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FUZZY STABILITY OF A CUBIC-QUADRATIC FUNCTIONAL EQUATION: A FIXED POINT APPROACH

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Abstract. Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following cubic-quadratic functional equation

$$\begin{aligned}(0.1)\quad & \frac{1}{2}\left(f(2x+y)+f(2x-y)-f(-2x-y)-f(y-2x)\right)\\ &= 2f(x+y)+2f(x-y)+4f(x)-8f(-x)-2f(y)-2f(-y)\end{aligned}$$

in fuzzy Banach spaces.

1. Introduction and preliminaries

Katsaras [23] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [13, 25, 52]. In particular, Bag and Samanta [3], following Cheng and Mordeson [8], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [24]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [4].

We use the definition of fuzzy normed spaces given in [3, 29, 30] to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (0.1) in the fuzzy normed vector space setting.

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DEFINITION 1.1. [3, 29, 30, 31] *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

(N_1) $N(x, t) = 0$ for $t \leq 0$;

(N_2) $x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;

(N_3) $N(cx, t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;

(N_4) $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

(N_5) $N(x, \cdot)$ is a non-decreasing function of $\mathbb{R}$ and $\lim_{t\to\infty} N(x, t) = 1$.

The pair (X, N) is called a *fuzzy normed vector space*.

The properties of fuzzy normed vector spaces and examples of fuzzy norms are given in [29, 28].

DEFINITION 1.2. [3, 29, 30, 31] *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is said to be convergent or converge if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ and we denote it by N-$\lim_{n\to\infty} x_n = x$.*

DEFINITION 1.3. [3, 29, 30] *Let (X, N) be a fuzzy normed vector space. A sequence $\{x_n\}$ in X is called Cauchy if for each $\varepsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \varepsilon$.*

It is well-known that every convergent sequence in a fuzzy normed vector space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be *complete* and the fuzzy normed vector space is called a *fuzzy Banach space*.

We say that a mapping $f : X \to Y$ between fuzzy normed vector spaces X and Y is continuous at a point $x_0 \in X$ if for each sequence $\{x_n\}$ converging to x_0 in X, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be *continuous* on X (see [4]).

The stability problem of functional equations originated from a question of Ulam [51] concerning the stability of group homomorphisms. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [41] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [41] has provided a lot of influence in the development of what we call *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [50] for mappings $f:X\to Y$, where X is a normed space and Y is a Banach space. Cholewa [9] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [10] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2], [17], [20]–[22], [32], [33], [36]–[39], [42]–[49]).

In [19], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x). \tag{1.1}$$

It is easy to show that the function $f(x)=x^3$ satisfies the functional equation (1.1), which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

We recall a fundamental result in fixed point theory.

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\le d(x,y)+d(y,z)$ for all $x,y,z\in X$.

Theorem 1.4. [5, 12] *Let (X,d) be a complete generalized metric space and let $J:X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either*

$$d(J^n x, J^{n+1}x)=\infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x)<\infty, \qquad \forall n\ge n_0;$

(2) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$*;*

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In 1996, G. Isac and Th.M. Rassias [18] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [6], [7], [28], [34], [35], [40]).

This paper is organized as follows: In Section 2, we prove the generalized Hyers-Ulam stability of the cubic-quadratic functional equation (0.1) in fuzzy Banach spaces for an odd case. In Section 3, we prove the generalized Hyers-Ulam stability of the cubic-quadratic functional equation (0.1) in fuzzy Banach spaces for an even case.

Throughout this paper, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

2. Generalized Hyers-Ulam stability of the functional equation (0.1): an odd case

One can easily show that an even mapping $f : X \to Y$ satisfies (0.1) if and only if the even mapping $f : X \to Y$ is a quadratic mapping, i.e.,

$$f(x + y) + f(x - y) = 2f(x) + 2f(y),$$

and that an odd mapping $f : X \to Y$ satisfies (0.1) if and only if the odd mapping mapping $f : X \to Y$ is a cubic mapping, i.e.,

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x).$$

For a given mapping $f : X \to Y$, we define

$$\begin{aligned} Df(x, y) : &= \frac{1}{2}\left(f(2x + y) + f(2x - y) - f(-2x - y) - f(y - 2x)\right) \\ &- 2f(x + y) - 2f(x - y) - 4f(x) + 8f(-x) \\ &+ 2f(y) + 2f(-y) \end{aligned}$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y)=0$ in fuzzy Banach spaces: an odd case.

THEOREM 2.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \le \frac{L}{8}\varphi(2x,2y)$$

for all $x,y\in X$. Let $f : X\to Y$ be an odd mapping satisfying

$$N\left(Df(x,y),t\right) \ge \frac{t}{t+\varphi(x,y)} \tag{2.1}$$

for all $x,y\in X$ and all $t>0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x\in X$ and defines a cubic mapping $C : X\to Y$ such that

$$N\left(f(x)-C(x),t\right) \ge \frac{(16-16L)t}{(16-16L)t+L\varphi(x,0)} \tag{2.2}$$

for all $x\in X$ and all $t>0$.

Proof. Letting $y=0$ in (2.1), we get

$$N\left(2f\left(2x\right)-16f(x),t\right) \ge \frac{t}{t+\varphi(x,0)} \tag{2.3}$$

for all $x\in X$.

Consider the set

$$S := \{g : X\to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu\in\mathbb{R}_+ : N(g(x)-h(x),\mu t) \ge \frac{t}{t+\varphi(x,0)},\ \forall x\in X, \forall t>0\},$$

where, as usual, $\inf\phi=+\infty$. It is easy to show that (S,d) is complete. (See the proof of Lemma 2.1 of [27].)

Now we consider the linear mapping $J : S\to S$ such that

$$Jg(x) := 8g\left(\frac{x}{2}\right)$$

for all $x\in X$.

Let $g,h\in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \ge \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(8g\left(\frac{x}{2}\right) - 8h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{8}\varepsilon t\right) \\ &\geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \varphi\left(\frac{x}{2}, 0\right)} \geq \frac{\frac{Lt}{8}}{\frac{Lt}{8} + \frac{L}{8}\varphi(x, 0)} \\ &= \frac{t}{t + \varphi(x, 0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$N\left(f(x) - 8f\left(\frac{x}{2}\right), \frac{L}{16}t\right) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{16}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C\left(\frac{x}{2}\right) = \frac{1}{8}C(x) \tag{2.4}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.4) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{L}{16 - 16L}.$$

This implies that the inequality (2.2) holds.

By (2.1),

$$N\left(8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), 8^n t\right) \geq \frac{t}{t + \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. So

$$N\left(8^n Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right), t\right) \geq \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x, y)}$$

for all $x, y \in X$, all $t > 0$ and all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} \frac{\frac{t}{8^n}}{\frac{t}{8^n} + \frac{L^n}{8^n}\varphi(x,y)} = 1$ for all $x, y \in X$ and all $t > 0$,

$$N(DC(x, y), t) = 1$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $C : X \to Y$ is cubic, as desired. $\square$

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 3$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an odd mapping satisfying*

$$N(Df(x, y), t) \geq \frac{t}{t + \theta(\|x\|^p + \|y\|^p)} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then $C(x) := N\text{-}\lim_{n\to\infty} 8^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that

$$N(f(x) - C(x), t) \geq \frac{2(2^p - 8)t}{2(2^p - 8)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{3-p}$ and we get the desired result. $\square$

THEOREM 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \le 8L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an odd mapping satisfying (2.1). *Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N(f(x) - C(x), t) \ge \frac{(16-16L)t}{(16-16L)t + \varphi(x,0)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S, d) be the generalized metric space defined in the proof of Theorem 2.1.

Consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{8} g(2x)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$N(g(x) - h(x), \varepsilon t) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(\frac{1}{8}g(2x) - \frac{1}{8}h(2x), L\varepsilon t\right) \\ &= N(g(2x) - h(2x), 8L\varepsilon t) \\ &\ge \frac{8Lt}{8Lt + \varphi(2x,0)} \ge \frac{8Lt}{8Lt + 8L\varphi(x,0)} \\ &= \frac{t}{t + \varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.3) that

$$N\left(f(x) - \frac{1}{8}f(2x), \frac{1}{16}t\right) \ge \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{1}{16}$.

By Theorem 1.4, there exists a mapping $C : X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.,

$$C(2x) = 8C(x) \tag{2.7}$$

for all $x \in X$. Since $f : X \to Y$ is odd, $C : X \to Y$ is an odd mapping. The mapping C is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that C is a unique mapping satisfying (2.7) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - C(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)}$$

for all $x \in X$;

(2) $d(J^n f, C) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x) = C(x)$$

for all $x \in X$;

(3) $d(f, C) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, C) \leq \frac{1}{16 - 16L}.$$

This implies that the inequality (2.6) holds.

The rest of the proofm is similar to the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 3$. Let X be a normed vector space with norm $\| \cdot \|$. Let $f : X \to Y$ be an odd mapping satisfying (2.5). Then $C(x) := N\text{-}\lim_{n\to\infty} \frac{1}{8^n} f(2^n x)$ exists for each $x \in X$ and defines a cubic mapping $C : X \to Y$ such that*

$$N(f(x) - C(x), t) \geq \frac{2(8 - 2^p)t}{2(8 - 2^p)t + \theta \|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-3}$ and we get the desired result. □

3. Generalized Hyers-Ulam stability of the functional equation (0.1): an even case

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equation $Df(x,y)=0$ in fuzzy Banach spaces: an even case.

THEOREM 3.1. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(x,y) \le \frac{L}{4}\varphi(2x,2y)$$

for all $x,y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0)=0$ and (2.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x)-Q(x),t\right) \ge \frac{(8-8L)t}{(8-8L)t+L\varphi(x,x)} \tag{3.1}$$

for all $x \in X$ and all $t>0$.

Proof. Letting $y=x$ in (2.1), we get

$$N\left(2f\left(2x\right)-8f(x),t\right) \ge \frac{t}{t+\varphi(x,x)} \tag{3.2}$$

for all $x \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\{\mu \in \mathbb{R}_+ : N(g(x)-h(x),\mu t) \ge \frac{t}{t+\varphi(x,x)}, \forall x \in X, \forall t>0\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete. (See the proof of Lemma 2.1 of [27].)

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g,h \in S$ be given such that $d(g,h)=\varepsilon$. Then

$$N(g(x)-h(x),\varepsilon t) \ge \frac{t}{t+\varphi(x,x)}$$

for all $x \in X$ and all $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\varepsilon t) &= N\left(4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right), L\varepsilon t\right) \\ &= N\left(g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right), \frac{L}{4}\varepsilon t\right) \geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \varphi\left(\frac{x}{2}, \frac{x}{2}\right)} \\ &\geq \frac{\frac{Lt}{4}}{\frac{Lt}{4} + \frac{L}{4}\varphi(x, x)} = \frac{t}{t + \varphi(x, x)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. So $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq L\varepsilon$. This means that

$$d(Jg, Jh) \leq Ld(g, h)$$

for all $g, h \in S$.

It follows from (3.2) that

$$N\left(f(x) - 4f\left(\frac{x}{2}\right), \frac{L}{8}t\right) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$ and all $t > 0$. So $d(f, Jf) \leq \frac{L}{8}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(\frac{x}{2}\right) = \frac{1}{4}Q(x) \tag{3.3}$$

for all $x \in X$. Since $f : X \to Y$ is even, $Q : X \to Y$ is an even mapping. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.3) such that there exists a $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \geq \frac{t}{t + \varphi(x, x)}$$

for all $x \in X$;

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f,Q) \leq \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,Q) \leq \frac{L}{8-8L}.$$

This implies that the inequality (3.1) holds.

The rest of the proof m is similar to the proof of Theorem 2.1. $\square$

COROLLARY 3.2. *Let $\theta \geq 0$ and let p be a real number with $p > 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.5). Then $Q(x) := N\text{-}\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that*

$$N\left(f(x) - Q(x), t\right) \geq \frac{(2^p-4)t}{(2^p-4)t + \theta\|x\|^p}$$

for all $x \in X$ and all $t > 0$.

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{2-p}$ and we get the desired result. $\square$

Similarly, we can obtain the following. We will omit the proof.

THEOREM 3.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 4L\varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f\left(2^n x\right)$ exists for each $x \in X$ and defines a quadratic mapping $Q : X \to Y$ such that

$$N\left(f(x) - Q(x), t\right) \geq \frac{(8-8L)t}{(8-8L)t + \varphi(x,x)}$$

for all $x \in X$ and all $t > 0$.

COROLLARY 3.4. *Let $\theta \geq 0$ and let p be a real number with $0 < p < 2$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$*

be an even mapping satisfying $f(0) = 0$ *and* (2.5). *Then* $Q(x) := N\text{-}\lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$ *exists for each* $x \in X$ *and defines a quadratic mapping* $Q : X \to Y$ *such that*

$$N(f(x) - Q(x), t) \geq \frac{(4-2^p)t}{(4-2^p)t + \theta\|x\|^p}$$

for all $x \in X$ *and all* $t > 0$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$. Then we can choose $L = 2^{p-2}$ and we get the desired result. □

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