J. Korean Math. Soc. **46** (2009), No. 1, pp. 171–185

FIBRED RIEMANNIAN SPACE WITH ALMOST COMPLEX STRUCTURES

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Abstract. We study fibred Riemannian spaces with almost complex structures which are induced by the almost complex structure or the almost contact structure on the base and fibre. We show that if the total space is a complex space form, then the total space is locally Euclidean. Moreover, we deal with the fibred Riemannian space with various Kaehlerian structures.

1. Introduction

The study of fibred space goes back to the unified field theory in a 5-dimensional Riemannian space due to T. Kaluza and O. Klein. Fibred Riemannian space was first considered by Y. Muto [10] and treated by B. L. Reinhart [14] in the name of foliated Riemannian manifolds. B. O'Neill [13] called such a foliation a Riemannian submersion and gave its structure equations and at the almost same time K. Yano and S. Ishihara [21] developed an extensive theory of fibred Riemannian space. These work were systematically reported in [5]. M. Ako [1], T. Okubo [12] and B. Watson [17, 18] studied fibred space with almost complex or almost Hermitian structure.

In connection with almost contact structures, S. Tanno [15] and Y. Ogawa [11] investigated principal bundles over almost complex spaces with a 1-dimensional structure group. Generalizing Calabi-Eikmann's example, S. Morimoto [9] defined an almost complex structure in the product of two almost contact spaces and obtained a condition on the normality, and S. I. Goldberg and K. Yano [4] studied similar properties for the product of two framed manifolds.

On the other hand Tashiro and Kim [16] have studied fibred Riemannian spaces with almost Hermitian or almost contact metric structure. They applied these results to the study of tangent bundles of Riemannian spaces. Kim [6,

Received July 2, 2007.

2000 *Mathematics Subject Classification.* Primary 53B35, 53C55, 53D15.

Key words and phrases. fibred Riemannian space, almost complex structure, Kaehlerian space, complex space forms.

This work was supported by the Research Grant 2001 of Kyung Hee University and Korea Research Council of Fundamental Science & Technology (KRCF) Grant No. C-RESEARCH-2006-11 NIMS.

7] has studied fibred Riemannian spaces with various geometric structures and has constructed model space and counter examples.

In this point of view, the purpose of this paper is to study the fibred Riemannian space with almost complex structure which is naturally induced by the almost complex structures or an almost contact structures for the base space and each fibre. We investigate whether the base or a fibre inherits the same structure from the total space endowed with a suitable structure. We show that if the total space is nearly Kaehlerian, the normal connection of each fibre vanishes and a fibre is a minimal submanifold of the total space. We also obtain a condition for the base and fibre to be Einstein when the total space is Einstein. Moreover, we shall deal with the fibred Riemannian space with various Kaehlerian structures and complex space forms.

2. Fibred Riemannian space

Let $\{M, B, \tilde{g}, \pi\}$ be a fibred Riemannian space, that is, M an m-dimensional total space with projectable Riemannian metric $\tilde{g}, B$ an n-dimensional base space, and $\pi : M \to B$ a projection with maximal rank n. The fibre passing through a point $\tilde{P} \in M$ is denoted by $F(\tilde{P})$ or generally F, which is a p-dimensional submanifold of M, where $p = m - n$.

Throughout this paper, manifolds, geometric objects and mappings are supposed to be of C^∞ class and manifolds are assumed to be connected. Also, unless stated otherwise, the ranges of indices are as follows;

$$\begin{aligned}
&A, B, C, D, E : 1, 2, \ldots, m,\\
&h, i, j, k, l : 1, 2, \ldots, m,\\
&a, b, c, d, e : 1, 2, \ldots, n,\\
&x, y, z, w, u : n+1, \ldots, n+p = m.
\end{aligned}$$

If we take coordinate neighborhoods $(\tilde{U},\ z^h)$ in M and $(U,\ x^a)$ in B such that $\pi(\tilde{U})=U$, then the projection π is expressed by equations

$$x^a = x^a(z^h) \tag{2.1}$$

with Jacobian $(\frac{\partial x^a}{\partial z^h})$ of maximum rank n. There is a local coordinate system y^x in $F \cap \tilde{U} \neq \emptyset$, (x^a, y^x) form a coordinate system in $\tilde{U}$ and each fibre $F(\tilde{P})$ at $\tilde{P}$ in $F \cap \tilde{U}$ is parametrized as $z^h = z^h(x^a, y^x)$. Then we can choose a local frame (E_a, C_x) and its dual frame (E^a, C^x) in $\tilde{U}$, where the components of E^a and C^x are given by

$$E_i{}^a = \frac{\partial x^a}{\partial z^i} \quad \text{and} \quad C^x{}_h = \frac{\partial y^x}{\partial z^h}. \tag{2.2}$$

The vector fields E_a span the horizontal distribution and C_x the tangent space of each fibre. The metric tensor g in the base space B is given by

$$g_{cb} = \tilde{g}(E_c, E_b) \tag{2.3}$$

and the induced metric tensor $\bar{g}$ in each fibre F by

$$\bar{g}_{xy} = \tilde{g}(C_x, C_y). \tag{2.4}$$

We write (E_B) for the frame (E_b, C_x) in all, if necessary. Let $h = (h_{xy}{}^a)$ be components of the second fundamental tensor with respect to the normal vector E_a and $L = (L_{cb}{}^x)$ the normal connection of each fibre F. Then we have

$$h_{xy}{}^a = h_{yx}{}^a \qquad \text{and} \qquad L_{cb}{}^x + L_{bc}{}^x = 0. \tag{2.5}$$

Denoting by $\tilde{\nabla}$ the Riemannian connection of the total space M, we have the following equations.

$$\tilde{\nabla}_j E^h{}_b = \Gamma_{cb}{}^a E_j{}^c E^h{}_a - L_{cb}{}^x E_j{}^c C^h{}_x + L_b{}^a{}_y C_j{}^y E^h{}_a - h_y{}^x{}_b C_j{}^y C^h{}_x, \tag{2.6.1}$$

$$\begin{aligned}\tilde{\nabla}_j C^h{}_x = {} & L_c{}^a{}_x E_j{}^c E^h{}_a - (h_x{}^y{}_c - P_{cx}{}^y) E_j{}^c C^h{}_y + h_{zx}{}^a C_j{}^z E^h{}_a \\ & + \bar{\Gamma}_{zx}{}^y C_j{}^z C^h{}_y,\end{aligned} \tag{2.6.2}$$

$$\tilde{\nabla}_j E_i{}^a = -\Gamma_{cb}{}^a E_j{}^c E_i{}^b - L_c{}^a{}_x (E_j{}^c C_i{}^x + C_j{}^x E_i{}^c) - h_{yx}{}^a C_j{}^y C_i{}^x, \tag{2.6.3}$$

$$\begin{aligned}\tilde{\nabla}_j C_i{}^x = {} & L_{cb}{}^x E_j{}^c E_i{}^b + (h_y{}^x{}_c - P_{cy}{}^x) E_j{}^c C_i{}^y \\ & + h_z{}^x{}_b C_j{}^z E_i{}^b \bar{\Gamma}_{zy}{}^x C_j{}^z C_i{}^y,\end{aligned} \tag{2.6.4}$$

where $\Gamma_{cb}{}^a$ are connection coefficients of the projection $\nabla = p\tilde{\nabla}$ in B, $\bar{\Gamma}_{zy}{}^x$ those of the induced connection $\bar{\nabla}$ in F, $L_c{}^a{}_y = L_{cb}{}^x g^{ba} \bar{g}_{xy}, h_y{}^x{}_b = h_{yz}{}^a \bar{g}^{zx} g_{ba}$ and $P_{cy}{}^x$ are local functions in $\tilde{U}$ defined by $[E_b, C_y] = P_{by}{}^x C_x$.

From (2.6.1), we see that $[E_c, E_b] = -2L_{cb}{}^x C_x$, and so the horizontal distribution is integrable if and only if the structure tensor L vanishes identically.

Let γ be a curve through a point P in the base space B and X be the tangent vector field of γ. There is a unique curve $\tilde{\gamma}$ through a point $\tilde{P} \in \pi^{-1}(P)$ such that its tangent vector field is the lift X^L. The curve $\tilde{\gamma}$ is called the *horizontal lift* of γ passing through $\tilde{P}$. If a curve γ joins points P and Q in B, then the horizontal lifts of γ through all points of the fibre $F(P)$ define a fibre mapping $\Phi_\gamma : F(P) \to F(Q)$, called the *horizontal mapping covering* γ.

If the horizontal mapping covering any curve in B is an isometry of fibres, then $\{M, B, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *isometric fibres*. A necessary and sufficient condition for M to have isometric fibres is $(\mathcal{L}_{X^L}\tilde{g}^V)^V = 0$ for any vector field X in B, or equivalently $h_{xy}{}^a = 0$. Here and hereafter A^H and A^V indicate the horizontal and vertical parts of A respectively. The model space of the fibred Riemannian space with isometric fibre can be seen in [3].

If the horizontal mapping covering any curve in B is conformal mapping of fibres, then $\{M, B, \tilde{g}, \pi\}$ is called a fibred Riemannian space with *conformal fibres*. A condition for M to have conformal fibres is $h_{xy}{}^a = \bar{g}_{xy} A^a$, where

$A = A^a E_a$ is the mean curvature vector along each fibre in M. The following theorem is well known [8].

Theorem 2.1. *If the normal connection $L = (L_{cb}{}^x)$ and second fundamental form $h = (h_{xy}{}^a)$ vanish identically in a fibred Riemannian space, then the fibred space is locally the Riemannian product of the base and a fibre.*

The curvature tensor of a fibred Riemannian space M is defined by

$$\tilde{K}(\tilde{X},\tilde{Y})\tilde{Z} = \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z} \tag{2.7}$$

for any vector fields $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ in M. We put

$$\tilde{K}(E_D, E_C)E_B = \tilde{K}_{DCB}{}^A E_A = \tilde{K}_{DCB}{}^a E_a + \tilde{K}_{DCB}{}^x C_x, \tag{2.8}$$

then $\tilde{K}_{DCB}{}^A$ are components of the curvature tensor with respect to the basis (E_B). Denoting the components of the curvature tensor in $(\tilde{U}, z^h)$ by $\tilde{K}_{kji}{}^h$, we have the relations

$$\tilde{K}_{DCB}{}^A = \tilde{K}_{kji}{}^h E^k{}_D E^j{}_C E^i{}_B E_h{}^A. \tag{2.9}$$

Substituting (2.6) into the definition (2.7) of the curvature tensor, we have the structure equations of a fibred Riemannian space as follows [2, 5, 7, 13, 16]:

$$\tilde{K}_{dcb}{}^a = K_{dcb}{}^a - L_d{}^a{}_x L_{cb}{}^x + L_c{}^a{}_x L_{db}{}^x + 2L_{dc}{}^x L_b{}^a{}_x, \tag{2.10}$$

$$\tilde{K}_{dcb}{}^x = -{}^*\nabla_d L_{cb}{}^x + {}^*\nabla_c L_{db}{}^x - 2L_{dc}{}^y h_y{}^x{}_b, \tag{2.11}$$

$$\begin{aligned}\tilde{K}_{dcy}{}^x = {}& {}^*\nabla_c h_y{}^x{}_d - {}^*\nabla_d h_y{}^x{}_c + 2{}^{**}\nabla_y L_{dc}{}^x + L_{de}{}^x \\ & L_c{}^e{}_y - L_{ce}{}^x L_d{}^e{}_y - h_z{}^x{}_d h_y{}^z{}_c + h_z{}^x{}_c h_y{}^z{}_d,\end{aligned} \tag{2.12}$$

$$\tilde{K}_{dzb}{}^a = {}^*\nabla_d L_b{}^a{}_z - L_d{}^a{}_x h_z{}^x{}_b + L_{db}{}^x h_{zx}{}^a - L_d{}^a{}_x h_z{}^x{}_d, \tag{2.13}$$

$$\tilde{K}_{dzb}{}^x = -{}^*\nabla_d h_z{}^x{}_b + {}^{**}\nabla_z L_{db}{}^x + L_d{}^e{}_z L_{eb}{}^x + h_z{}^y{}_d h_y{}^x{}_b, \tag{2.14}$$

$$\tilde{K}_{zyb}{}^a = L_{zyb}{}^a + h_z{}^x{}_b h_{yx}{}^a - h_y{}^x{}_b h_{zx}{}^a, \tag{2.15}$$

$$\tilde{K}_{zyx}{}^a = {}^{**}\nabla_z h_{yx}{}^a - {}^{**}\nabla_y h_{zx}{}^a, \tag{2.16}$$

$$\tilde{K}_{zyx}{}^w = \bar{K}_{zyx}{}^w + h_{zx}{}^e h_y{}^w{}_e - h_{yx}{}^e h_z{}^w{}_e, \tag{2.17}$$

where we have put

$$K_{dcb}{}^a = \partial_d \Gamma_{cb}{}^a - \partial_c \Gamma_{db}{}^a + \Gamma_{de}{}^a \Gamma_{cb}{}^e - \Gamma_{ce}{}^a \Gamma_{db}{}^e, \tag{2.18}$$

$${}^*\nabla_d L_{cb}{}^x = \partial_d L_{cb}{}^x - \Gamma_{dc}{}^e L_{eb}{}^x - \Gamma_{db}{}^e L_{ce}{}^x + Q_{dy}{}^x L_{cb}{}^y, \tag{2.19}$$

$${}^*\nabla_d L_c{}^a{}_y = \partial_d L_c{}^a{}_y + \Gamma_{de}{}^a L_c{}^e{}_y - \Gamma_{dc}{}^e L_e{}^a{}_y - Q_{dy}{}^z L_c{}^a{}_z, \tag{2.20}$$

$${}^*\nabla_d h_{zy}{}^a = \partial_d h_{zy}{}^a + \Gamma_{de}{}^a h_{zy}{}^e - Q_{dz}{}^x h_{xy}{}^a - Q_{dy}{}^x h_{zx}{}^a, \tag{2.21}$$

$${}^*\nabla_d h_y{}^x{}_b = \partial_d h_y{}^x{}_b - \Gamma_{db}{}^e h_y{}^x{}_e + Q_{dz}{}^x h_y{}^z{}_b - Q_{dy}{}^z h_z{}^x{}_b, \tag{2.22}$$

$Q_{cy}{}^x$ being defined by

$$Q_{cy}{}^x = P_{cy}{}^x - h_y{}^x{}_c$$

and

(2.23) $$^{**}\nabla_y L_{cb}{}^x = \partial_y L_{cb}{}^x + \bar{\Gamma}_{yz}{}^x L_{cb}{}^z - L_c{}^e{}_y L_{eb}{}^x - L_b{}^e{}_y L_{ce}{}^x,$$

(2.24) $$^{**}\nabla_y L_b{}^a{}_x = \partial_y L_b{}^a{}_x - \bar{\Gamma}_{yx}{}^z L_b{}^a{}_z + L_e{}^a{}_y L_b{}^e{}_x - L_b{}^e{}_y L_e{}^{ax},$$

(2.25) $$^{**}\nabla_z h_{yx}{}^a = \partial_z h_{yx}{}^a - \bar{\Gamma}_{zy}{}^w h_{wx}{}^a - \bar{\Gamma}_{zx}{}^w h_{yw}{}^a + L_e{}^a{}_z h_{yx}{}^e,$$

(2.26) $$^{**}\nabla_z h_y{}^x{}_b = \partial_z h_y{}^x{}_b + \bar{\Gamma}_{zw}{}^x h_y{}^w{}_b - \bar{\Gamma}_{zw}{}^y h_w{}^x{}_b - L_b{}^e{}_z h_y{}^x{}_e,$$

(2.27) $$L_{yxb}{}^a = \partial_y L_b{}^a{}_x - \partial_x L_b{}^a{}_y + L_e{}^a{}_w L_b{}^e{}_x - L_e{}^a{}_x L_b{}^e{}_y,$$

(2.28) $$\bar{K}_{zyx}{}^w = \partial_z \bar{\Gamma}_{yx}{}^w - \partial_y \bar{\Gamma}_{zx}{}^w + \bar{\Gamma}_{zu}{}^w \bar{\Gamma}_{yx}{}^u - \bar{\Gamma}_{yu}{}^w \bar{\Gamma}_{zx}{}^u.$$

Among these, the functions $K_{dcb}{}^a$ are projectable in $\tilde{U}$ and its projections, denoted by $K_{dcb}{}^a$ too, are components of the curvature tensor of the base space $\{B, g\}$. On each fibre F, the functions $\bar{K}_{zyx}{}^w$ are components of the curvature tensor of the induced Riemannian metric $\bar{g}$ and $L_{yxb}{}^a$ those of the curvature tensor of the normal connection of F in M. The components $\tilde{K}_{DCB}{}^A$ satisfy the same algebraic equations as those $\tilde{K}_{kji}{}^h$ satisfy. Denote by $\tilde{K}_{CB}$ components of the Ricci tensor of $\{M, \tilde{g}\}$ with respect to the basis (E_B) in $\tilde{U}$, and by K_{cb} and $\bar{K}_{yx}$ components of the Ricci tensors of the base space $\{B, g\}$ in (U, x^a) and each fibre $\{F, \bar{g}\}$ in $(\bar{U}, y^x)$ respectively. Then we have

(2.29) $$\tilde{K}_{cb} = K_{cb} - 2L_{ce}{}^x L_b{}^e{}_x - h_y{}^x{}_c h_x{}^y{}_b + \tfrac{1}{2}({}^*\nabla_c h_x{}^x{}_b + {}^*\nabla_b h_x{}^x{}_c),$$

(2.30) $$\tilde{K}_{xb} = {}^{**}\nabla_x h_y{}^y{}_b - {}^{**}\nabla_y h_x{}^y{}_b + {}^*\nabla_e L_b{}^e{}_x - 2h_x{}^y{}_e L_b{}^e{}_y,$$

(2.31) $$\tilde{K}_{yx} = \bar{K}_{yx} - h_{yx}{}^e h_z{}^z{}_e + {}^*\nabla_e h_{yx}{}^e - L_a{}^e{}_y L_e{}^a{}_x.$$

Denoting by $\tilde{K}, K$ and $\bar{K}$ the scalar curvatures of M, B and each fibre F respectively, we have the relation

(2.32) $$\tilde{K} = K^L + \bar{K} - L_{cbz}L^{cbz} - h_{yxe}h^{yxe} - h_y{}^y{}_e h_u{}^{ue} + 2{}^*\nabla_e h_z{}^{ze}.$$

3. Almost complex manifold

An almost complex structure on a Riemannian manifold M is a linear endomorphism J of TM such that $J^2 = -I$, where I stands for the identity transformation of TM. A Riemannian manifold M with an almost complex structure is called an almost complex manifold. Since $J^2 = -I$, we easily see that every almost complex manifold is of even dimension.

A Hermitian metric on an almost complex manifold M is a Riemannian metric g which is invariant under the action of the almost complex structure J, i.e.,

$g(JX, JY) = g(X, Y)$ for any vector fields X and Y on M. A Hermitian metric thus defines a Hermitian inner product on each tangent space $T_x(M)$ with respect to the complex structure defined by J. An almost complex manifold (resp. a complex manifold) with a Hermitian metric is called an almost Hermitian manifold (resp. a Hermitian manifold). That is, an almost Hermitian manifold is a triple (M, g, J), where (M, g) is a Riemannian manifold and

(i) J is a tensor field of type (1, 1) on M satisfying $J \circ J = -I$,
(ii) g is a Hermitian metric of M, i.e., $g(JX, JY) = g(X, Y)$ for all vector fields X, Y on M.

The fundamental 2-form Φ of an almost Hermitian manifold M is defined by $\Phi(X, Y) = g(JX, Y)$. A Hermitian metric on an almost complex manifold is called a Kaehler metric if the fundamental 2-form is closed. An almost complex manifold with a Kaehler metric is called an almost Kaehler manifold [8, 16, 20]. An almost Hermitian manifold M with almost complex structure J is called a nearly Kaehler manifold (or almost Tachibana manifold, or K-space) if $(\nabla_X J)Y + (\nabla_Y J)X = 0$ for any vector fields X and Y on M or equivalently $(\nabla_X J) = 0$ for every vector field X on M.

The complex torsion or Nijenhuis tensor N of an almost complex structure J is the tensor field of type (1,2) given by

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \tag{3.1}$$

for every vector fields X, Y on M.

A well-known theorem of Newlander and Nirenberg [8, 9, 20] states that J is the almost complex structure associated to a complex structure on M if and only if the Nijenhuis tensor of J vanishes, in which case we say that J is integrable.

A complex manifold with a Kaehler metric is called a Kaehler manifold. The Kaehler manifolds are characterized by $\nabla_X J = 0$ for every vector field X on M.

4. Fibred Riemannian space with almost complex structure

In this chapter, we investigate the fibred Riemannian space with almost complex structures induced by the almost complex structure A in the base space and H in each fibre. Let us put

$$J_j{}^i = A_b{}^a E_j{}^b E^i{}_a + H_x{}^y C_j{}^x C^i{}_y. \tag{4.1}$$

Then we can see that $J^2 = -I$ and the horizontal and vertical subspaces are invariant by J. Conversely we can easily see that $A^2 = -I$ and $H^2 = -I$ due to $J^2 = -I$.

Theorem 4.1. *Let B and F be almost complex manifolds. Then the fibred Riemannian space M admits an almost complex structure.*

Using equations (2.6.1)-(2.6.4), and (4.1), we obtain the following equations.

(4.2.1) $(\tilde{\nabla}_k J_j{}^i)E^k{}_c E^j{}_d E_i{}^e = \nabla_c A_d{}^e,$

(4.2.2) $(\tilde{\nabla}_k J_j{}^i)E^k{}_c E^j{}_d C_i{}^z = L_{cd}{}^x H_x{}^z - L_{cb}{}^z A_d{}^b,$

(4.2.3) $(\tilde{\nabla}_k J_j{}^i)E^k{}_c C^j{}_w E_i{}^e = L_c{}^e{}_x H_w{}^x - L_c{}^b{}_w A_b{}^e,$

(4.2.4) $(\tilde{\nabla}_k J_j{}^i)E^k{}_c C^j{}_w C_i{}^z = Q_{cy}{}^z H_w{}^y - Q_{cw}{}^y H_y{}^z,$

(4.2.5) $(\tilde{\nabla}_k J_j{}^i)C^k{}_z E_j{}^d E_i{}^e = L_c{}^e{}_z A_d{}^c - L_d{}^c{}_z A_c{}^e,$

(4.2.6) $(\tilde{\nabla}_k J_j{}^i)C^k{}_y E^j{}_d C_i{}^z = h_y{}^x{}_d H_x{}^z - h_y{}^z{}_b A_d{}^b,$

(4.2.7) $(\tilde{\nabla}_k J_j{}^i)C^k{}_z C^j{}_w E_i{}^e = h_{zy}{}^e H_w{}^y - h_{zw}{}^b A_b{}^e,$

(4.2.8) $(\tilde{\nabla}_k J_j{}^i)C^k{}_y C^j{}_w C_i{}^z = \nabla_y H_w{}^z.$

From the equations (4.2.1)-(4.2.8), we get

(4.3.1) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c E^j{}_d E^i{}_e = \nabla_c A_{de} + \nabla_d A_{ec} + \nabla_e A_{cd},$$

(4.3.2) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c E^j{}_d C^i{}_z = 2L_{cd}{}^x H_{xz},$$

(4.3.3) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c C^j{}_z E^i{}_d = 2L_{cdx} H_z{}^x,$$

(4.3.4) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})E^k{}_c C^j{}_z C^i{}_w = Q_{cyw} H_z{}^y - Q_{cz}{}^y H_{yw} + h_{zyc} H_w{}^y + h_w{}^x{}_c H_{xz},$$

(4.3.5) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})C^k{}_z E^j{}_c E^i{}_d = L_{cd}{}^x H_{xz} + L_{dcx} H_z{}^x,$$

(4.3.6) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})C^k{}_z E^j{}_c C^i{}_w = h_z{}^x{}_c H_{xw} + Q_{cyz} H_w{}^y - Q_{cw}{}^y H_{yz} + h_{wyc} H_z{}^y,$$

(4.3.7) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})C^k{}_z C^j{}_w E^i{}_c = h_{zyc} H_w{}^y + h_w{}^x{}_c H_{xz} + Q_{cyw} H_z{}^y - Q_{cz}{}^y H_{yw},$$

(4.3.8) $$(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj})C^k{}_z C^j{}_w C^i{}_y = \nabla_z H_{wy} + \nabla_w H_{yz} + \nabla_y H_{zw}.$$

Let M be almost Kaehlerian, then the fundamental 2-form $\Phi(X, Y) = g(JX, Y)$ is closed, i.e.,$\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ik} + \tilde{\nabla}_i J_{kj} = 0$. Hence we see that $L = 0$ from the equations (4.3.2) and the base space and each fibre are almost Kaehler manifolds from the equations (4.3.1) and (4.3.8) respectively. Therefore we can state

Theorem 4.2. *If a fibred almost Hermitian space $\{M, B, \tilde{g}, \pi\}$ with invariant fibers is almost Kaehlerian, then B and F are almost Kaehlerian and $L = 0$.*

Using the equations (4.2.1)-(4.2.8), we get

$$(4.4.1)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c E^j{}_d E^i{}_e = \nabla_c A_{de} + \nabla_d A_{ce},$$

$$(4.4.2)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c E^j{}_d C^i{}_z = -L_{cbz} A_d{}^b - L_{dbz} A_c{}^b,$$

$$(4.4.3)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c C^j{}_z E^i{}_e = L_{cex} H_w{}^x - 2L_c{}^b{}_w A_{BE} + L_{dew} A_c{}^d,$$

$$(4.4.4)\qquad \begin{aligned} &(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) E^k{}_c C^j{}_w C^i{}_z \\ &= Q_{cyz} H_w{}^y - Q_{cw}{}^y H_{yz} + h_w{}^x{}_c H_{xz} - h_{wzb} A_c{}^b, \end{aligned}$$

$$(4.4.5)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_w E^j{}_c E^i{}_d = L_{edw} A_c{}^e - 2L_c{}^e{}_w A_{ed} + L_{cdx} H_w{}^x,$$

$$(4.4.6)\qquad \begin{aligned} &(\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_y E^j{}_d C^i{}_z \\ &= h_y{}^x{}_d H_{xz} - h_{yzb} A_d{}^b + Q_{dxz} H_y{}^x - Q_{dy}{}^x H_{xz}, \end{aligned}$$

$$(4.4.7)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_y C^j{}_w E^i{}_d = h_{zyd} H_w{}^z - 2h_{yw}{}^b A_{bd} + h_{wxd} H_y{}^x,$$

$$(4.4.8)\qquad (\tilde{\nabla}_k J_{ji} + \tilde{\nabla}_j J_{ki}) C^k{}_y C^j{}_x C^i{}_z = \nabla_w H_{yz} + \nabla_y H_{zw}.$$

Now we suppose that the total space M is nearly Kaehlerian. Then the right hand sides of the equations (4.4.1)-(4.4.8) vanish identically.

From the equations (4.4.1) and (4.4.8), we see that the base space and each fibre are nearly Kaehlerian. Using the skew symmetry of L and the equation (4.4.2), we get

$$(4.4.9)\qquad L_{cbz} A_d{}^b + L_{dbz} A_c{}^b = 0.$$

The equations (4.4.3) and (4.4.9) imply

$$(4.4.10)\qquad L_{cdw} A_e{}^d + L_{cex} H_w{}^x = 0.$$

From equations (4.4.9) and (4.4.10), we get

$$(4.4.11)\qquad L_{edw} A_c{}^e - L_c{}^e{}_w A_{ed} + L_{cdx} H_w{}^x - L_c{}^e{}_w A_{ed} = 0.$$

Hence we get $L_c{}^e{}_w A_{ed} = 0$, which implies $L = 0$. By contracting indices y and w in the equation (4.4.7), we obtain $h^b = h_{uu}{}^b = 0$. Now each fibre F is a minimal submanifold of M. Thus we can state

Theorem 4.3. *Let $\{M, B, \tilde{g}, \pi\}$ be a fibred almost Hermitian space with almost complex structure J and invariant fibres. If M is nearly Kaehlerian, then B and F are nearly Kaehlerian, $L = 0$ and F is a minimal submanifold of M.*

The Nijenhuis tensor defined by equation (3.1) can be rewritten as

$$\tilde{N}_{ji}{}^h = J_j{}^k(\partial_k J_i{}^h - \partial_i J_k{}^h) - J_i{}^k(\partial_k J_j{}^h - \partial_j J_k{}^h). \tag{4.5}$$

Using the equation (4.1.1)-(4.1.4), and (4.5), we get

$$\tilde{N}_{cb}{}^a = J_c{}^k(\partial_k J_b{}^a - \partial_b J_k{}^a) - J_b{}^k(\partial_k J_c{}^a - \partial_c J_k{}^a) = N_{cb}{}^a, \tag{4.5.1}$$

$$\begin{aligned}\tilde{N}_{cx}{}^y &= J_c{}^k(\partial_k J_x{}^y - \partial_x J_k{}^y) - J_x{}^k(\partial_k J_c{}^y - \partial_c J_k{}^y)\\ &= A_c{}^d(\partial_d H_x{}^y) + H_x{}^z(\partial_c H_z{}^y),\end{aligned} \tag{4.5.2}$$

$$\tilde{N}_{xy}{}^z = J_x{}^k(\partial_k J_y{}^z - \partial_y J_k{}^z) - J_y{}^k(\partial_k J_x{}^z - \partial_x J_k{}^z) = \tilde{N}_{xy}{}^z, \tag{4.5.3}$$

and all other components vanish. An almost complex structure J is integrable if and only if Nigenhuis tensor of J vanish. From equations (4.5.1)-(4.5.3), we get the following.

Theorem 4.4. *An almost complex structure J on the fibred Riemannian space $\{M, B, \tilde{g}, \pi\}$ is integrable if and only if the almost complex structures A on B and H on F are integrable and*

$$A_c{}^d(\partial_d H_x{}^y) + H_x{}^z(\partial_c H_z{}^y) = 0. \tag{4.6}$$

If the total space is Kaehlerian, then $\tilde{\nabla} J = 0$. Since a Kaehler manifold is nearly Kaehlerian, we see that $L = 0$ and F is a minimal submanifold of M from Theorem 4.3. Moreover, the equations (4.2.1)-(4.2.8) and $\tilde{\nabla} J = 0$ imply that B and F are Kaehlerian. Thus we can state

Theorem 4.5. *If a fibred almost Hermitian space $\{M, B, \tilde{g}, \pi\}$ with almost complex structure J and invariant fibres is Kaehlerian, then B and F are Kaehlerian, $L = 0$ and each fibre F is a minimal submanifold of M and the equation* (4.6) *holds.*

Let M be a Kaehler Einstein manifold. Then

$$\tilde{K}_{ji} = \frac{\tilde{K}}{m} g_{ji} \tag{4.7}$$

and $\tilde{K}$ is a constant if $m > 2$.

Using equations (2.26)-(2.33), and (4.7), we have

$$\tilde{K}_{cb} = K_{cb} - h_y{}^x{}_c h_x{}^y{}_b, \tag{4.8.1}$$

$$^{**}\nabla_x h_y{}^x{}_b = 0, \tag{4.8.2}$$

$$\bar{K}_{yx} = \frac{\tilde{K}}{m}\bar{g}_{yx} -{}^{*} \nabla_e h_{yx}{}^e, \tag{4.8.3}$$

$$\tilde{K} = K + \bar{K} - \|h\|^2. \tag{4.8.4}$$

The equations (4.8.1), (4.8.3), and (4.8.4) imply

$$rK - nK - r\|h\|^2 = 0. \tag{4.8.5}$$

These equations lead to the following theorem.

Theorem 4.6. *If a fibred Kaehler space is Einstein, then its base space B is Einstein if and only if* $h_z{}^w{}_c h_w{}^z{}_b = \lambda g_{cb}$ *for some* λ.

Next, if each fibre is Einstein, then equation (4.8.3) implies

$$\frac{\bar{K}}{r}\bar{g}_{yx} = \frac{\tilde{K}}{m}\bar{g}_{yx} -{}^* \nabla_e h_{yx}{}^e, \tag{4.8.6}$$

which, together with the equation (4.8.5), implies

$$^*\nabla_e h_{yx}{}^e = 0. \tag{4.8.7}$$

The converse is evident. Hence we have

Theorem 4.7. *If a fibred Kaehler space is Einstein, then the fibre F is Einstein if and only if* $^*\nabla_e h_{yx}{}^e = 0$.

It is well known that in an almost Hermitian manifold the vector fields X and JX are orthogonal. It follows that, at a point p, the vectors X and JX determine a 2-dimensional subspace and hence a sectional curvature defined by this subspace. This is called the *holomorphic sectional curvature* at p in the direction X. If X is a unit vector, the holomorphic sectional curvature is given by $R(X, JX, X, JX)$. It may happen that at the point p the holomorphic sectional curvature is independent of the vector X. In that case, we say that the manifold has constant holomorphic sectional curvature at p. Indeed, we have a result which is analogous to Schur's theorem in the case of real Riemannian manifolds, namely, if the holomorphic sectional curvature is constant at each point p of a Kaehler manifold, then it has the same constant value over the whole manifold. Such a manifold is said to have a constant holomorphic sectional curvature or it is said to be a complex space form.

The following theorem is well known [19, 20].

Theorem 4.8. *A Kaehler manifold M has a constant holomorphic sectional curvature c if and only if*

$$\tilde{K}_{kji}{}^h = \tfrac{c}{4}[\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki} + J_k{}^h J_{ji} - J_j{}^h J_{ki} - 2J_{kj}J_i{}^h]. \tag{4.9}$$

Here non-trivial components of $\tilde{K}_{kji}{}^h$ *are*

$$K_{dcb}{}^a = \tfrac{c}{4}[\delta_d{}^a g_{cb} - \delta_c{}^a g_{db} - A_c{}^a A_{db} + A_d{}^a A_{cb} - 2A_{dc}A_b{}^a], \tag{4.10.1}$$

$$\tilde{K}_{dcy}{}^x = -\tfrac{c}{2}A_{dc}H_y{}^x, \tag{4.10.2}$$

$$\tilde{K}_{dyb}{}^x = \tfrac{c}{4}[-\delta_y{}^x g_{db} - H_y{}^x A_{db}], \tag{4.10.3}$$

$$\tilde{K}_{zyb}{}^a = \tfrac{c}{4}[-H_{zy}A_b{}^a], \tag{4.10.4}$$

$$\tilde{K}_{zyx}{}^a = 0, \tag{4.10.5}$$

$$\tilde{K}_{zyx}{}^w = \tfrac{c}{4}[\delta_z{}^w g_{yx} - \delta_y{}^w g_{zx} + H_z{}^w H_{yx} - H_y{}^w H_{zx} - 2H_{zy}H_x{}^w]. \tag{4.10.6}$$

Since M is Kaehlerian, $L = 0$ and F is a minimal submanifold of M. Using these facts and the equations (2.10)-(2.17), we get

$$\frac{c}{4}[\delta_k{}^h g_{ji} - \delta_j{}^h g_{ki} + J_k{}^h J_{ji} - J_j{}^h J_{ki} - 2J_{kj}J_i{}^h] = K_{dcb}{}^a, \tag{4.11.1}$$

$$-\frac{c}{2}A_{dc}H_y{}^x =^* \nabla_c h_y{}^x{}_d -^* \nabla_d h_y{}^x{}_c - h_w{}^x{}_d h_y{}^w{}_c + h_w{}^x{}_c h_y{}^w{}_d, \tag{4.11.2}$$

$$\frac{c}{4}[-\delta_y{}^x g_{db} - H_y{}^x A_{db}] =^* \nabla_d h_y{}^x{}_b + h_y{}^w{}_d h_w{}^x{}_b, \tag{4.11.3}$$

$$\frac{c}{4}[-H_{zy}A_b{}^a] = h_z{}^w{}_b h_{yw}{}^a - h_y{}^w{}_b h_{zw}{}^a, \tag{4.11.4}$$

$$0 =^* \nabla_z h_{yx}{}^a -^{**} \nabla_y h_{zx}{}^a, \tag{4.11.5}$$

$$\begin{aligned}&\frac{c}{4}[\delta_z{}^w g_{yx} - \delta_y{}^w g_{zx} + H_z{}^w H_{yx} - H_y{}^w H_{zx} - 2H_{zy}H_x{}^w]\\ &= \tilde{K}_{zyx}{}^w + h_{zx}{}^e h_y{}^w{}_e - h_{yx}{}^e h_z{}^w{}_e.\end{aligned} \tag{4.11.6}$$

Contracting the indices z and x in the equation (4.11.6), we obtain

$$\bar{K}_{xy} + h_{zy}{}^e h_x{}^z{}_e = \frac{c}{4}(r+1)\bar{g}_{xy} \tag{4.12}$$

using the minimality of the fibred Kaehler space. By taking the skew symmetric part of the equation (4.12), we get

$$h_{zy}{}^e h_x{}^z{}_e - h_{zx}{}^e h_y{}^z{}_e = 0. \tag{4.13}$$

The equations (4.10.4) and (4.13) imply $cH_{xy}A_b{}^a = 0$, i.e., $c = 0$. Hence we have

Theorem 4.9. *If a fibred Kaehler space $\{M, B, \tilde{g}, \pi\}$ with invariant fibre has a constant holomorphic sectional curvature, then M is locally Euclidean.*

5. Fibred Kaehlerian space of another type

In this chapter, we consider a fibred Riemannian space M such that the base space B and each fibre F are almost contact spaces with almost contact structures $(\phi_b{}^a, \eta_b, \xi^a)$ and $(\bar{\phi}_x{}^y, \bar{\eta}_x, \bar{\xi}^y)$ respectively. The structure (ϕ, η, ξ) satisfies $\phi^2 = -I + \eta\otimes\xi$, $\phi(\xi) = 0$, $\eta\otimes\phi = 0$, $\eta(\xi) = 1$, where I is the identity map. If we define

$$J_j{}^i = \phi_b{}^a E_j{}^b E^i{}_a - \eta_b \bar{\xi}^y E_j{}^b C^i{}_y + \bar{\eta}_x \xi^a C_j{}^x E^i{}_a + \bar{\phi}_x{}^y C_j{}^x C^i{}_y, \tag{5.1}$$

then we can easily see that $J^2 = -I$ and that we can construct an almost Hermitian structure with almost complex structure J on the total space M, which will be called fibred a almost Hermitian space. Thus we can state

Proposition 5.1. *Let B and F be almost contact metric spaces. Then the fibred Riemannian space M admits an almost Hermitian structure.*

The equation (5.1) is rewritten as

$$J_j{}^i E^j{}_d = \phi_d{}^a E^i{}_a - \eta_d \bar{\xi}^y C^i{}_y, \tag{5.2}$$

$$J_j{}^i C^j{}_z = \bar{\eta}_z \xi^a E^i{}_a + \bar{\phi}_z{}^y C^i{}_y. \tag{5.3}$$

Components of the covariant derivative $\tilde{\nabla} J$ with respect to the frame $(E_B) = (E_b, C_y)$ are given by $(\tilde{\nabla}_j J_{ih}) E^j{}_C E^i{}_B E^h{}_A$ and we can obtain the following expressions by means of (2.6), (5.2), and (5.3).

$$(\tilde{\nabla}_k J_{ji}) E^k{}_c E^j{}_d E^i{}_e = \nabla_c \phi_{de} + L_{cd}{}^x \bar{\eta}_x \eta_e - L_{cey} \eta_d \bar{\xi}^y, \tag{5.4.1}$$

$$(\tilde{\nabla}_k J_{ji}) E^k{}_c E^j{}_d C^i{}_z = -(\nabla_c \eta_d)\bar{\eta}_z - \eta_d ({}^*\nabla_c \bar{\eta}_z) + L_{cd}{}^x \bar{\phi}_{xz} - L_{caz}\phi_d{}^a, \tag{5.4.2}$$

$$(\tilde{\nabla}_k J_{ji}) E^k{}_c C^j{}_z E^i{}_d = ({}^*\nabla_c \bar{\eta}_z)\eta_d + (\nabla_c \eta_d)\bar{\eta}_z + L_{cdy}\bar{\phi}_z{}^y - L_c{}^a{}_z \phi_{ad}, \tag{5.4.3}$$

$$(\tilde{\nabla}_k J_{ji}) E^k{}_c C^j{}_z C^i{}_w = {}^* \nabla_c \bar{\phi}_{zw} - L_{caw}\bar{\eta}_z \xi^a + L_c{}^a{}_z \eta_a \bar{\eta}_w + (\nabla_z \bar{\eta}_w)\eta_c, \tag{5.4.4}$$

$$(\tilde{\nabla}_k J_{ji}) C^k{}_z E_j{}^d E^i{}_e = {}^{**} \nabla_z \phi_{de} - \eta_d \bar{\xi}^y h_{zye} + h_z{}^y{}_d \bar{\eta}_y \eta_e, \tag{5.4.5}$$

$$\begin{aligned} &(\tilde{\nabla}_k J_{ji}) C^k{}_z E^j{}_d C^i{}_w \\ &= -({}^{**}\nabla_z \eta_d)\bar{\eta}_w - \eta_d \bar{\nabla}_z \bar{\eta}_w + h_z{}^y{}_d \bar{\phi}_{yw} - h_{zwa}\phi_d{}^a, \end{aligned} \tag{5.4.6}$$

$$\begin{aligned} &(\tilde{\nabla}_k J_{ji}) C^k{}_z C^j{}_x E^i{}_c \\ &= (\bar{\nabla}_z \bar{\eta}_x)\eta_c + \bar{\eta}_x ({}^{**}\nabla_z \eta_c) + h_{zyc}\bar{\phi}_x{}^y - h_{zx}{}^b \phi_{bc}, \end{aligned} \tag{5.4.7}$$

$$(\tilde{\nabla}_k J_{ji}) C^k{}_z C^j{}_x C^i{}_y = \bar{\nabla}_z \bar{\phi}_{xy} + h_{zx}{}^b \eta_b \bar{\eta}_y - h_{zyb}\xi^b \bar{\eta}_x. \tag{5.4.8}$$

Suppose that the induced structure J on M is Kaehlerian, i.e., $\tilde{\nabla}_k J_{ji} = 0$. Then we get

$$\nabla_c \phi_{de} + L_{cd}{}^x \bar{\eta}_x \eta_e - L_{ce}{}^x \bar{\eta}_x \eta_d = 0, \tag{5.5.1}$$

$$(\nabla_c \eta_d)\bar{\eta}_z + ({}^*\nabla_c \bar{\eta}_z)\eta_d - L_{cd}{}^x \bar{\phi}_{xz} + L_{caz}\phi_d{}^a = 0, \tag{5.5.2}$$

$$({}^{**}\nabla_z \bar{\eta}_w)\eta_d + ({}^{**}\nabla_z \eta_d)\bar{\eta}_w + h_{zwa}\phi_d{}^a - h_z{}^y{}_d \bar{\phi}_{yw} = 0, \tag{5.5.3}$$

$${}^{**}\nabla_z \bar{\phi}_{xy} + h_{zx}{}^b \eta_b \bar{\eta}_y - h_{zy}{}^b \eta_b \bar{\eta}_x = 0. \tag{5.5.4}$$

If the base space B is a contact manifold, i.e., $2\phi_{cb} = \nabla_c \eta_b - \nabla_b \eta_c$, then transvecting (5.5.2) with $\bar{\eta}^z$, we have

$$2\phi_{cd} = (L_{daz}\phi_c{}^a - L_{caz}\phi_d{}^a)\bar{\eta}^z. \tag{5.6}$$

Transvecting (5.6) with $\phi_e{}^c$, we obtain

$$2(-g_{de} + \eta_d \eta_e) = (-L_{caz}\phi_d{}^a \phi_e{}^c - L_{dez})\bar{\eta}^z + L_{da}{}^z \eta^a \eta_e.$$

From this equation, we can see that $n = 1$. Therefore we have

Theorem 5.2. *Let M be a fibred Kaehlerian manifold and the base space B a contact manifold. Then the dimension of B should be one.*

If we assume that M is a Kaehlerian space with conformal fibres, i.e., $h_{xy}{}^a = \bar{g}_{xy}h^a$, then transvecting (5.5.3) with η^d,

$$(5.7) \qquad {}^{**}\nabla_z \bar{\eta}_x + (h_d \eta^d)\bar{\phi}_{xz} = 0.$$

From the equations (5.5.4) and (5.7) we have,

Theorem 5.3. *Let M be a fibred Kaehlerian manifold with conformal fibres. Then the manifold F is cosymplectic if and only if $h_d\eta^d = 0$.*

If we assume that M is a Kaehlerian space, then transvecting (5.5.2) with $\bar{\eta}^z$ gives

$$(5.8) \qquad \nabla_c \eta_d + L_{ca}{}^z \bar{\eta}_z \phi_d{}^a = 0.$$

Using (5.5.1) and (5.8), we get

Theorem 5.4. *Let M be a fibred Kaehlerian manifold. Then the manifold B is cosymplectic if and only if $L \bigotimes \bar{\eta} = 0$.*

If we assume that the total space M is a Kaehlerian manifold of constant holomorphic sectional curvature c, then the equations (2.10)-(2.17) turn to

$$(5.9.1) \qquad \begin{aligned} K_{dcb}{}^a = {} & \frac{c}{4}[\delta_d{}^a g_{cb} - \delta_c{}^a g_{db} + \phi_{cb}\phi_d{}^a - \phi_{db}\phi_c{}^a - 2\phi_{dc}\phi_b{}^a) \\ & + L_d{}^a{}_\epsilon L_{cb}{}^\epsilon - L_c{}^a{}_\epsilon L_{db}{}^\epsilon - 2L_{dc}{}^\epsilon L_b{}^a{}_\epsilon, \end{aligned}$$

$$(5.9.2) \qquad \begin{aligned} & {}^*\nabla_d L_b{}^a{}_z - L_d{}^a{}_\epsilon h_{z}{}^\epsilon{}_b + L_{db}{}^\epsilon h_{z\epsilon}{}^a - L_b{}^a{}_\epsilon h_{z}{}^\epsilon{}_d \\ & = \frac{c}{4}(\phi_d{}^a \bar{\eta}_z \xi_b + \phi_{db}\bar{\eta}_z \xi^a + 2\phi_b{}^a \eta_d \bar{\xi}_z), \end{aligned}$$

$$(5.9.3) \qquad L_{wzb}{}^a = -\tfrac{c}{2}(\bar{\phi}_{wz}\phi_b{}^a + h_z{}^\epsilon{}_b h_{w\epsilon}{}^a - h_w{}^\epsilon{}_b h_{z\epsilon}{}^a),$$

$$(5.9.4) \qquad {}^{**}\nabla_w h_{zy}{}^a - {}^{**}\nabla_z h_{wy}{}^a = \tfrac{c}{4}(\bar{\eta}_w \bar{\phi}_{yz} - \bar{\eta}_z \bar{\phi}_{yw} - 2\bar{\phi}_{zw}\bar{\eta}_y)\xi^a,$$

$$(5.9.5) \qquad \begin{aligned} \bar{K}_{wzy}{}^x = {} & \frac{c}{4}(\delta_w{}^x \bar{g}_{zy} - \delta_z{}^x \bar{g}_{wy} + \bar{\phi}_{zy}\bar{\phi}_w{}^x - \bar{\phi}_{wy}\bar{\phi}_z{}^x - 2\bar{\phi}_{wz}\bar{\phi}_y{}^x) \\ & - h_{wy}{}^e h_z{}^x{}_e + h_{zy}{}^e h_w{}^x{}_e \end{aligned}$$

by means of (4.9). If M have conformal fibres, then the equation (5.9.4) is reduced to

$$(5.10) \qquad \bar{g}_{zy}({}^{**}\nabla_w h^a) - \bar{g}_{wy}({}^{**}\nabla_z h^a) = \frac{c}{4}(\bar{\eta}_w \bar{\phi}_{yz} - \bar{\eta}_z \bar{\phi}_{yw} - 2\bar{\phi}_{zw}\bar{\eta}_y)\xi^a.$$

Transvecting $\bar{g}^{zy}$ to (5.10), it is reduced to

$$(5.11) \qquad (p-1)^{**}\nabla_w h^a = 0.$$

Hence the mean curvature vector is parallel with respect to the normal connection along each fibre if $p > 1$. Moreover, we get $c = 0$ from (5.10) and

(5.11), that is, M is locally Euclidean. Hence the equations (5.9.1), (5.9.2), (5.9.3), and (5.9.5) are reduced to

$$K_{dcb}{}^{a} = L_{d}{}^{a}{}_{x} L_{cb}{}^{x} - L_{c}{}^{a}{}_{x} L_{db}{}^{x} - 2L_{dc}{}^{x} L_{b}{}^{a}{}_{x}, \tag{5.12.1}$$

$$^{*}\nabla_{d} L_{ba}{}^{z} = L_{ba}{}^{z} h_{b} - L_{db}{}^{z} h_{a} + L_{ba}{}^{z} h_{d}, \tag{5.12.2}$$

$$L_{wzb}{}^{a} = 0, \tag{5.12.3}$$

$$\bar{K}_{wzy}{}^{x} = \| h_{e} \|^{2} (\bar{g}_{zy} \delta_{w}{}^{x} - \bar{g}_{wy} \delta_{z}{}^{x}). \tag{5.12.4}$$

Thus we have

Theorem 5.5. *Let M be a fibred Kaehlerian manifold with conformal fibres where $p > 1$. If the induced almost complex structure J defined by* (5.1) *is Kaehlerian one of constant holomorphic sectional curvature, then we have*

(a) *M is locally Euclidean,*
(b) *the base space B has the curvature tensor of the form* (5.12.1),
(c) *each fibre is a space of constant curvature,*
(d) *the mean curvature vector is parallel with respect to the normal connection along each fibre,*
(e) *the scalar curvature of base space is equal to $K = 3\|L_{cb}{}^{x}\|^{2}$,*
(f) *the scalar curvature of each fibre is equal to $p(p-1)\|\|h_{e}\|\|^{2}$.*

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