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A NOTE ON REAL QUATERNION

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Abstract. We consider pm-ring with the property such that every prime ideal is contained in only one maximal ideal. Orsatti[4] characterized pm-rings by means of the retraction. Contessa[1] found algebraic condition, by using that direct product of pm-rings is a pm-ring. We show that $C(X, H)$ and $C(X, \mathbb{C})$ are pm-rings and we extend a quasi pm-domain.

1. Introduction

We consider the ring with the property such that every arbitrary prime ideal is contained in only one maximal ideal. Orsatti[4] characterized pm-rings by means of the retraction. We define a quasi pm-ring as the ring whose every nonzero prime ideal is contained in only one maximal ideal[2]. Thus a quasi pm-domain is a quasi pm-ring which is a domain. An interesting example of a pm domain is the entire functions ring. We show that $C(X, H)$, which is the ring of continuous functions from a topological space X to the ring H of real quaternion which is $\mathbb{R}^4$ as a metric space, is a pm-ring. We denote by $Spec(\mathbb{R})$ the set of all the prime ideals of $\mathbb{R}$ with the Zarisky topology and $Max\mathbb{R}$ is the subspace of $Spec\mathbb{R}$[1]. In this paper all rings are commutative rings with identity, unless stated.

2. Results

Let $\mathbb{R}$ be an integral domain with quotient field K. H is real quaternion. Recall that a proper ideal P of a ring $\mathbb{R}$ is called prime if $a\mathbb{R}b \subset P$ implies that either $a \in P$ or $b \in P$; and P is called completely prime if $ab \in P$ implies that either $a \in P$ or $b \in P$. The two notions coincide for

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commutative rings. F is an inclusion, that is, $F : C(X,\mathbb{R}) \Rightarrow C(X,H)$. $F^*(Q) = F^{-1}(Q) = P$. $P = Q \cap C(X,\mathbb{R})$ [5.p13.ex, 21].

FACT 1. Whenever the maximal ideal space $Max(\mathbb{R})$ is a retract of the prime spectrum $Spec(\mathbb{R})$, $\mathbb{R}$ is a pm-ring[3, p182. Prop. 1.5].

THEOREM 2.1. *Let $\mathbb{R}$ be an integral domain with the quotient field K such that $\mathbb{R} \neq K$. Then $\mathbb{R}$ is a quasi pm-domain if and only if for any $M \neq M' \in Max(\mathbb{R})$ and for the set $S = (\mathbb{R} - M)(\mathbb{R} - M')$, we have $S^{-1}\mathbb{R} \cong K$.*

Proof. Let $\mathbb{R}$ be a quasi pm-domain. Since $\mathbb{R}$ is a domain, $0 \notin S$. Suppose that there exists nonzero prime ideal P of $\mathbb{R}$ such that $S \cap P = \phi$. Thus $P \subset \mathbb{R} - S$. Since $\mathbb{R} - M$ and $\mathbb{R} - M'$ contain 1, $\mathbb{R} - S \subset M \cap M'$ and hence $P \subset M \cap M'$. It contradicts the fact that $\mathbb{R}$ is a quasi pm domain. So $S^{-1}\mathbb{R}$ has no nonzero prime ideal and hence $S^{-1}\mathbb{R}$ is a field containing $\mathbb{R}$. Thus it is isomorphic to K.

For the converse, suppose that $S^{-1}\mathbb{R}$ is isomorphic to K. If $\mathbb{R}$ has a nonzero prime ideal P, which is contained in two distinct maximal ideals M, M' of $\mathbb{R}$. Since $S^{-1}R$ is a field, $S \cap P \neq \phi$. Thus there exist $a \in \mathbb{R} - M$ and $a' \in \mathbb{R} - M'$ such that $aa' \in P$. Hence we have that $a \in P$ or $a' \in P$. Then $a \in M$ or $a' \in M'$. It is a contradiction. Thus $\mathbb{R}$ is a quasi pm-domain. □

We will prove that $F^* : Spec(C(X,H)) \Rightarrow Spec(C(X,\mathbb{R}))$ is a bijection.

$f : A \Rightarrow B$. (A a commutative ring B a noncommutative ring.) $f^* : Spec(B) \Rightarrow Spec(A)$ is not into, that is, $f^{-1}(Q) \notin Spec(A)$ ($Q \in Spec(B)$). When $f(A) \neq B$, $ab \in P \Rightarrow a \in P$ $b \in P$, $f(a)f(b) \in Q \not\Rightarrow f(a) \in Q$, $f(b) \in Q$. And $ab \in P \Rightarrow arb \in P$ and $\{f(a)f(r)f(b)\} \neq B$ that is, $f(A) \neq B$. But if $P + Pi + Pj + Pk$ is a prime, P is a prime.

THEOREM 2.2. *Let $f : C(X,\mathbb{R}) \rightarrow C(X,H)$ be an inclusion. Then f^*: $Spec(C(X,H)) \Rightarrow Spec(C(X,\mathbb{R}))$ is a homeomorphism and thus $C(X,H)$ is a pm-ring.*

Proof. For each $Q \in Spec(C(X,H))$, since $C(X,\mathbb{R})$ is contained in the center of $C(X,H)$, $f^*(Q) = f^{-1}(Q) \in Spec(C(X,\mathbb{R}))$. For each ideal I of $C(X,\mathbb{R})$, $I = f^{-1}(I + Ii + Ij + Ik)$ and $I + Ii + Ij + Ik$ is an ideal of $C(X,H)$. For each $P \in Spec(C(X,\mathbb{R}))$, $P^e = P + Pi + Pj + Pk$ is a completely prime ideal of a non commutative ring $C(X,H)$. Let us prove this. Let $fg \in P^e$, $f = f_1 + f_2 i + f_3 j + f_4 k$,

$g = g_1 + g_2 i + g_3 j + g_4 k$. Then $|f|^2|g|^2 \in P^e \cap C(X,\mathbb{R}) = P$. $|f|^2 \in P$ or $|g|^2 \in P$. We define F_i such that $f_i^3 = F_i\ |f|^2$ and $f_i^3 = |f|^2 \frac{f_i^3}{|f|^2} = |f|^2$ F_i. $F_i = \frac{f_i(x)^3}{|f(x)|^2}$, $f_i(x) \neq 0$. $F_i = 0$, $f_i(x) = 0$. Then $|F_i| \leqq |f_i|$, and hence $F_i \in C(X,\mathbb{R})$. Therefore $\{\ f_i^3\ |i = 1,2,3,4\ \} \subset P$ or $\{\ g_i^3\ |i = 1,2,3,4\ \}$ $\subset P$. Hence $\{\ f_i\ |i = 1,2,3,4\ \} \subset P$ or $\{\ g_i\ |i = 1,2,3,4\ \} \subset P$. Then $f \in P^e$ or $g \in P^e$. Since a completely prime ideal is a prime ideal and $f^*(P^e) = P$, f^* is a surjection.

For an ideal J of $C(X,H)$, there exist ideals $I_i(i = 1,2,3,4)$ of $C(X,\mathbb{R})$ such that $J = I_1 + I_2 i + I_3 j + I_4 k$. Let us prove this. Let J be an arbitrary ideal of $C(X,H)$. Let $f \in J$. Then there exist $f_i \in C(X,\mathbb{R})$ such that $f = f_1+f_2i+f_3j+f_4k$. Then $if = if_1 - f_2 i + f_3 k - f_4 j$ $\in J$. $fi = if_1 - f_2 - f_3 k + f_4 j \in J$. Therefore $\frac{fi+if}{2} = if_1$ - $f_2 \in J$. Thus $f_1 + f_2 i \in J$ and $(f_1+f_2i)j = f_1 j + f_2 k \in J$. $j(f_1+f_2i) = f_1 j - f_2 k \in J$. $\therefore$ $f_1 j \in J$. Thus $f_1 \in J$. Similarly $f_i \in J$. Thus $J = I_1 + I_2 i + I_3 j + I_4 k$ (I_i $= \{\ f_i|\ f_1+f_2i+f_3j+f_4k \in J\ \}$). And we know $I_i = I_j$ $(i,j = 1,2,3,4)$. Thus $J = I + Ii + Ij + Ik = I^e$. That is, every ideal of $C(X,H)$ is an extended ideal. Thus we conclude that f^* is an injection.

Since $A \subset B$ implies $f^*(A) \subset f^*(B)$, $Max(C(X,\mathbb{R})) = f^*(Max(C(X$ $,H)))$. Retraction γ_1 : $Spec(C(X,\mathbb{R})) \to Max(C(X,\mathbb{R}))$ exists, because $C(X,\mathbb{R})$ is a pm-ring[2.14, 4]. Thus we can define retraction γ_2 : $Spec(C(X,H)) \to Max(C(X,H))$ such that $\gamma_2 = f^{*-1} \circ \gamma_1 \circ f^*$. That is, $C(X,H)$ is a pm-ring by fact 1 (or [Proposition 1.5, 3]). □

Theorem 2.3. *Let $g : C(X,\mathbb{R}) \to C(X,\mathbb{C})$ be an inclusion. Then g^*: $Spec(C(X,\mathbb{C})) \Rightarrow Spec(C(X,\mathbb{R}))$ is a homeomorphism and thus $C(X,\mathbb{C})$ is a pm-ring.*

Proof. The above theorem says that $Spec(C(X,H)) = \{P^e|\ P \in$ $Spec(C(X,\mathbb{R})\ \}$. Let $Q \in Spec(C(X,\mathbb{C}))$. And let us denote by $P^{e'}$ an extended ideal of P from $C(X,\mathbb{R})$ to $C(X,\mathbb{C})$. Let $P = Q^c$ and $P^{e'}$ $= P + Pi \subset Q$. We want to prove that $Q = P + Pi = P^{e'}$. Assume Q - $(P + Pi) \neq \emptyset$. Then there exists $f \in Q$ - $(P + Pi)$. Let $f = f_1 + f_2 i$. $\Rightarrow (f - f_1)^2 = (f_2 i)^2 = -f_2^2$. Then there exist $\alpha, \beta \in C(X,R)$ such that $f^2 + \alpha f + \beta = 0$. Since $\beta = -f^2 - \alpha\ f \in Q \bigcap C(X,\mathbb{R}) = P \subset P + Pi$. Hence $P^{e'} \supset Q \bigcap C(X,\mathbb{R}) \ni \beta$. Thus $f^2 + \alpha f = f(f+\alpha) \in P^{e'}$. Then $f + \alpha \in P^{e'} \subset Q$ since $f \notin P^{e'}$ and $P^{e'} = P + Pi$ prime ideal, because $P^{e'} = P^{ec}$ from $C(X,\mathbb{R})$ to $C(X,H)$. Thus $\alpha \in Q$ since $f \in Q$. Hence $\alpha \in Q \bigcap C(X,\mathbb{R}) = P$. Then $\alpha \in P \subset P^{e'}$. Thus $f \in P + Pi = P^{e'}$

. It contradicts the fact $f \in Q$ - $(P + Pi)$. Thus we proved $Q = P + Pi$, and thus $Spec(C(X, \mathbb{C})) = \{P^{e'} | \ P \in Spec(C(X, \mathbb{R}))\ \}$. That is, g^* is bijective, when g is an inclusion map from $C(X, \mathbb{R})$ to $C(X, \mathbb{C})$. The domain of g^*, $Spec(C(X, \mathbb{C}))$ is a compact space and $Spec(C(X, \mathbb{R}))$ is a Hausdorff space. Hence continuous bijective map g^* is a homeomorphism. Thus $Spec(C(X, \mathbb{C})) \cong Spec(C(X, \mathbb{R}))$. And we conclude that $C(X, \mathbb{C})$ is a pm-ring. □

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