m\varphi}$. \hfill \Box

**Lemma 2.7.** A map $H_m(x, \cdot, \varphi, \cdot) : [0, 1]^2 \to \mathbb{T}$ is a homotopy between a loop $\theta \mapsto \Gamma_m(x, \theta, \varphi)$ and a constant loop $\theta \mapsto e^{2\pi i \cdot 2m\varphi}$ for any $(x, \varphi) \in [0, 1]^2$.
Proof. It is clear since \( H_m(x, \cdot, \varphi, 0) = \Gamma_m(x, \cdot, \varphi) \) and \( H_m(x, \cdot, \varphi, 1) = e^{2\pi i \cdot 2m\varphi} \).

**Lemma 2.8.** A map \( H_m(x, \theta, \cdot, \cdot) : [0, 1]^2 \to \mathbb{T} \) is a homotopy between a loop \( \varphi \mapsto \Gamma_m(x, \theta, \varphi) \) and a loop \( \varphi \mapsto e^{2\pi i \cdot 2m\varphi} \) for any \( (x, \theta) \in [0, 1]^2 \).

**Proof.** It is clear since \( H_m(x, \theta, \cdot, 0) = \Gamma_m(x, \theta, \cdot) \) and \( H_m(x, \theta, \cdot, 1) = e^{2\pi i \cdot 2m\varphi} \).

Lemma 2.6 and Lemma 2.7 imply that
\[
\arg(\Gamma_m(x + 1, \theta, \varphi)) = \arg(\Gamma_m(x, \theta, \varphi)) = \arg(\Gamma_m(x, \theta + 1, \varphi))
\]
and Lemma 2.8 implies that
\[
\frac{1}{2\pi} \arg(\Gamma_m(x, \theta, \varphi + 1)) = \frac{1}{2\pi} \arg(\Gamma_m(x, \theta, \varphi)) + 2m.
\]

**Theorem 2.9.** Let \( \alpha \in [0, 1] \) and let \( \mu = re^{2\pi i \omega} \in \mathbb{D} \). Besides let \( a = a(\theta, \varphi) \) and \( b = b(\theta, \varphi) \) be as in Theorem 2.1. Then for the Blaschke product
\[
B_{\theta, \varphi, m}(z) = e^{2\pi i m \theta} z \left( \frac{z - a}{1 - \bar{a}z} \right)^m \left( \frac{z - b}{1 - \bar{b}z} \right)^m,
\]
\( B_{\theta, \varphi, m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \) is an orientation preserving homeomorphism. Moreover
(a) In the case that \( 0 \leq r < 1 \), there exists \( (\theta_0, \varphi_0) \in [0, 1]^2 \) such that
\[
\rho(B_{\theta_0, \varphi_0, m}|_{\mathbb{T}}) = \alpha.
\]
(b) In the case that \( r = 1 \), if \( \alpha + m\omega \equiv 0 \pmod{1} \), then there exists \( (\theta_0, \varphi_0) \in [0, 1]^2 \) such that \( \rho(B_{\theta_0, \varphi_0, m}|_{\mathbb{T}}) = \alpha \) and \( 2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod{1} \).

**Proof.** In the case that \( r = 1 \) and \( 2\varphi + \theta + \omega \equiv 0 \pmod{1} \),
\[
B_{\theta, \varphi, m}(z) = e^{2\pi i m (2\varphi + \theta)} z = e^{2\pi i m \omega} z.
\]
Therefore \( B_{\theta, \varphi, m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \) is an orientation preserving homeomorphism and its rotation number satisfies that \( \rho(B_{\theta, \varphi, m}|_{\mathbb{T}}) \equiv -m\omega \pmod{1} \). In the other cases, we consider a lift
\[
\tilde{B}_{\theta, \varphi, m}(x) = m\theta + x + \frac{1}{2\pi} \arg(\Gamma_m(x, \theta, \varphi))
\]
of \( B_{\theta, \varphi, m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \) via \( x \mapsto e^{2\pi i x} \). By Lemma 2.6,
\[
\tilde{B}_{\theta, \varphi, m}(x + 1) = m\theta + x + 1 + \frac{1}{2\pi} \arg(\Gamma_m(x + 1, \theta, \varphi)) = \tilde{B}_{\theta, \varphi, m}(x) + 1
\]
for all \( x \in \mathbb{R} \). This implies that \( B_{\theta, \varphi, m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T} \) is an orientation preserving homeomorphism. Consequently the rotation number of \( \rho(\tilde{B}_{\theta, \varphi, m}) \) is well defined. By Lemma 2.7, we obtain that \( \tilde{B}_{1, \varphi, m}(x) = \tilde{B}_{0, \varphi, m}(x) + mn \) and hence
\[
\rho(\tilde{B}_{1, \varphi, m}) = \rho(\tilde{B}_{0, \varphi, m}) + m.
\]
Moreover by Lemma 2.8, we obtain that $\tilde{B}_{\theta,1,m}^n(x) = \tilde{B}_{\theta,0,m}^n(x) + 2mn$ and hence

$$\rho(\tilde{B}_{\theta,1,m}) = \rho(\tilde{B}_{\theta,0,m}) + 2m. \tag{14}$$

These two equations (13) and (14) imply that

$$\rho(\tilde{B}_{1,1,m}) = \rho(\tilde{B}_{0,0,m}) + 3m.$$

Therefore in the case that $0 \leq r < 1$, there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_T) \equiv \rho(\tilde{B}_{\theta_0,\varphi_0,m}) \quad (\text{mod } 1)$$

since the rotation number function $(\theta, \varphi) \mapsto \rho(B_{\theta,\varphi,m}|_T)$ is continuous. In the case that $r = 1$, if $2\varphi + \theta + \omega \equiv 0 \pmod{1}$, then $\rho(B_{\theta,\varphi,m}|_T) \equiv -m\omega \pmod{1}$. Hence if $\alpha + m\omega \not\equiv 0 \pmod{1}$, then there exists $(\theta_0, \varphi_0) \in [0, 1]^2$ such that

$$\alpha = \rho(B_{\theta_0,\varphi_0,m}|_T) \equiv \rho(\tilde{B}_{\theta_0,\varphi_0,m}) \quad (\text{mod } 1)$$

and $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod{1}$. \hfill \Box

**Remark 2.10.** By Theorem 2.1, the degree of $B_{\theta_0,\varphi_0,m}$ is $2m + 1$.

### 3. Rational functions with Siegel disks

In this section, we show Theorem 1.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. If there exists $k \geq 1$ such that

$$\frac{1}{k} \leq \left| \frac{f(x + t) - f(x)}{f(x) - f(x - t)} \right| \leq k$$

for all $x \in \mathbb{R}$ and all $t \geq 0$, then $f$ is called $k$-quasisymmetric. A homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is $k$-quasisymmetric if its lift $\tilde{h} : \mathbb{R} \to \mathbb{R}$ is $k$-quasisymmetric. By the theorem of Beurling and Ahlfors, any $k$-quasisymmetric homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is extended to a $K$-quasiconformal map $F : \mathbb{H} \to \mathbb{H}$, where $\mathbb{H}$ is the upper half plane (More precisely $F : \mathbb{C} \to \mathbb{C}$). The dilatation $K$ of $F$ depends only on $k$. Therefore if a homeomorphism $h : \mathbb{T} \to \mathbb{T}$ is $k$-quasisymmetric, then we can extend $h$ to a $K$-quasiconformal map $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ whose dilatation depends only on $k$.

**Theorem 3.1** (Herman–Świątek). *The rotation number $\rho(f)$ of a real analytic orientation preserving homeomorphism $f : \mathbb{T} \to \mathbb{T}$ is of bounded type if and only if $f$ is quasisymmetrically linearizable, that is, there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ f \circ h^{-1}(z) = e^{2\pi i \rho(f)}z$.*

Recall that

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} \left( \frac{z - a}{1 - \bar{a}z} \right)^m \left( \frac{z - b}{1 - \bar{b}z} \right)^m$$

and

$$F_{\lambda,\mu,m}(z) = z \left( \frac{z + \lambda}{\mu z + 1} \right)^m.$$
Figure 2. Golden Siegel disks of $F_{\lambda, \mu, 1}$ centered at the origin, where $\lambda = e^{2\pi i (\sqrt{5}-1)/2}$ and $\mu = re^{2\pi i (\sqrt{5}-1)/2}$. In the case that $r = 1$, the point at infinity is the center of another golden Siegel disk.

Proof of Theorem 1.5. By Theorem 2.9, there exist $(\theta, \varphi) \in [0, 1]^2$ such that the degree of $B_{\theta, \varphi, m}$ is $2m + 1$ and $\rho(B_{\theta, \varphi, m}|_{\Gamma}) = \alpha$. By Theorem 3.1, there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ B_{\theta, \varphi, m}|_{\Gamma} \circ h^{-1}(z) = R_\alpha(z) = e^{2\pi i \alpha}z$ since $\alpha$ is of bounded type. By the theorem of Beurling and Ahlfors, $h$ has a quasiconformal extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $H(0) = 0$. We
define a new map $\mathcal{B}_{\theta, \varphi, m}$ as

$$\mathcal{B}_{\theta, \varphi, m} = \begin{cases} B_{\theta, \varphi, m} & \text{on } \hat{\mathcal{C}} \setminus \mathbb{D}, \\ H^{-1} \circ R_\alpha \circ H & \text{on } \mathbb{D}. \end{cases}$$

The map $\mathcal{B}_{\theta, \varphi, m}$ is quasiregular on $\hat{\mathcal{C}}$ since $\mathbb{T}$ is an analytic curve. Moreover $\mathcal{B}_{\theta, \varphi, m}$ is a degree $m + 1$ branched covering of $\hat{\mathcal{C}}$. We define a conformal structure $\sigma_{\theta, \varphi, m}$ as

$$\sigma_{\theta, \varphi, m} = \begin{cases} H^*(\sigma_0) & \text{on } \mathbb{D}, \\ \left( \mathcal{B}_{\theta, \varphi, m}^n \right)^* \circ H^*(\sigma_0) & \text{on } \mathcal{B}_{\theta, \varphi, m}^{-n}(\mathbb{D}) \setminus \mathbb{D} \text{ for all } n \in \mathbb{N}, \\ \sigma_0 & \text{on } \hat{\mathcal{C}} \setminus \bigcup_{n=1}^\infty \mathcal{B}_{\theta, \varphi, m}^{-n}(\mathbb{D}), \end{cases}$$

where $\sigma_0$ is the standard conformal structure on $\hat{\mathcal{C}}$. The conformal structure $\sigma_{\theta, \varphi, m}$ is invariant under $\mathcal{B}_{\theta, \varphi, m}$ and its maximal dilatation is the dilatation of $H$ since $H$ is quasiconformal and $B_{\theta, \varphi, m}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\Psi : \hat{\mathcal{C}} \to \hat{\mathcal{C}}$ such that $\Psi^*\sigma_0 = \sigma_{\theta, \varphi, m}$. Therefore $\Psi \circ \mathcal{B}_{\theta, \varphi, m} \circ \Psi^{-1}$ is a rational map of degree $m + 1$. We normalize $\Psi = \Psi_j$ by $\Psi_j(0) = 0$, $\Psi_j(b) = -\lambda_j$ and $\Psi_j(\infty) = \infty$, where $\lambda_j = e^{2\pi i (\alpha + j)/m}$ for $j \in \{1, \ldots, m\}$.

**Lemma 3.2.** If $\mu \neq 0$, then there exists $\mu_j^m = \mu^m$ such that

$$F_{\lambda_j, \mu_j, m} = \Psi_j \circ \mathcal{B}_{\theta, \varphi, m} \circ \Psi_j^{-1}.$$  

**Proof of Lemma 3.2.** Define $\xi_j$ as $\xi_j = -\Psi_j(1/\bar{u})$. Note that $\lambda_j \neq \xi_j$ since such $\Psi_j$ is unique. Since orders of zeros and poles are invariant under conjugation, we obtain that

$$\left( \Psi_j \circ \mathcal{B}_{\theta, \varphi, m} \circ \Psi_j^{-1} \right)'(0) = \frac{\eta_j \lambda_j^m}{\xi_j^m} = e^{2\pi i \alpha}$$

and

$$\frac{1}{(\Psi_j \circ \mathcal{B}_{\theta, \varphi, m} \circ \Psi_j^{-1})(\infty)} = \frac{1}{\eta_j} = \mu^m.$$
By the equations (15) and (16), we obtain that \((\xi_j \mu)^m = 1\). Then there exists an \(m\)-th root of unity \(\nu_j\) such that \(\xi_j = \nu_j / \mu\). Therefore

\[
\Psi_j \circ \mathbf{B}_{\theta, \varphi, m} \circ \Psi_j^{-1}(z) = \frac{z}{\mu^m} \left( \frac{z + \lambda_j}{z + \nu_j / \mu} \right)^m = z \left( \frac{z + \lambda_j}{\mu z + \nu_j} \right)^m
\]

\[
= \frac{z}{\nu_j^m} \left( \frac{z + \lambda_j}{(\mu / \nu_j) z + 1} \right)^m = z \left( \frac{z + \lambda_j}{\mu_j z + 1} \right)^m = F_{\lambda_j, \mu_j, m}(z),
\]

where \(\mu_j = \mu / \nu_j\).

Let \(\mu_j = 0\) for all \(j \in \{1, \ldots, m\}\) if \(\mu = 0\). It is easy to check that the pairs \(\{ (\lambda_j, \mu_j) \}_{j=1}^m\) satisfies (i) and (ii). The map \(F_{\lambda_j, \mu_j, m}\) has a Siegel disk \(\Delta = \Psi_j(\mathbb{D})\) with a critical point \(\Psi_j(e^{2\pi i \varphi}) \in \partial \Delta\). Moreover \(\partial \Delta = \Psi_j(\mathbb{T})\) is a quasicircle since \(\Psi_j\) is quasiconformal.

\[\square\]

Proof of Corollary 1.6. Let \(\mathcal{I}(z) = 1/z\). Then \(F_{\lambda_j, \mu_j, m} = \mathcal{I} \circ F_{\mu_j, \lambda_j, m} \circ \mathcal{I}\). Let \(\Delta\) and \(\Delta_\infty\) be Siegel disks of \(F_{\lambda_j, \mu_j, m}\) centered at the origin and the point at infinity respectively. By Theorem 1.5, the boundary of \(\Delta\) contains a critical point of \(F_{\lambda_j, \mu_j, m}\). On the other hand, \(\mathcal{I}(\Delta_\infty)\) is the Siegel disk of \(F_{\mu_j, \lambda_j, m}\) centered at the origin. By Theorem 1.5, the boundary of \(\mathcal{I}(\Delta_\infty)\) contains a critical point of \(F_{\mu_j, \lambda_j, m}\). Therefore the boundary of \(\Delta_\infty\) contains a critical point of \(F_{\lambda_j, \mu_j, m}\).

\[\square\]

References


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