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CRITICAL POINT THEORY AND AN ASYMMETRIC BEAM EQUATION WITH TWO JUMPING NONLINEAR TERMS

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Abstract. We investigate the multiple nontrivial solutions of the asymmetric beam equation $u_{tt} + u_{xxxx} = b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3]$ with Dirichlet boundary condition and periodic condition on t. We reduce this problem into a two-dimensional problem by using variational reduction method and apply the Mountain Pass theorem to find the nontrivial solutions of the equation.

1. Introduction

In ([3], [6]) the authors investigated the multiplicity of solutions a nonlinear suspension bridge equation in an interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\begin{aligned}
&u_{tt} + u_{xxxx} + bu^+ = f(x) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\
&u(\pm\frac{\pi}{2}, t) = u_{xx}(\pm\frac{\pi}{2}, t) = 0, \qquad (1.1)\\
&u \text{ is } \pi - \text{periodic in } t \text{ and even in } x
\end{aligned}$$

where the nonlinearity $-(bu^+)$ crosses an eigenvalues. This equation represents a bending beam supported by cables under a load f. The constant b represents the restoring force if the cables stretch. The nonlinearity u^+ models the fact that cables resist expansion but do not resist compression.

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In this paper we investigate the existence of nontrivial solutions $u(x,t)$ for a perturbation $g(u) = b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3]$ of the asymmetric beam equation

$$\begin{aligned} u_{tt} + u_{xxxx} &= g(u) \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ u(\pm\frac{\pi}{2}, t) &= u_{xx}(\pm\frac{\pi}{2}, t) = 0, \\ u(x, t+\pi) &= u(x,t) = u(-x,t), \end{aligned} \tag{1.2}$$

where $u^+ = \max\{u, 0\}$, b_1, b_2 are constants. This equation satisfies Dirichlet boundary condition on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ and periodic condition on the variable t.

In [6] Lazer and McKenna point out that this kind of nonlinearity $b[(u+1)^+ - 1]$ can furnish a model to study traveling waves in suspension bridges. So the nonlinear equation with jumping nonlinearity have been extensively studied by many authors. For fourth order elliptic equation Tarantello [11], Micheletti and Pistoia [8][9] proved the existence of nontrivial solutions used degree theory and critical points theory separately. For one-dimensional case Lazer and McKenna [7] proved the existence of nontrivial solution by the global bifurcation method. For this jumping nonlinearity we are interest in the multiple nontrivial solutions of the equation. Here we used variational reduction method to find the nontrivial solutions of problem (1.1).

The organization of this paper is as following. In section 2, we investigate some properties of the Hilbert space spanned by eigenfunctions of the beam operator. We show that only the trivial solution exists for problem (1.2) when $-3 < b_1, b_2 < 1$ and $-3 < b_1 + b_2 < 1$. In section 3 we state the Mountain Pass Theorem. In section 4 we use the variational reduction method to apply mountain pass theorem in order to get the main result that (1.2) has at least three periodic solutions for $-15 < b_1, b_2 < -3$, $-15 < b_1 + b_2 < -3$ and two of them are nontrivial.

2. Preliminaries

Let L be the differential operator, $Lu = u_{tt} + u_{xxxx}$. Then the eigenvalue problem

$$\begin{aligned} &Lu = \lambda u \quad \text{in} \quad (-\frac{\pi}{2}, \frac{\pi}{2}) \times \mathbb{R}, \\ &u(\pm\frac{\pi}{2}, t) = 0, \quad u(x, t+\pi) = u(x,t) = u(-x,t) \end{aligned} \tag{2.1}$$

has infinitely many eigenvalues $\lambda_{mn} = (2n+1)^4 - 4m^2$ $(m, n = 0, 1, 2, ...)$ and corresponding normalized eigenfunctions $\phi_{mn}, \psi_{mn} (m, n \geq 0)$ given by

$$\begin{aligned} \phi_{0n} &= \frac{\sqrt{2}}{\pi} \cos{(2n+1)x} \quad \text{for} \quad n \geq 0, \\ \phi_{mn} &= \frac{2}{\pi} \cos 2mt \cdot \cos{(2n+1)x} \quad \text{for} \quad m > 0, \quad n \geq 0, \\ \psi_{mn} &= \frac{2}{\pi} \sin 2mt \cdot \cos{(2n+1)x} \quad \text{for} \quad m > 0, \quad n \geq 0. \end{aligned}$$

We note that all eigenvalues in the interval $(-19, 45)$ are given by

$$\lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \lambda_{41} = 17.$$

Let Ω be the square $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and H_0 the Hilbert space defined by

$$H_0 = \{u \in L^2(\Omega) : u \text{ is even in } \quad x\}.$$

Then the set of functions $\{\phi_{mn}, \psi_{mn}\}$ is an orthonormal basis in H_0. Let us denote an element u in H_0 as

$$u = \sum (h_{mn}\phi_{mn} + k_{mn}\psi_{mn}),$$

and we define a subspace H of H_0 as

$$H = \{u \in H_0 : \sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2) < \infty\}.$$

Then this is a complete normed space with a norm

$$\|u\|_H = [\sum |\lambda_{mn}|(h_{mn}^2 + k_{mn}^2)]^{\frac{1}{2}}.$$

Since $|\lambda_{mn}| \geq 1$ for all m, n, we have that

(i) $\|u\|_H \geq \|u\|$, where $\|u\|$ denotes the L^2 norm of u,

(ii)$\|u\| = 0$ if and only if $\|u\|_H = 0$.

Define $L_\beta u = Lu + \beta u$. Then we have the following lemma(cf. [4]).

LEMMA 2.1. *Let $\beta \in \mathbb{R}$, $\beta \neq -\lambda_{mn}$ $(m, n \geq 0)$. Then we have:*

$$L_\beta^{-1} \quad \text{is a bounded linear operator from} \quad H_0 \text{ into } H.$$

THEOREM 2.1. *Let $-3 < b_1, b_2 < 1$ and $-3 < b_1 + b_2 < 1$. Then the equation, with Dirichlet boundary condition,*

$$Lu = b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3]$$

has only the trivial solution in H_0.

Proof. Since $\lambda_{10} = -3$ and $\lambda_{00} = 1$, let $\beta = -\frac{1}{2}(\lambda_{00} + \lambda_{10}) = -\frac{1}{2}(-3 + 1) = 1$. The equation is equivalent to

$$u = (L+\beta)^{-1}(b_1[(u+2)^+ - 2] + b_2[(u+3)^+ - 3] + \beta u). \tag{2.2}$$

By Lemma 2.1 $(L+\beta)^{-1}$ is a compact linear map from H_0 into H_0. Therefore its L^2 norm $\frac{1}{2}$. We note that

$$\begin{aligned}
&\|b_1[(u_1+2)^+ - (u_2+2)^+] + b_1[(u_1+3)^+ - (u_2+3)^+] \\
&+\beta(u_1 - u_2)\| \\
&\leq \max\{|b_1+\beta|, |b_2+\beta|, |b_1+b_2+\beta|, |\beta|\}\|u_1 - u_2\| \\
&< \frac{1}{2}(\lambda_{00} - \lambda_{10})\|u_1 - u_2\| \\
&= 2\|u_1 - u_2\|.
\end{aligned}$$

So the right hand side of (2.2) defines a Lipschitz mapping of H_0 into H_0 with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in H_0$. Since $u \equiv 0$ is a solution of equation (2.2), $u \equiv 0$ is the unique solution. □

3. Mountain pass theorem

The mountain pass theorem concerns itself with proving the existence of critical points of functional $I \in C^1(E, \mathbb{R})$ which satisfy the Palais-Smale(PS) condition, which occurs repeatedly in critical point theory.

DEFINITION 1. *We say that I satisfies the Palais-Smale condition if any sequence $\{u_m\} \subset E$ for which $I(u_m)$ is bounded and $I'(u_m) \to 0$ as $m \to \infty$ possesses a convergent sequence.*

The following deformation theorem is stated in [10].

Theorem 3.1. *Let E be a real Banach space and $I \in C^1(E, R)$. Suppose I satisfies Palais-Smale condition. Let N be a given neighborhood of the set K_c of the critical points of I at a given level c. Then there exists $\epsilon > 0$, as small as we want, and a deformation $\eta : [0,1] \times E \to E$ such that, denoting by A_b the set $\{x \in E : I(x) \le b\}$:*
(i) $\eta(0, x) = x \quad \forall x \in E,$
(ii) $\eta(t, x) = x \quad \forall x \in A_{c-2\epsilon} \cup (E \backslash A_{c+2c}), \forall t \in [0,1],$
(iii) $\eta(1, \cdot)(A_{c+\epsilon} \backslash N) \subset A_{c-\epsilon}.$

We state the Mountain Pass Theorem.

Theorem 3.2. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS) condition. suppose*
(I_1) there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \ge I(0) + \alpha$, and
(I_2) there is an $e \in E \backslash \bar{B}_\rho$ such that $I(e) \le I(0)$.
Then I possesses a critical value $c \ge \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} I(u),$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0, g(1) = e\}.$$

4. Critical point theory and multiple nontrivial solutions

We investigate the existence of multiple solutions of (1.2) when $-7 < b_1, b_2 < -3$ $-7 < b_1 + b_2 < -3$. We define a functional on H by

$$\begin{aligned} J(u) = & \int_\Omega [\frac{1}{2}(-|u_t|^2 + |u_{xx}|^2) - \frac{b_1}{2}|(u+2)^+|^2 + 2b_1 u \\ & - \frac{b_2}{2}|(u+3)^+|^2 + 3b_2 u] dxdt. \end{aligned} \tag{4.1}$$

Then the functional J is well-defined in H and the solutions of (1.2) coincide with the critical points of $J(u)$. Now we investigate the property of functional J.

Lemma 4.1. **(cf.**[4]**)** *$J(u)$ is continuous and Frechet differentiable at each $u \in H$ with*

$$DJ(u)v = \int_\Omega (Lu - b_1(u+2)^+ + 2b_1 - b_2(u+3)^+ + 3b_2) v dxdt, \ v \in H.$$

We shall use a variational reduction method to apply the mountain pass theorem.

Let $V =$ closure of $span\{\phi_{10}, \psi_{10}\}$ be the two-dimensional subspace of H. Both of them have the same eigenvalue λ_{10}. Then $\|v\|_H = \sqrt{3}\|v\|$ for $v \in V$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ denote that of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in H$ is expressed by

$$u = v + w,$$

where $v = Pu$, $w = (I - P)u$.

Lemma 4.2. *Let $-15 < b_1, b_2 < -3$ and $-15 < b_1 + b_2 < -3$. Let $v \in V$ be given. Then we have: there exists a unique solution $z \in W$ of equation*

(4.2)

$$Lz + (I - P)[-b_1(v + z + 2)^+ + 2b_1 - b_2(v + z + 3)^+ + 3b_2] = 0 \text{ in } W.$$

Let $z = \theta(v)$, then θ satisfies a uniform Lipschitz continuous on V with respect to the L^2 norm(also the norm $\|\cdot\|_H$).

Proof. Choose $\beta = 7$ and let $g(\xi) = b_1(\xi+2)^+ + b_2(\xi+3)^+ + \beta\xi$. Then equation (4.2) can be written as

$$z = (L + \beta)^{-1}(I - P)[g(v + z) - (b_1 + b_2)]. \tag{4.3}$$

Since $(L + \beta)^{-1}(I - P)$ is a self-adjoint, compact, linear map from $(I - P)H$ into itself, the eigenvalues of $(L + \beta)^{-1}(I - P)$ in W are $(\lambda_{mn}+\beta)^{-1}$, where $\lambda_{mn} > 1$ or $\lambda_{mn} \leq -15$. Therefore $\|(L+\beta)^{-1}(I-P)\|$ is $\frac{1}{4}$. Since

$|g(\xi_1) - g(\xi_2)| \leq \max\{|b_1 + \beta|, |b_2 + \beta|, |b_1 + b_2 + \beta|, |\beta|\}|\xi_1 - \xi_2|$
$< 8|\xi_1 - \xi_2|$,

the right-hand side of equation (4.3) defines a Lipschitz mapping if $(I - P)H_0$ into itself for fixed $v \in V$. By the contraction mapping principle there exists a unique $z \in (I - P)H_0$ (also $z \in (I - P)H$) for fixed $v \in V$. Since $(L + \beta)^{-1}$ is bounded from H to W there exsits a unique solution $z \in W$ of (4.2) for given $v \in V$.

Let

$$\gamma = \frac{\max\{|b_1 + \beta|, |b_2 + \beta|, |b_1 + b_2 + \beta|, |\beta|\}}{8}.$$

Then $0 < \gamma < 1$. If $z_1 = \theta(v_1)$ and $z_2 = \theta(v_2)$ for any v_1, $v_2 \in V$, then

$$\begin{aligned}\|z_1 - z_2\| &\leq \|(L+\beta)^{-1}(I-P)\| \, \|(g(v_1+z_1) - g(v_2+z_2))\| \\ &< \gamma(\|v_1 - v_2\| + \|z_1 - z_2\|).\end{aligned}$$

Hence

$$\|z_1 - z_2\| \leq \frac{\gamma}{1-\gamma}\|v_1 - v_2\|.$$

Since $\|(L+\beta)^{-1}(I-P)\|_H \leq \frac{1}{8}\|u\|$,

$$\begin{aligned}\|z_1 - z_2\|_H &= \|(L+\beta)^{-1}(I-P)(g(v_1+z_1) - g(v_2+z_2))\|_H \\ &\leq (\|z_1 - z_2\| + \|v_1 - v_2\|) \\ &\leq \frac{1}{1-\gamma}\|v_1 - v_2\|_H.\end{aligned}$$

Therefore θ is continuous on V with respect to norm $\|\cdot\|$ (also, to $\|\cdot\|_H$). ☐

LEMMA 4.3. *If $\tilde{J} : V \to \mathbb{R}$ is defined by $\tilde{J}(v) = J(v + \theta(v))$, then $\tilde{J}$ is a continuous Frechet derivative $D\tilde{J}$ with respect to V and*

$$D\tilde{J}(v)s = DJ(v+\theta(v))(s) \text{ for all } s \in V.$$

If v_0 is a critical point of $\tilde{J}$, then $v_0 + \theta(v_0)$ is a solution of (1.2) and conversely every solution of (1.2) is of this form.

Proof. Let $v \in V$ and set $z = \theta(v)$. If $w \in W$, then from (4.2)

$$\begin{aligned}&\int_\Omega (-\theta(v)_t w_t + \theta(v)_x w_x - b_1(v+\theta(v)+2)^+ w + 2b_1 w \\ &-b_2(v+\theta(v)+3)^+ w + 3b_2 w) dt dx = 0.\end{aligned}$$

Since $\int_\Omega v_t w_t = 0$ and $\int_\Omega v_x w_x = 0$,

$$DJ(v+\theta(v))(w) = 0 \quad \text{for all } \ w \in W.$$

Let W_1, W_2 be the two subspaces of H as defining following:

$$\begin{aligned}W_1 &= \text{closure of span}\{\phi_{mn}, \psi_{mn} | \lambda_{mn} \leq -15\}, \\ W_2 &= \text{closure of span}\{\phi_{mn}, \psi_{mn} | \lambda_{mn} \geq 1\}.\end{aligned}$$

Given $v \in V$ and consider the function $h : W_1 \times W_2 \to$ defined by

$$h(w_1, w_2) = J(v + w_1 + w_2).$$

The function h has continuous partial Fréchet derivatives D_1h and D_2h with respect to its first and second variables given by

$$D_1h(w_1, w_2)(y_1) = DJ(v + w_1 + w_2)(y_1) \text{ for } y_1 \in W_1,$$
$$D_2h(w_1, w_2)(y_2) = DJ(v + w_1 + w_2)(y_2) \text{ for } y_2 \in W_2.$$

Therefore let $\theta(v) = \theta_1(v) + \theta_2(v)$ with $\theta_1(v) \in W_1$ and $\theta_2(v) \in W_2$. Then by Lemma 4.2

$$\begin{aligned} D_1h(\theta_1(v), \theta_2(v))(y_1) = 0, &\quad \text{for } y_1 \in W_1 \\ D_2h(\theta_1(v), \theta_2(v))(y_2) = 0, &\quad \text{for } y_2 \in W_2. \end{aligned} \tag{4.4}$$

If $w_2, y_2 \in W_2$ and $w_1 \in W_1$, then

$$\begin{aligned} & [Dh(w_1, w_2) - Dh(w_1, y_2)](w_2 - y_2) \\ = \;& (DJ(v + w_1 + w_2) - DJ(v + w_1 + y_2))(w_2 - y_2) \\ = \;& \int_\Omega -|(w_2 - y_2)_t|^2 + |(w_2 - y_2)^2_{xx}| - b_1[(v + w_1 + w_2 + 2)^+ \\ & -(v + w_1 + y_2 + 2)^+ - b_2(v + w_1 + w_2 + 3)^+ \\ & -(v + w_1 + y_2 + 3)^+](w_2 - y_2)dtdx. \end{aligned}$$

Since $(s^+ - t^+)(s - t) \geq 0$ for any $s, t \in \mathbb{R}$ and $-7 < b_1, b_2, b_1 + b_2 < -3$, it is easy to know that

$$\begin{aligned} & \int_\Omega -b_1[(v + w_1 + w_2 + 2)^+ - (v + w_1 + y_2 + 2)^+](w_2 - y_2) \\ & -b_2[(v + w_1 + w_2 + 3)^+ - (v + w_1 + y_2 + 3)^+](w_2 - y_2)dxdt \geq 0. \end{aligned}$$

And

$$\int_\Omega [-|(w_2 - y_2)_t|^2 + (w_2 - y_2)^2_{xx}]dtdx = \|w_2 - y_2\|^2_H,$$

it follows that

$$(Dh(w_1, w_2) - Dh(w_1, y_2))(w_2 - y_2) \geq \|w_2 - y_2\|^2_H.$$

Therefore, h is strictly convex with respect to the second variable.

Similarly, using the fact that $-b_j(s^+ - t^+)(s - t) \leq -b_j(s - t)^2$ for any $s, t \in \mathbb{R}$,if w_1 and y_1 are in W_1 and $w_2 \in W_2$, then

$$\begin{aligned} & (D_1h(w_1, w_2) - D_1h(y_1, w_2))(w_1 - y_1) \\ \leq \;& -\|w_1 - y_1\|^2_H - b_1\|w_1 - y_1\|^2 - b_2\|w_1 - y_1\|^2 \\ \leq \;& (-1 - \frac{b_1 + b_2}{7})\|w_1 - y_1\|^2_H, \end{aligned}$$

where $-15 < b_1 + b_2 < -3$. Therefore, h is strictly concave with respect to the first variable. From equation (4.4) it follows that

$$J(v + \theta_1(v) + \theta_2(v)) \leq J(v + \theta_1(v) + y_2) \text{ for any } y_2 \in W_2,$$

$$J(v + \theta_1(v) + \theta_2(v)) \geq J(v + y_1 + \theta_2(v)) \text{ for any } y_1 \in W_1,$$

with equality if and only if $y_1 = \theta_1(v), y_2 = \theta_2(v)$.
Since h is strictly concave (convex) with respect to its first (second) variable, Theorem 2.3 of [1] implies that $\tilde{J}$ is C^1 with respect to v and

$$D\tilde{J}(v)(s) = DJ(v + \theta(v))(s), \quad \text{any} \quad s \in V. \tag{4.5}$$

Suppose that there exists $v_0 \in V$ such that $D\tilde{J}(v_0) = 0$. From (4.5) it follows that $DJ(v_0 + \theta(v_0))(v) = 0$ for all $v \in V$. Then by Lemma 4.2 it follows that $DJ(v_0+\theta(v_0))v = 0$ for any $v \in H$. Therefore, $u = v_0+\theta(v_0)$ is a solution of (1.2).

Conversely if u is a solution of (1.2) and $v_0 = Pu$, then $D\tilde{J}(v_0)v = 0$ for any $v \in H$. □

LEMMA 4.4. *Let $-15 < b_1, b_2 < -3$ and $-15 < b_1 + b_2 = b < -3$. Then there exists a small open neighborhood B of 0 in V such that $v = 0$ is a strict local minimum of $\tilde{J}$.*

Proof. For $-15 < b_1, b_2 < -3$ and $-15 < b_1 + b_2 = b < -3$, problem (1.2) has a trivial solution $u_0 = 0$. Thus we have $0 = u_0 = v + \theta(v)$. Since the subspace W is orthogonal complement of subspace V, we get $v = 0$ and $\theta(v) = 0$. Furthermore $\theta(0)$ is the unique solution of equation (4.2) in W for $v = 0$. The trivial solution u_0 is of the form $u_0 = 0+\theta(0)$ and $I+\theta$, where I is an identity map on V, is continuous, it follows that there exists a small open neighborhood B of 0 in V such that if $v \in B$ then $v + \theta(v) + 2 > 0$, $v + \theta(v) + 3 > 0$. By Lemma 4.2, $\theta(0) = 0$ is the solution of (4.3) for any $v \in B$. Therefore, if $v \in B$, then for $z = \theta(v)$

we have $z = 0$. Thus

$$\begin{aligned}
&\tilde{J}(v) = J(v+z)\\
= &\int_\Omega [\frac{1}{2}(-|(v+z)_t|^2 + |(v+z)_{xx}|^2) - \frac{b_1}{2}|(v+z+2)^+|^2\\
&+2b_1(v+z) - \frac{b_2}{2}|(v+z+3)^+|^2 + 3b_2(v+z)]dtdx\\
= &\int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) - \frac{b_1}{2}(v+1)^2 + b_1 v\\
&-\frac{b_2}{2}(v+2)^2 + 2b_2 v]dtdx\\
= &\int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_{xx}|^2) - \frac{b_1}{2}v^2 - 2b_1 - \frac{b_2}{2}v^2 - \frac{9}{2}b_2]dtdx.
\end{aligned}$$

If $v \in V$, then $Lv = -3v$. Therefore in B,

$$\begin{aligned}
\tilde{J}(v) &= \tilde{J}(v) - \tilde{J}(0)\\
&= \int_\Omega [\frac{1}{2}(-|v_t|^2 + |v_x|^2) - \frac{b}{2}v^2]dtdx\\
&= \frac{1}{2}(-3-b)\int_\Omega v^2 dtdx \geq 0,
\end{aligned}$$

where $-15 < b = b_1 + b_2 < -3$. It follows that $v = 0$ is a strict local point of minimum of $\tilde{J}$. □

PROPOSITION 1. *If $-15 < b < 1$, then the equation $Lu - bu^+ = 0$ admits only the trivial solution $u = 0$ in H_0.*

Proof. $H_1 = span\{\cos x \cos 2mt, m \geq 0\}$ is invariant under L and under the map $u \mapsto bu^+$. So the spectrum σ_1 of L retracted to H_1 contains $\lambda_{10} = -3$ in $(-15, 1)$. the spectrum σ_2 of L retracted to $H_2 = H_1^\perp$ contains $\lambda_{10} = -3$ in $(-15, 1)$. From the symmetry theorem in [5] , any solution $y(t)cosx$ of this equation satisfies $y'' + y - by^+ = 0$.This nontrivial periodic solution is periodic with periodic $\pi + \dfrac{\pi}{\sqrt{-b+1}} \neq \pi$. This shows that there is no nontrivial solution of $Lv - bv^+ = 0$. □

LEMMA 4.5. *Let $b = b_1 + b_2$ and $-15 < b_1, b_2, b < -3$. Then the functional $\tilde{J}$, defined on V, satisfies the Palais-Smale condition.*

Proof. Let $\{v_n\} \subset V$ be a Palais-Smale sequence that is $\tilde{J}(v_n)$ is bounded and $D\tilde{J}(v_n) \to 0$ in V. since V is two-dimensional it is enough to prove that $\{v_n\}$ is bounded in V.

Let u_n be the solution of (1.2) with $u_n = v_n + \theta(v_n)$ where $v_n \in V$. So

$$Lu_n - b_1(u_n+2)^+ + 2b_1 - b_2(u_n+3)^+ + 3b_2 = DJ(u_n) \quad \text{in} \quad H.$$

By contradiction we suppose that $\|v_n\| \to +\infty$, also $\|u_n\| \to +\infty$. Dividing by $\|u_n\|$ and taking $w_n = \frac{u_n}{\|u_n\|}$ we get

(4.6)

$$Lw_n - b_1(w_n + \frac{2}{\|u_n\|})^+ + \frac{2b_1}{\|u_n\|} - b_2(w_n + \frac{3}{\|u_n\|})^+ + \frac{3b_2}{\|u_n\|} = \frac{(DJ(u_n))}{\|u_n\|} \to 0.$$

Since $\|w_n\| = 1$ we get : $w_n \to w_0$ weakly in H_0. By L^{-1} is a compact operator, passing to a subsequence we get : $w_n \to w_0$ strongly in H_0. Taking the limit of both sides of (4.6), it follows

$$Lw_0 - bw_0^+ = 0,$$

with $\|w_0\| \neq 0$. This contradicts to the fact that for $-15 < b < -3$ the following equation

$$Lu - bu^+ = 0 \quad \text{in} \quad H_0$$

has only the trivial solution by Proposition 1. Hence $\{v_n\}$ is bounded in V. $\square$

We now define the functional on H, for $-15 < b < -3$,

$$J^*(u) = \int_\Omega [-\frac{1}{2}(-|u_t|^2 + |u_x|^2) - \frac{b}{2}|u^+|^2 dxdt.$$

The critical points of $J^*(u)$ coincide with solutions of the equation

$$Lu - bu^+ = 0 \quad \text{in} \quad H_0$$

The above equation $(-15 < b < -3)$has only the trivial solution and hence $J^*(u)$ has only one critical point $u = 0$.

Given $v \in V$, let $\theta^*(v) = \theta(v) \in W$ be the unique solution of the equation

$$Lz + (I-P)[-b_1(v+z+2)^+ + 2b_1 - b_2(v+z+3)^+ + 3b_2] = 0 \text{ in } W,$$

where $-15 < b_1, b_2, b_1+b_2 = b < -3$. Let us define the reduced functional $\tilde{J}^*(v)$ on V by $J(v+\theta^*(v))$. We note that we can obtain the same results as Lemma 4.1 and Lemma 4.2 when we replace $\theta(v)$ and $\tilde{J}(v)$ by $\theta^*(v)$ and $\tilde{J}^*(v)$. We also note that, for $-15 < b < -3$, $\tilde{J}^*(v)$ has only the critical point $v = 0$.

Lemma 4.6. *Let* $-15 < b_1, b_2 < -3$, $b = b_1 + b_2$ *and* $-15 < b < -3$. *Then we have:* $\tilde{J}^*(v) < 0$ *for all* $v \in V$ *with* $v \neq 0$.

The proof of the lemma can be found in [3].

Lemma 4.7. *Let* $-15 < b_1, b_2 < -3$, $b = b_1 + b_2$ *and* $-15 < b < -3$. *Then we have*

$$\lim_{\|v\| \to \infty} \tilde{J}(v) \to -\infty$$

for all $v \in V$ *(certainly for also the norm* $\|\cdot\|_H$*).*

Proof. Suppose that it is not true that

$$\lim_{\|v\| \to \infty} \tilde{J}(v) \to -\infty.$$

Then there exists a sequence (v_n) in V and a constant C such that

$$\lim_{n \to \infty} \|v_n\| \to \infty$$

and

$$\begin{aligned} \tilde{J}(v_n) &= \int_\Omega (\frac{1}{2} L(v_n + \theta(v_n)) \cdot (v_n + \theta(v_n) - \frac{b_1}{2}|(v_n + \theta(v_n) + 2)^+|^2 \\ &\quad + 2b_1(v_n + \theta(v_n)) - \frac{b_2}{2}|(v_n + \theta(v_n) + 3)^+|^2 \\ &\quad + 3b_1(v_n + \theta(v_n)))dtdx \geq C. \end{aligned}$$

For given $v_n \in V$ let $w_n = \theta(v_n)$ be the unique solution of the equation (4.7)

$$Lw + (I-P)[-b_1(v_n + w + 2)^+ + 2b_1 - b_2(v_n + w + 3)^+ + 3b_2] = 0 \quad \text{in} \quad W.$$

Let $z_n = v_n + w_n$, $v_n^* = \frac{v_n}{\|v_n\|}$, $z_n^* = \frac{z_n}{\|v_n\|}$. Then $z_n^* = v_n^* + w_n^*$. By dividing $\|v_n\|$ we have

$$\begin{aligned} w_n^* &= L^{-1}(I-P)\left(b_1\left(\frac{v_n + w_n + 2}{\|v_n\|}\right)^+ - \frac{2b_1}{\|v_n\|}\right) \\ &\quad + L^{-1}(I-P)\left(b_2\left(\frac{v_n + w_n + 3}{\|v_n\|}\right)^+ - \frac{3b_2}{\|v_n\|}\right) \quad \text{in} \quad W. \end{aligned}$$

By lemma 4.2 $w_n = \theta(v_n)$ is Lipschitz continuous on V. So the sequence $\left\{\frac{w_n + v_n}{\|v_n\|}\right\}$ is bounded in H. Since $\lim_{n \to \infty} \frac{1}{\|v_n\|} = 0, \lim_{n \to \infty} \frac{b_j}{\|v_n\|} = 0$ $(j =$

$1, 2)$, it follows that $b_1\left(\frac{v_n + w_n + 2}{\|v_n\|}\right)^+ - \frac{2b_1}{\|v_n\|}$, $b_2\left(\frac{v_n + w_n + 3}{\|v_n\|}\right)^+ - \frac{3b_2}{\|v_n\|}$ are bounded in H. Since L^{-1} is a compact operator there is a subsequence of w_n^* converge to some w^* in W, denote by itself. Since V is two-dimensional space, assume that sequence (v_n^*) converges to $v^* \in V$ with $\|v^*\| = 1$. Therefore, we can get that the sequence (z_n^*) converges to an element z^* in H.

On the other hand, since $\tilde{J}(v_n) \geq C$, dividing this inequality by $\|v_n\|^2$, we get

$$\begin{aligned}\int_\Omega \frac{1}{2}L(z_n^*)\cdot z_n^* - \frac{b_1}{2}((z_n^* + \frac{2}{\|v_n\|})^+)^2 + 2b_1\frac{z_n^*}{\|v_n\|} \\ - \frac{b_2}{2}((z_n^* + \frac{3}{\|v_n\|})^+)^2 + 3b_3\frac{z_n^*}{\|v_n\|}dtdx \geq \frac{C}{\|v_n\|^2}.\end{aligned} \tag{4.8}$$

By lemma 4.2 it follows that for any $y \in W$

$$\int_\Omega [-(z_n)_t y_t + (z_n)_x y_x - b_1(z_n+2)^+ y + 2b_1 y - b_2(z_n+3)^+ y + 3b_2 y]dtdx = 0. \tag{4.9}$$

If we set $y = w_n$ in (4.9) and divide by $\|v_n\|^2$, then we obtain

$$\int_\Omega [-|(w_n^*)_t|^2 + |(w_n^*)_{xx}|^2 - b_1(z_n^*)^+ w_n^* + \frac{2b_1}{\|v_n\|}w_n^* - b_2(z_n^*)^+ w_n^* + \frac{3b_2}{\|v_n\|}w_n^*]dtdx = 0. \tag{4.10}$$

Let $y \in W$ be arbitrary. Dividing (4.9) by $\|v_n\|$ and letting $n \to \infty$, we obtain

$$\int_\Omega [-(z^*)_t y_t + (z^*)_{xx} y_{xx} - b(z^*)^+ y]dtdx = 0, \tag{4.11}$$

Where $b = b_1 + b_2$. Then (4.11) can be written in the form $D\tilde{J}^*(v^* + w^*)(y) = 0$ for all $y \in W$. Put $w^* = \theta(v^*)$. Letting $n \to \infty$ in (4.10), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega(-|(w_n^*)_t|^2+|(w_n^*)_{xx}|^2)dtdx\\
=&\lim_{n\to\infty}\int_\Omega b(z_n^*)^+w_n^*-\frac{b}{\|v_n\|}w_n^*dtdx\\
=&\int_\Omega b(z^*)^+w^*dtdz\\
=&\int_\Omega(-(z^*)_t(w^*)_t+(z^*)_x(w^*)_{xx})dtdx\\
=&\int_\Omega(-|(w^*)_t|^2+|(w^*)_{xx}|^2)dtdx,
\end{aligned}$$

where we have used (4.11). Hence

$$\lim_{n\to\infty}\int_\Omega[-|(z_n^*)_t|^2+|(z_n^*)_{xx}|^2]dtdx=\int_\Omega[-|(z^*)_t|^2+|(z^*)_{xx}|^2]dtdx.$$

Letting $n\to\infty$ in (4.8), we obtain

$$\tilde{J}^*(v^*)=\int_\Omega[\frac{1}{2}(-|(z^*)_t|^2+|(z^*)_{xx}|^2)+\frac{b}{2}|(z^*)^+|^2]dtdx\geq 0.$$

Since $\|v^*\|=1$, this contradicts to the fact that $\tilde{J}^*(v)<0$ for all $v\neq 0$. This proves that $\lim_{\|v\|\to\infty}\tilde{J}(v)\to-\infty$. □

Now we state the main result in this paper:

THEOREM 4.1. *Let $-15<b_1,b_2<-3$, $b=b_1+b_2$ and $-15<b<-3$. Then there exists at least three solutions of the equation*

$$\begin{aligned}
u_{tt}-u_{xx} &= b_1[(u+2)^+-2]+b_2[(u+3)^+-3]\quad \text{in}\quad (-\frac{\pi}{2},\frac{\pi}{2})\times\mathbb{R},\\
u(\pm\frac{\pi}{2},t) &= 0,\quad u(x,t+\pi)=u(x,t).
\end{aligned}$$

and two of them are nontrivial solutions.

Proof. We remark that $u=0$ is the trivial solution of problem (1.2). Then $v=0$ is a critical point of functional $\tilde{J}$. Next we want to find others critical points of $\tilde{J}$ which are corresponding to the solutions of problem (1.2).

By Lemma 4.4, there exists a small open neighborhood B of 0 in V such that $v=0$ is a strict local point of minimum of $\tilde{J}$. Since

$\lim_{\|v\|_H \to \infty} \tilde{J}(v) \to -\infty$ from lemma 4.7 and V is a two-dimensional space, there exists a critical point $v_0 \in V$ of $\tilde{J}$ such that

$$\tilde{J}(v_0) = \max_{v \in V} \tilde{J}(v).$$

Let B_{v_0} be an open neighborhood of v_0 in V such that $B \cap B_{v_0} = \varnothing$. Since $\lim_{\|v\|_H \to \infty} \tilde{J}(v) \to -\infty$, we can choose $v_1 \in V \backslash (B \cup B_{v_0})$ such that $\tilde{J}(v_1) < \tilde{J}(0)$. Since $\tilde{J}$ satisfies the Palais-Smale condition, by the Mountain Pass Theorem (Theorem 3.2) there is a critical value

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v)$$

where $\Gamma = \{\gamma \in C([0,1], E) | \gamma(0) = 0, \gamma(1) = v_0\}$.

If $\tilde{J}(v_0) \neq c$, then there exists a critical point v of $\tilde{J}$ at level c such that $v \neq v_0$, 0 (since $c \neq \tilde{J}(v_0)$ and $c > \tilde{J}(0)$). Therefore, in case $\tilde{J}(v_0) \neq c$, the functional $\tilde{J}(v)$ has also at least 3 critical points $0, v_0, v$.

If $\tilde{J}(v_0) = c$, then define

$$c' = \inf_{\gamma \in \Gamma'} \sup_{\gamma} \tilde{J}(v)$$

where $\Gamma' = \{\gamma \in \Gamma : \gamma \cap B_{v_0} = \varnothing\}$. Hence

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v) \leq \inf_{\gamma \in \Gamma'} \sup_{\gamma} \tilde{J}(v) \leq \max_{v \in V} \tilde{J}(v) = c.$$

That is $c = c'$. By contradiction assume $K_c = \{v \in V | \tilde{J}(v) = c, D\tilde{J}(v) = 0\} = \{v_0\}$. Use the functional $\tilde{J}$ for the deformation theorem (Theorem 4.1) and taking $\epsilon < \frac{1}{2}(c - \tilde{J}(0))$. We choose $\gamma \in \Gamma'$ such that $\sup_{\gamma} \tilde{J} \leq c$. From the deformation theorem (Theorem 3.1) $\eta(1, \cdot) \circ \gamma \in \Gamma$ and

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{J}(v) \leq \sup_{\eta(1,\cdot)\circ\gamma} \tilde{J}(v) \leq c - \epsilon,$$

which is a contradiction. Therefore, there exists a critical point v of $\tilde{J}$ at level c such that $v \neq v_0$, 0, which means that the equation (1.2) has at least three critical points. Since $\|v\|_H, \|v_0\|_H \neq 0$, these two critical points coincide with two nontrivial period solutions of problem (1.2). $\square$

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