P-I-OPEN MAPPINGS, P-I-CONTINUOUS MAPPINGS AND P-I-IRRESOLUTE MAPPINGS

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ABSTRACT. The notions of P-I-open (closed) mappings, P-I-continuous mappings, P-I-neighborhoods, P-I-irresolute mappings and I-irresolute mappings are introduced. Relations between P-I-open (closed) mappings and I-open (closed) mappings are given. Characterizations of P-I-open (closed) mappings are provided. Relations between a P-I-continuous mapping and an I-continuous mapping are discussed, and characterizations of a P-I-continuous mapping are considered. Conditions for a mapping to be an I-irresolute mapping (resp. P-I-irresolute mapping) are provided.

1. INTRODUCTION

In 1990, D. Janković, and T.R. Hamlett have introduced the notion of I-open sets in topological spaces. Since then, several kinds of I-openness, that is, (weakly) semi-I-open set, δ-I-open sets, β-I-open sets, α-I-open sets, b-I-open sets, (weakly) pre-I-open sets, etc. are introduced, and several properties and relations are investigated (see [2, 3, 8, 9, 10, 11, 12, 25, 28]). In [18], Kang and Kim first introduced the notions of pre-local function, semi-local function and α-local function with respect to a topology and an ideal, and investigated several properties. They next introduced the concept of P-I-open set and P-I-closed set in ideal topological spaces, and investigated related properties. They discussed relations between I-open sets and P-I-open sets. Finally they introduced the notion of P-∗-closure, and investigated many properties related to P-I-open set, pre-local function, semi-local function and α-local function with respect to a topology and an ideal.

In this paper, we deal with P-I-open mappings, P-I-continuous mappings and P-I-irresolute mappings. In section 3, we define the notion of P-I-open (closed) mappings, and give relations between P-I-open (closed) mappings and I-open (closed)
mappings. We provide characterizations of $P$-I-open (closed) mappings. In section 4, we define a $P$-I-continuous mapping and a $P$-I-neighborhood, and then we investigate relations between a $P$-I-continuous mapping and an I-continuous mapping. We discuss characterizations of a $P$-I-continuous mapping. In the final section, we introduce the notions of $P$-I-irresolute mappings and I-irresolute mappings. We give conditions for a mapping to be an I-irresolute mapping (resp. $P$-I-irresolute mapping).

2. PRELIMINARIES

Through this paper, $(X, \tau)$ and $(Y, \kappa)$ (simply $X$ and $Y$) always mean topological spaces. A subset $A$ of $X$ is said to be semi-open [19] (respectively, $\alpha$-open [26] and pre-open [24]) if $A \subset \text{Cl}(\text{Int}(A))$ (respectively, $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ and $A \subset \text{Int}(\text{Cl}(A)))$. The complement of a pre-open set (respectively, an $\alpha$-open set and a semi-open set) is called a pre-closed set (respectively, an $\alpha$-closed set and a semiclosed set). The intersection of all pre-closed sets (respectively, $\alpha$-closed sets and semi-closed sets) containing $A$ is called the pre-closure (respectively, $\alpha$-closure and semi-closure) of $A$, denoted by $p\text{Cl}(A)$ (respectively, $\alpha\text{Cl}(A)$ and $s\text{Cl}(A)$). A subset $A$ is also pre-closed (respectively, $\alpha$-closed and semi-closed) if and only if $A = p\text{Cl}(A)$ (respectively, $A = \alpha\text{Cl}(A)$ and $A = s\text{Cl}(A)$). We denote the family of all pre-open sets (respectively, $\alpha$-open sets and semi-open sets) of $(X, \tau)$ by $\tau^p$ (respectively, $\tau^\alpha$ and $\tau^s$).

An ideal is defined as a nonempty collection $\mathcal{I}$ of subsets of $X$ satisfying the following two conditions.

1. If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$. (heredity)
2. If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. (finite additivity)

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathcal{I}$ on $X$, and it is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subset X$, the set

$$A^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$$

is called the local function of $A$ with respect to $\tau$ and $\mathcal{I}$, where

$$\tau(x) = \{U \in \tau : x \in U\}.$$ 

We will use $A^*$ and/or $A^*(\mathcal{I})$ instead of $A^*(\tau, \mathcal{I})$.

**Lemma 2.1** ([16]). Let $(X, \tau)$ be a topological space with ideals $\mathcal{I}$ and $\mathcal{J}$ on $X$. For subsets $A$ and $B$ of $X$, we have the following assertions.
(i) $A \subseteq B \Rightarrow A^* \subseteq B^*$.
(ii) $\mathcal{I} \subseteq \mathcal{J} \Rightarrow A^*(\mathcal{J}) \subseteq A^*(\mathcal{I})$.
(iii) $A^* = \text{Cl}(A^*) \subseteq \text{Cl}(A)$ ($A^*$ is a closed subset of Cl(A)).
(iv) $(A^*)^* \subseteq A^*$.
(v) $(A \cup B)^* = A^* \cup B^*$.
(vii) $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subseteq (U \cap A)^*$.
(viii) $B \in \mathcal{I} \Rightarrow (A \cup B)^* = A^* = (A \setminus B)^*$.

**Definition 2.2.** Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset $A$ of $X$ is said to be $\mathcal{I}$-open [1] if $A \subseteq \text{Int}(A^*)$.

The set of all $\mathcal{I}$-open sets in ideal topological space $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{IO}(X, \tau, \mathcal{I})$ or written simply as $\mathcal{IO}(X)$ when there is no chance for confusion.

**Definition 2.3** ([18]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ be a subset of $X$. Then the set

$$A^*_p(\tau, \mathcal{I}) = \{ x \in X : U \cap A \not\subseteq \mathcal{I} \text{ for each } U \in \tau^p(x) \}$$

is called the pre-local function with respect to $\tau$ and $\mathcal{I}$, where

$$\tau^p(x) = \{ U \in \tau^p : x \in U \}.$$

We will use $A^*_p$ and/or $A^*_p(\mathcal{I})$ instead of $A^*_p(\tau, \mathcal{I})$.

**Lemma 2.4** ([18]). Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let $A$ be a subset of $X$. Then

(i) If $\mathcal{I} = \{ \emptyset \}$, then $A^*_p = p\text{Cl}(A)$, $A^*_s = s\text{Cl}(A)$ and $A^*_\alpha = \alpha\text{Cl}(A)$.
(ii) If $\mathcal{I} = \mathcal{P}(X)$, then $A^*_p = A^*_s = A^*_\alpha = \emptyset$.

**Lemma 2.5** ([18]). Let $(X, \tau)$ be a topological space with ideals $\mathcal{I}$ and $\mathcal{J}$ on $X$, and let $A, B$ be subsets of $X$. Then

(i) $A \subseteq B \Rightarrow A^*_p \subseteq B^*_p$.
(ii) $\mathcal{I} \subseteq \mathcal{J} \Rightarrow A^*_p(\mathcal{J}) \subseteq A^*_p(\mathcal{I})$.
(iii) $A^*_p = p\text{Cl}(A^*_p) \subseteq p\text{Cl}(A)$ ($A^*_p$ is a pre-closed subset of $p\text{Cl}(A)$).
(iv) $(A^*_p)^*_p \subseteq A^*_p$.
(v) $B \in \mathcal{I} \Rightarrow B^*_p = \emptyset$.
(vi) $U \in \tau^\alpha \Rightarrow U \cap A^*_p = U \cap (U \cap A)^*_p \subseteq (U \cap A)^*_p$.
(vii) $B \in \mathcal{I} \Rightarrow (A \cup B)^*_p = A^*_p = (A \setminus B)^*_p$.
(viii) $A^*_p(\mathcal{I} \cap \mathcal{J}) \supset A^*_p(\mathcal{I}) \cup A^*_p(\mathcal{J})$. 

Definition 2.6 ([18]). Let \((X, \tau, \mathcal{I})\) be an ideal topological space. A subset \(A\) of \(X\) is said to be \(P-\mathcal{I}\)-open if \(A \subset \text{pInt}(A^*_p)\). A subset \(B\) of \(X\) is said to be \(P-\mathcal{I}\)-closed if the complement of \(B\) is \(P-\mathcal{I}\)-open.

The set of all \(P-\mathcal{I}\)-open sets in \((X, \tau, \mathcal{I})\) is denoted by \(\text{PIO}(X, \tau, \mathcal{I})\). Simply \(\text{PIO}(X, \tau, \mathcal{I})\) is written as \(\text{PIO}(X)\) or \(\text{PIO}(X, \tau)\) when there is no chance for confusion.

Definition 2.7 ([1]). A mapping \(f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{J})\) is said to be \(\mathcal{I}\)-open (resp. \(\mathcal{I}\)-closed) if for each \(U \in \tau\) (resp. \(U^c \in \tau\)), \(f(U)\) is an \(\mathcal{I}\)-open (resp. \(\mathcal{I}\)-closed) set.

Theorem 2.8 ([18]). Let \(A \in \text{PIO}(X, \tau)\). Then \(A\) is \(\mathcal{I}\)-open.

Remark 2.9. By Theorem 2.8, we know that \(P-\mathcal{I}\)-open set implies \(\mathcal{I}\)-open set. By [1, Remark 2.2], we know that \(\mathcal{I}\)-open set implies pre-open set. Hence we can deduce that \(P-\mathcal{I}\)-open set implies pre-open set. The converse is not true, in general.

Theorem 2.10 ([18]). Let \(\{U_i \in \text{PIO}(X) : i \in \Lambda\}\) be a class of \(P-\mathcal{I}\)-open sets in an ideal topological space \((X, \tau, \mathcal{I})\). Then \(\bigcup_{i \in \Lambda} \{U_i \in \text{PIO}(X) : i \in \Lambda\}\) is \(P-\mathcal{I}\)-open.

Theorem 2.11 ([18]). If \(A\) is \(P-\mathcal{I}\)-closed in an ideal topological space \((X, \tau, \mathcal{I})\), then \(A \supset (\text{pInt}(A))^*_p\).

Lemma 2.12 ([17]). Let \(A\) be a subset of a topological space \((X, \tau)\). Then the following assertions are satisfied.

(i) \((\text{pInt}(A))^c = \text{pCl}(A^c)\).

(ii) \((\text{pCl}(A))^c = \text{pInt}(A^c)\).

3. \(P-\mathcal{I}\)-OPEN MAPPINGS AND \(P-\mathcal{I}\)-CLOSED MAPPINGS

Definition 3.1. A mapping \(f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{J})\) is said to be \(P-\mathcal{I}\)-open (resp. \(P-\mathcal{I}\)-closed) if for each \(U \in \tau\) (resp. \(U^c \in \tau\)), \(f(U)\) is a \(P-\mathcal{I}\)-open set (resp. \(P-\mathcal{I}\)-closed set).

Example 3.2. Consider a topological space \((X, \tau)\) with \(X = \{a, b, c\}\) and \(\tau = \{\emptyset, X, \{a\}, \{b, c\}\}\), and consider an ideal topological space \((Y, \kappa, \mathcal{I})\) where \(Y = \{1, 2, 3, 4\}\), \(\kappa = \{\emptyset, Y, \{3\}, \{1, 2\}, \{1, 2, 3\}\}\), and \(\mathcal{I} = \{\emptyset, \{1\}\}\). Then

\[\text{PIO}(Y, \kappa) = \{\emptyset, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4\}\}.
\]

Let \(f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I})\) be a mapping given by \(f(a) = 2 = f(b)\) and \(f(c) = 3\). Then \(f(\{a\}) = \{2\}, f(\{b, c\}) = \{2, 3\}, f(X) = \{2, 3\}\) and \(f(\emptyset) = \emptyset\). Hence \(f\) is a \(P-\mathcal{I}\)-open mapping. Let \(g : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I})\) be a mapping given by \(g(a) = 1 = g(b)\)
and \( g(c) = 4 \). Then \( g(\{b, c\}) = \{1, 4\} = g(X) \), \( g(\{a\}) = \{1\} \) and \( g(\emptyset) = \emptyset \). Hence \( g \) is a \( P-I \)-closed mapping.

**Theorem 3.3.** Let \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) be a \( P-I \)-open (resp. \( P-I \)-closed) mapping. Then \( f \) is an \( I \)-open (resp. \( I \)-closed) mapping.

**Proof.** Suppose that \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) be a \( P-I \)-open (resp. \( P-I \)-closed) mapping. Let \( G \in \tau \) (resp. \( G^c \in \tau \)). Then \( f(G) \) is a \( P-I \)-open set (resp. \( P-I \)-closed set) in \( Y \). Since \( P-I \)-open (resp. \( P-I \)-closed) set is an \( I \)-open (resp. \( I \)-closed) set by Theorem 2.8, \( f(G) \) is an \( I \)-open (resp. \( I \)-closed) set. Hence \( f \) is \( I \)-open (resp. \( I \)-closed).

The converse of Theorem 3.3 may not be true as seen in the following example.

**Example 3.4.** Consider a topological space \( (X, \tau) \) with \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\} \), and consider an ideal topological space \( (Y, \kappa, \mathcal{I}) \) where \( Y = \{1, 2, 3, 4\} \), \( \kappa = \{\emptyset, Y, \{3\}, \{1, 2\}, \{1, 2, 3\}\} \), and \( \mathcal{I} = \{\emptyset, \{1\}\} \). Then a mapping \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) given by \( f(a) = 1 \), \( f(b) = 2 = f(c) \), and \( f(d) = 3 \) is \( I \)-open. Since \( f(\{a, b\}) = \{1, 2\} \not\subseteq \{2\} = p\text{Int}(\{1, 2\}^\circ_\mathcal{I}) \), we know that \( f \) is not \( P-I \)-open.

**Corollary 3.5.** Let \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) be a \( P-I \)-open (resp. \( P-I \)-closed) mapping. Then \( f \) is a pre-open (resp. pre-closed) mapping.

**Proof.** Using Theorem 3.3 and Remark 2.9, we know that \( f \) is a pre-open (resp. pre-closed) mapping.

**Example 3.6.** Consider a topological space \( (X, \tau) \) with \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, X, \{1\}, \{2, 3\}\} \), and consider an ideal topological space \( (Y, \kappa, \mathcal{I}) \) where \( Y = \{a, b, c, d\} \), \( \kappa = \{\emptyset, Y, \{c\}, \{a, b\}, \{a, b, c\}\} \), and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then a mapping \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) given by \( f(1) = b = f(2) \) and \( f(3) = c \) is \( P-I \)-open. But \( f \) is not an open mapping because \( f(1) = \{b\} \not\subseteq \kappa \) for \( \{1\} \in \tau \).

**Example 3.7.** Consider a topological space \( (X, \tau) \) with \( X = \{1, 2, 3\} \) and \( \tau = \{\emptyset, X, \{1\}, \{2, 3\}\} \), and consider an ideal topological space \( (Y, \kappa, \mathcal{I}) \) where \( Y = \{a, b, c, d\} \), \( \kappa = \{\emptyset, Y, \{c\}, \{a, b\}, \{a, b, c\}\} \), and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then a mapping \( g : (X, \tau) \rightarrow (Y, \kappa, \mathcal{I}) \) given by \( g(1) = c, g(2) = a \), and \( g(3) = b \) is an open mapping. But \( g \) is not a \( P-I \)-open mapping since \( g(\{2, 3\}) = \{a, b\} \not\subseteq p\text{Int}((\{a, b\}^\circ_\mathcal{I}) = \{b\} \) for \( \{2, 3\} \in \tau \).

**Remark 3.8.** We know that the \( P-I \)-open mapping and the open mapping are independent notions as seen in Examples 3.6 and 3.7.
Theorem 3.9. Let \( f : (X, \tau) \to (Y, \kappa, \mathcal{I}) \) be a mapping. Then the following statements are equivalent:

(i) \( f \) is a \( P-I \)-open mapping.

(ii) For each \( x \in X \) and each open neighborhood \( U \) of \( x \), there exists a \( P-I \)-open set \( W \subset Y \) containing \( f(x) \) such that \( W \subset f(U) \)

Proof. (i) \( \Rightarrow \) (ii). Suppose that \( f \) is a \( P-I \)-open mapping. Let \( x \in X \). Then for each open set \( G \) containing \( x \), \( f(x) \in f(G) \). Since \( f \) is \( P-I \)-open, \( f(G) \) is a \( P-I \)-open set in \( Y \). Putting \( W := f(G) \), we obtain (ii).

(ii) \( \Rightarrow \) (i). Let \( G \) be an open set in \( X \). Then for any \( x \in G \), there exists \( W_x \in P\mathcal{O}(Y, \kappa) \) such that \( f(x) \in W_x \subset f(G) \). This implies that \( f(G) = \bigcup_{x \in G} f(x) \subset \bigcup_{x \in G} W_x \subset f(G) \). Hence \( \bigcup_{x \in G} W_x = f(G) \). By Theorem 2.10, \( f(G) \) is \( P-I \)-open. Therefore \( f \) is a \( P-I \)-open mapping. \( \square \)

Theorem 3.10. Let \( f : (X, \tau) \to (Y, \kappa, \mathcal{I}) \) be a mapping. Then \( f \) is \( P-I \)-open if and only if it satisfies the following assertion:

\[
(3.1) \quad f(\text{Int}(A)) \subset p\text{Int}(f(A)_{p}^*)
\]

for all \( A \) in \( (X, \tau) \).

Proof. Suppose that \( f \) is a \( P-I \)-open mapping. Let \( A \) be a subset of \( X \). Then \( \text{Int}(A) \) is an open set and \( f(\text{Int}(A)) \) is a \( P-I \)-open set. Hence

\[
f(\text{Int}(A)) \subset p\text{Int}(f(\text{Int}(A))_{p}^*) \subset p\text{Int}(f(A)_{p}^*).
\]

Conversely, suppose that \( f \) satisfies (3.1). Let \( G \) be an open subset of \( X \). Then \( f(G) = f(\text{Int}(G)) \subset p\text{Int}(f(G)_{p}^*) \). Hence \( f(G) \) is a \( P-I \)-open set in \( (Y, \kappa, \mathcal{I}) \). Therefore \( f \) is a \( P-I \)-open mapping. \( \square \)

Corollary 3.11. Let \( f : (X, \tau) \to (Y, \kappa, \mathcal{I}) \) be a mapping satisfying the inclusion \( f(\text{Int}(A)) \subset p\text{Int}(f(A)_{p}^*) \) for all \( A \) in \( (X, \tau) \). Then \( f \) is an \( I \)-open mapping.

Proof. Straightforward. \( \square \)

If \( f \) is an \( I \)-open mapping then is Theorem 3.10 true? The answer is negative as seen in the following example.

Example 3.12. Consider a topological space \( (X, \tau) \) with \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\} \), and consider an ideal topological space \( (Y, \kappa, \mathcal{I}) \) where \( Y = \{1, 2, 3, 4\} \), \( \kappa = \{\emptyset, Y, \{3\}, \{1, 2\}, \{1, 2, 3\}\} \), and \( \mathcal{I} = \{\emptyset, \{1\}\} \). Then a mapping \( f : (X, \tau) \to (Y, \kappa, \mathcal{I}) \) given by \( f(a) = 1 \), \( f(b) = 2 = f(c) \), and \( f(d) = 3 \) is an \( I \)-open
mapping. If \( A = \{a, b, d\} \), then \( f(\text{Int}(A)) = f(\{a, b\}) = \{1, 2\} \) and
\[
p\text{Int}(f(A)_p^*) = p\text{Int}(\{1, 2, 3\}_p^*) = p\text{Int}(\{2, 3, 4\}) = \{2, 3, 4\}.
\]
Hence we know that \( f(\text{Int}(A)) \not\subset p\text{Int}(f(A)_p^*) \).

**Theorem 3.13.** Let \( f : (X, \tau) \to (Y, \kappa, \mathcal{I}) \) be a mapping. Then \( f \) is \( P-\mathcal{I} \)-open if and only if it satisfies the following assertion:

\[
(3.2) \quad \text{Int}(f^{-1}(B)) \subset f^{-1}(p\text{Int}(B^*_p))
\]

for all \( B \) in \((Y, \kappa, \mathcal{I})\).

**Proof.** Suppose that \( f \) is \( P-\mathcal{I} \)-open. Let \( B \) be a subset of \( Y \). Then \( f^{-1}(B) \) is a subset of \((X, \tau)\). Since \( f \) is \( P-\mathcal{I} \)-open, we obtain
\[
f(\text{Int}(f^{-1}(B))) \subset p\text{Int}(f(f^{-1}(B))_p^*).
\]
It follows that
\[
\text{Int}(f^{-1}(B)) \subset f^{-1}(f(\text{Int}(f^{-1}(B))))
\subset f^{-1}(p\text{Int}(f(f^{-1}(B))_p^*))
\subset f^{-1}(p\text{Int}(B^*_p)).
\]

Conversely, suppose that \( f \) satisfies (3.2). Let \( G \) be an open set in \((X, \tau)\). Then \( \text{Int}(f^{-1}(f(G))) \subset f^{-1}(p\text{Int}(f(G)_p^*)) \) since \( f(G) \) is a set in \((Y, \kappa, \mathcal{I})\). Since \( G \subset f^{-1}(f(G)) \) and \( \text{Int}(G) = G \), we have
\[
G \subset \text{Int}(f^{-1}(f(G))) \subset f^{-1}(p\text{Int}(f(G)_p^*)�).
\]
This implies that \( f(G) \subset f(f^{-1}(p\text{Int}(f(G)_p^*)) \subset p\text{Int}(f(G)_p^*) \). Hence \( f(G) \) is a \( P-\mathcal{I} \)-open set in \((Y, \kappa, \mathcal{I})\). Therefore \( f \) is \( P-\mathcal{I} \)-open. \( \square \)

If \( f \) is an \( \mathcal{I} \)-open mapping then does Theorem 3.13 hold? The answer is negative as seen in the following example.

**Example 3.14.** In Example 3.12, let \( B = \{1, 2\} \). Then
\[
\text{Int}(f^{-1}(B)) = \text{Int}(\{a, b, c\}) = \{a, b, c\}
\]
and \( f^{-1}(p\text{Int}(B^*_p)) = f^{-1}(p\text{Int}(\{2\})) = \{b, c\} \). Hence we know that \( \text{Int}(f^{-1}(B)) \not\subset f^{-1}(p\text{Int}(B^*_p)) \).
Theorem 3.15. Let $f : (X, \tau) \to (Y, \kappa, \mathcal{I})$ be a mapping. Then $f$ is $P-I$-closed if and only if it satisfies the following assertion:

\[(3.3) \quad pCl(((f(Cl(A))^c)^*_p)^c) \subset f(Cl(A))\]

for $A$ in $X$.

Proof. Let $f$ be a $P-I$-closed mapping. Then

\[f(Cl(A))^c \subset pInt((f(Cl(A))^c)^*_p).\]

Hence $pCl(((f(Cl(A))^c)^*_p)^c) \subset f(Cl(A))$.

Conversely, assume that (3.3) is valid and let $B$ be a closed set in $X$. Then

\[pCl(((f(B))^c)^*_p) = pCl(((f(Cl(B))^c)^*_p)^c) \subset f(Cl(B)) = f(B).\]

This implies that $f(B)^c \subset pInt((f(B))^c)^*_p$. Hence $f$ is a $P-I$-closed mapping. \hfill \Box

If $f$ is an $I$-closed mapping then do $f$ satisfy the following assertion?

\[pCl(((f(Cl(A))^c)^*_p)^c) \subset f(Cl(A))\]

The answer is negative as seen in the following example.

Example 3.16. Let $(X, \tau)$ be a topological space with $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$, and consider an ideal topological space $(Y, \kappa, \mathcal{I})$ where $Y = \{1, 2, 3, 4\}$, $\kappa = \{\emptyset, Y, \{3\}, \{1, 2\}, \{1, 2, 3\}\}$, and $\mathcal{I} = \{\emptyset, \{1\}\}$. Then a mapping $f : (X, \tau) \to (Y, \kappa, \mathcal{I})$ given by $f(a) = 2 = f(b)$, $f(c) = 1$, and $f(d) = 4$ is an $I$-closed mapping. Let $A = \{d\}$. Then we know that $pCl(((f(Cl(A))^c)^*_p)^c) = \{1\}$ and $f(Cl(A)) = \{4\}$. Hence

\[pCl(((f(Cl(A))^c)^*_p)^c) \not\subset f(Cl(A)).\]

Theorem 3.17. Let $f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})$ be $P-I$-open such that

\[(3.4) \quad (\forall A \subset X)(f(A^*) \subset f(A)_p^* \text{ or } f(A^*) \subset f(A)).\]

Then the image of each $I$-open set is $P-I$-open.

Proof. Suppose that $f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})$ is a $P-I$-open mapping. Let $A$ be an $I$-open set in $X$. Then $A \subset Int(A^*)$. Since $f$ is a $P-I$-open mapping, $f(Int(A^*))$ is a $P-I$-open set in $Y$. It follows that

\[f(A) \subset f(Int(A^*)) \subset pInt(f(Int(A^*))^*_p) \subset pInt(f(A^*_p)).\]

Since $f(A^*) \subset f(A)_p^*$ or $f(A^*) \subset f(A)$, we have
\[ f(A) \subset p\text{Int}((f(A)^*_p)^*_p) \subset p\text{Int}(f(A)^*_p), \]

and so \( f(A) \subset p\text{Int}(f(A)^*_p) \).

The converse of Theorem 3.17 is not valid as seen in the following example.

**Example 3.18.** Consider two ideal topological spaces \((X, \tau, \mathcal{I})\) and \((Y, \kappa, \mathcal{J})\) where \(X = \{a, b, c, d\}, \ \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}, \ \mathcal{I} = \{\emptyset, \{a\}\}, \ Y = \{1, 2, 3, 4\}, \ \kappa = \{\emptyset, Y, \{1, 2\}, \{1, 2, 3\}\}, \) and \(\mathcal{J} = \{\emptyset, \{3\}\}\). Then a mapping \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})\) given by \(f(a) = 1, f(b) = 2 = f(c)\) and \(f(d) = 4\) is a \(P-\mathcal{I}\)-open mapping in which the image of each \(\mathcal{I}\)-open set is a \(P-\mathcal{I}\)-open set. But if \(A = \{b, c\}\) then 
\(f(A^*) = f(X) = \{1, 2, 4\} \not\subset \{2\} = f(A)^*_p = f(A)\).

**Corollary 3.19.** Let \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})\) be \(P-\mathcal{I}\)-open. Assume that every subset \(A\) of \(X\) satisfies \(f(A^*) \subset f(A)^*_p\) or \(f(A^*) \subset f(A)\). Then the image of each \(P-\mathcal{I}\)-open set is \(P-\mathcal{I}\)-open.

**Proof.** We can obtain the result by analogous way to Theorem 3.17.

We have a question: In Theorem 3.17, if we use the following condition

\[(\forall A \subset X)(f(A^*) \subset f(A)^*))\]

instead of the condition (3.4), then does Theorem 3.17 hold?

We provide a partial answer to the above question.

**Theorem 3.20.** Let \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})\) be \(P-\mathcal{I}\)-open such that

\[(\forall A \subset X)(f(A^*) \subset f(A)^*))\]

\[(\forall B \subset Y)((B^*)^*_p \subset B^*_p).\]

Then the image of each \(\mathcal{I}\)-open set is \(P-\mathcal{I}\)-open.

**Proof.** Suppose that \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa, \mathcal{J})\) is \(P-\mathcal{I}\)-open. Let \(A\) be an \(\mathcal{I}\)-open set in \(X\). Then \(A \subset \text{Int}(A^*)\). Since \(f\) is \(P-\mathcal{I}\)-open, \(f(\text{Int}(A^*))\) is a \(P-\mathcal{I}\)-open set in \(Y\). It follows that

\[f(A) \subset f(\text{Int}(A^*))\]

\[\subset p\text{Int}(f(\text{Int}(A^*))^*_p)\]

\[\subset p\text{Int}(f(A)^*_p)\]
\[ \subset p\text{Int}((f(A)^*)_p^*) \]
\[ \subset p\text{Int}(f(A)_p^*) \]

Hence \( f(A) \) is a \( P-I \)-open set in \( Y \).

**Theorem 3.21.** Let \( f : (X, \tau) \to (Y, \kappa, I) \) be a \( P-I \)-open mapping. If \( W \subset Y \) and \( F \) is a closed set in \( X \) containing \( f^{-1}(W) \), then there exists a \( P-I \)-closed set \( H \) in \( Y \) containing \( W \) such that \( f^{-1}(H) \subset F \).

**Proof.** Let \( f : (X, \tau) \to (Y, \kappa, I) \) be a \( P-I \)-open mapping. Suppose that \( W \subset Y \) and \( F \) is a closed set in \( X \) containing \( f^{-1}(W) \). Then \( F^c \) is open in \( X \) and \( f(F^c) \) is \( P-I \)-open in \( Y \). Putting \( H := f(F^c)^c \), we get

\[
\begin{align*}
  f^{-1}(W) \subset F & \Rightarrow f^{-1}(W^c) \supset F^c \\
& \Rightarrow f(f^{-1}(W^c)) \supset f(F^c) \\
& \Rightarrow W^c \supset f(f^{-1}(W^c)) \supset f(F^c) \\
& \Rightarrow W \subset f(F^c)^c = H,
\end{align*}
\]

and \( f^{-1}(H) = f^{-1}(f(F^c)^c) \subset (F^c)^c = F \). Hence \( H \) is a \( P-I \)-closed set containing \( W \) and \( f^{-1}(H) \subset F \).

**Lemma 3.22.** For any bijective mapping \( f : (X, \tau) \to (Y, \kappa, I) \), \( f \) is \( P-I \)-open if and only if \( f \) is \( P-I \)-closed.

**Proof.** Suppose that \( f \) is \( P-I \)-open. Let \( F \) be closed in \( X \). Then \( F^c \) is open in \( X \). This implies that \( f(F^c) = f(F)^c \) is \( P-I \)-open in \( Y \). Hence \( f(F) \) is \( P-I \)-closed in \( Y \). Therefore \( f \) is a \( P-I \)-closed mapping.

Conversely, we can obtain the result by analogous way.

**Theorem 3.23.** Let \( f : (X, \tau) \to (Y, \kappa, J) \) and \( g : (Y, \kappa, J) \to (Z, \delta, H) \) be two mappings, where \( I, J, H \) are ideals on \( X, Y \) and \( Z \) respectively. Then

(i) \( g \circ f \) is \( P-I \)-open if \( f \) is an open mapping and \( g \) is a \( P-I \)-open mapping.

(ii) Assume that \( g(V^*) \subset g(V)_p^* \) or \( g(V^*) \subset g(V) \) for every subset \( V \) of \( Y \). If \( f \) is \( I \)-open and \( g \) is \( P-I \)-open, then \( g \circ f \) is \( P-I \)-open.

**Proof.** (i) Straightforward.

(ii) Let \( A \subset X \) be an open set. Since \( f \) is \( I \)-open, \( f(A) \) is an \( I \)-open set. Since \( g \) is \( P-I \)-open, it follows from Theorem 3.17 that \( g(f(A)) \) is a \( P-I \)-open set. Hence \( g \circ f \) is a \( P-I \)-open mapping.
Corollary 3.24. Let \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{J}) \) and \( g : (Y, \kappa, \mathcal{J}) \rightarrow (Z, \delta, \mathcal{H}) \) be two mappings, where \( \mathcal{I}, \mathcal{J}, \mathcal{H} \) are ideals on \( X, Y \) and \( Z \) respectively. Assume that \( g(V^*) \subset g(V)^*_p \) or \( g(V^*) \subset g(V) \) for every subset \( V \) of \( Y \). If \( f \) is \( P\mathcal{I}\)-open and \( g \) is \( P\mathcal{I}\)-open, then \( g \circ f \) is \( P\mathcal{I}\)-open.

Proof. Straightforward. \( \Box \)

If \( f \) is \( P\mathcal{I}\)-open and \( g \) is \( P\mathcal{I}\)-open then is \( g \circ f \) \( P\mathcal{I}\)-open? The answer is negative as seen in the following example.

Example 3.25. Consider a topological space

\[
(X = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{1, 2\}, \{1, 2, 3\}\})
\]

and ideal topological spaces \((Y, \kappa, \mathcal{J})\) and \((Z, \delta, \mathcal{H})\) where \( Y = \{x, y, z\}, \kappa = \{\emptyset, Y, \{x\}\}, \mathcal{J} = \{\emptyset, \{y\}\}, Z = \{a, b, c, d\}, \delta = \{\emptyset, Z, \{c\}, \{a, b\}, \{a, b, c\}\}, \) and \( \mathcal{H} = \{\emptyset, \{a\}\} \). A mapping \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(1) = x, f(2) = y = f(3), \) and \( f(4) = z \) is a \( P\mathcal{I}\)-open mapping. And a mapping \( g : (Y, \kappa, \mathcal{J}) \rightarrow (Z, \delta, \mathcal{H}) \) given by \( g(x) = b, g(y) = d, \) and \( g(z) = c \) is a \( P\mathcal{I}\)-open mapping. Let \( A = \{1, 2\} \in \tau \). Then \( g \circ f(A) = \{b, d\} \) is not a \( P\mathcal{I}\)-open set in \((Z, \delta, \mathcal{H})\). Hence \( g \circ f \) is not a \( P\mathcal{I}\)-open mapping.

Remark 3.26. From Theorem 3.3 and Example 3.25, we know that the answers to the following questions are negative.

(i) If a mapping \( f \) is \( P\mathcal{I}\)-open and a mapping \( g \) is \( \mathcal{I}\)-open, then is \( g \circ f \) \( P\mathcal{I}\)-open?

(ii) If a mapping \( f \) is \( \mathcal{I}\)-open and a mapping \( g \) is \( P\mathcal{I}\)-open, then is \( g \circ f \) \( P\mathcal{I}\)-open?

(iii) If a mapping \( f \) is \( \mathcal{I}\)-open and a mapping \( g \) is \( \mathcal{I}\)-open, then is \( g \circ f \) \( P\mathcal{I}\)-open?

If a mapping \( f \) is \( P\mathcal{I}\)-open and a mapping \( g \) is open, then is \( g \circ f \) \( P\mathcal{I}\)-open? The answer is negative as seen in the following example.

Example 3.27. Consider the example as presented in Example 3.25. A mapping \( f : (X, \tau) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(1) = x, f(2) = y = f(3), \) and \( f(4) = z \) is a \( P\mathcal{I}\)-open mapping. And a mapping \( g : (Y, \kappa, \mathcal{J}) \rightarrow (Z, \delta, \mathcal{H}) \) given by \( g(x) = c, g(y) = a, \) and \( g(z) = b \) is an open mapping. Let \( A = \{1, 2\} \in \tau \). Then \( g \circ f(A) = \{a, c\} \) is not a \( P\mathcal{I}\)-open set in \((Z, \delta, \mathcal{H})\). Hence \( g \circ f \) is not a \( P\mathcal{I}\)-open mapping.
Let \( f : (X, \tau) \to (Y, \kappa, \mathcal{J}) \) and \( g : (Y, \kappa, \mathcal{J}) \to (Z, \delta, \mathcal{H}) \) be two mappings. We have two questions as follow.

(i) If \( g \circ f \) is \( P-\mathcal{I} \)-open and \( g \) is \( P-\mathcal{I} \)-open, then is \( f \) an open mapping?

(ii) If \( g \circ f \) is \( P-\mathcal{I} \)-open and \( f \) is open, then is \( g \) a \( P-\mathcal{I} \)-open mapping?

The answers to these questions are negative as seen in the following two examples.

**Example 3.28.** Let \( X = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{1, 2\}, \{1, 2, 3\}\}. \) Let \( Y = \{x, y, z\} \)
\( \kappa = \{\emptyset, Y, \{x\}\}, \mathcal{J} = \{\emptyset, \{y\}\} \) and let \( Z = \{a, b, c, d\}, \delta = \{\emptyset, Z, \{c\}, \{a, b\}, \{a, b, c\}\}, \)
\( \mathcal{H} = \{\emptyset, \{a\}\}. \) Consider mappings \( f : (X, \tau) \to (Y, \kappa, \mathcal{J}) \) given by \( f(1) = x = f(2), \)
\( f(3) = z = f(4) \) and \( g : (Y, \kappa, \mathcal{J}) \to (Z, \delta, \mathcal{H}) \) given by \( g(x) = b, g(y) = d \) and \( g(z) = c. \) Then \( g \circ f \) and \( g \) are \( P-\mathcal{I} \)-open. But \( f \) is not an open mapping because \( f(A) = \{x, z\} \notin \kappa \) for \( A = \{1, 2, 3\} \in \tau. \)

**Example 3.29.** Let \( X = \{1, 2, 3, 4\}, \tau = \{\emptyset, X, \{1, 2\}, \{1, 2, 3\}\}. \) Let \( Y = \{x, y, z\} \)
\( \kappa = \{\emptyset, Y, \{x\}, \{x, y\}\}, \mathcal{J} = \{\emptyset, \{y\}\} \) and let \( Z = \{a, b, c, d\}, \delta = \{\emptyset, Z, \{c\}, \{a, b\}, \{a, b, c\}\}, \)
\( \mathcal{H} = \{\emptyset, \{a\}\}. \) Consider mappings \( f : (X, \tau) \to (Y, \kappa, \mathcal{J}) \) given by \( f(1) = f(2) = f(3) = x, f(4) = y \) and \( g : (Y, \kappa, \mathcal{J}) \to (Z, \delta, \mathcal{H}) \) given by \( g(x) = b, g(y) = c, \)
\( g(z) = a. \) Then \( g \circ f \) is \( P-\mathcal{I} \)-open and \( f \) is open. But \( g \) is not a \( P-\mathcal{I} \)-open mapping because \( g(Y) = \{a, b, c\} \) is not a \( P-\mathcal{I} \)-open set in \( Z. \)

4. **\( P-\mathcal{I} \)-continuous mappings**

**Definition 4.1.** A mapping \( f : (X, \tau, \mathcal{I}) \to (Y, \kappa) \) is said to be \( P-\mathcal{I} \)-continuous if \( f^{-1}(V) \in PIO(X, \tau, \mathcal{I}) \) for all \( V \in \kappa. \)

**Example 4.2.** Let \( X = \{a, b, c, d\} \) with topology \( \tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\} \) and an ideal \( \mathcal{I} = \{\emptyset, \{c\}\}. \) Then we know that

\[
PIO(X, \tau, \mathcal{I}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}.
\]

Let \( Y = \{1, 2, 3, 4, 5\} \) with topology \( \kappa = \{\emptyset, Y, \{2\}, \{3\}, \{2, 3\}\}. \) Then a mapping \( f : (X, \tau, \mathcal{I}) \to (Y, \kappa) \) given by \( f(a) = 2, f(b) = 3, \) and \( f(c) = 5 = f(d) \) is a \( P-\mathcal{I} \)-continuous mapping.

**Theorem 4.3.** If a mapping \( f : (X, \tau, \mathcal{I}) \to (Y, \kappa) \) is \( P-\mathcal{I} \)-continuous, then it is \( \mathcal{I} \)-continuous.

**Proof.** It follows from Theorem 2.8. \( \square \)

**Corollary 4.4.** If a mapping \( f : (X, \tau, \mathcal{I}) \to (Y, \kappa) \) is \( P-\mathcal{I} \)-continuous, then it is \( \text{pre-continuous}. \)
Proof. It follows from Remark 2.9.

Is any \( \mathcal{I} \)-continuous mapping a \( P-\mathcal{I} \)-continuous mapping? The answer to this question is negative as seen in the following example.

**Example 4.5.** Consider an ideal topological space \((X, \tau, \mathcal{I})\) where \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Then

\[
P\mathcal{I}O(X, \tau, \mathcal{I}) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, c, d\}\},
\]

\[
I\mathcal{O}(X, \tau, \mathcal{I}) = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}.
\]

Let \((Y, \kappa)\) be a topological space where \(Y = \{1, 2, 3, 4\}\) and \(\kappa = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}\).

Consider a mapping \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa)\) given by \(f(a) = 3 = f(d), f(b) = 1\) and \(f(c) = 2\). Then \(f^{-1}(\{1\}) = \{b\}, f^{-1}(\{2\}) = \{c\}, f^{-1}(\{1, 2\}) = \{b, c\} \) and \(f^{-1}(\{1, 2, 3\}) = X = f^{-1}(Y)\). Hence \(f\) is \(\mathcal{I}\)-continuous. But \(f\) is not \(P-\mathcal{I}\)-continuous because \(f^{-1}(\{1, 2, 3\}) = X\) is not \(P-\mathcal{I}\)-open.

Is any \(P-\mathcal{I}\)-continuous mapping a continuous mapping and vice versa? The following examples show that the answer to this question is negative.

**Example 4.6.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space with \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}\), and \( \mathcal{I} = \{\emptyset, \{a\}\} \). Consider a topological space \((Y, \kappa)\) with \(Y = \{1, 2, 3\}\) and \(\kappa = \{\emptyset, Y, \{1\}, \{2\}, \{1, 2\}\}\). Let \(f : (X, \tau, \mathcal{I}) \to (Y, \kappa)\) be defined by \(f(a) = f(b) = f(c) = 1\) and \(f(d) = 3\). Then \(f^{-1}(\{1\}) = \{a, b, c\} = f^{-1}(\{1, 2\}), f^{-1}(\{2\}) = \emptyset \) and \(f^{-1}(Y) = X\). Hence \(f\) is continuous. But \(f\) is not \(P-\mathcal{I}\)-continuous because \(f^{-1}(Y) = X\) is not \(P-\mathcal{I}\)-open.

**Example 4.7.** Consider an ideal topological space \((X, \tau, \mathcal{I})\) with \(X = \{a, b, c\}\), \(\tau = \{\emptyset, X, \{a\}\}\), and \( \mathcal{I} = \{\emptyset, \{b\}\} \). Let \(Y = \{1, 2, 3, 4\}\) with topology \(\kappa = \{\emptyset, Y, \{1, 2\}, \{1, 2, 3\}\}\). Define a mapping \(g : (X, \tau, \mathcal{I}) \to (Y, \kappa)\) by \(g(a) = 1, g(b) = 2\) and \(g(c) = 4\). Then \(g^{-1}(\{1, 2\}) = \{a, b\} = f^{-1}(\{1, 2, 3\}) \) and \(f^{-1}(Y) = X\). Hence \(f\) is \(P-\mathcal{I}\)-continuous. However, \(f\) is not continuous because \(f^{-1}(\{1, 2\}) = \{a, b\}\) is not open.

**Definition 4.8.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. A subset \(S\) of \(X\) is called a \(P-\mathcal{I}\)-neighborhood of \(x\) if \(S\) is a superset of a \(P-\mathcal{I}\)-open set \(G\) containing \(x\).

**Example 4.9.** Let \(X = \{a, b, c, d\}\) with a topology \(\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}\) and an ideal \(\mathcal{I} = \{\emptyset, \{c\}\}\). Then
\[ \text{PIO}(X, \tau, I) = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X \} \]
and the set \( S = \{a, c, d\} \) is a \( P-I \)-neighborhood of \( a \) because there exists a \( P-I \)-open set \( \{a, d\} \) such that \( a \in \{a, d\} \subseteq S \). But \( S \) is not a \( P-I \)-neighborhood of \( c \).

**Theorem 4.10.** For a mapping \( f : (X, \tau, I) \to (Y, \kappa) \), the following statements are equivalent.

(i) \( f \) is \( P-I \)-continuous.

(ii) For each \( x \in X \) and each \( V \in \kappa \) containing \( f(x) \), there exists
\[
W \in \text{PIO}(X, \tau, I)
\]
containing \( x \) such that \( f(W) \subset V \).

(iii) For each \( x \in X \) and each \( V \in \kappa \) containing \( f(x) \), \( f^{-1}(V) \) is a \( P-I \)-neighborhood of \( x \).

(iv) For each \( x \in X \) and each \( V \in \kappa \) containing \( f(x) \), \( f^{-1}(V) \) is a pre-neighborhood of \( x \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( x \in X \) and \( V \in \kappa \) containing \( f(x) \). Since \( f \) is \( P-I \)-continuous, \( f^{-1}(V) \) is a \( P-I \)-open set. Putting \( W := f^{-1}(V) \), we have \( f(W) \subset V \).

(ii) \( \Rightarrow \) (i) Let \( A \) be an open set in \( Y \). If \( f^{-1}(A) = \emptyset \) then \( f^{-1}(A) \) is clearly \( P-I \)-open. Assume that \( f^{-1}(A) \neq \emptyset \). Let \( x \in f^{-1}(A) \). Then \( f(x) \in A \), which implies that there exist \( P-I \)-open \( W \) containing \( x \) such that \( f(W) \subset A \). Thus \( W \subset f^{-1}(f(W)) \subset f^{-1}(A) \). Since \( W \) is \( P-I \)-open, \( x \in W \subset p\text{Int}(W^*_p) \subset p\text{Int}(f^{-1}(A)^*_p) \) and so \( f^{-1}(A) \subset p\text{Int}(f^{-1}(A)^*_p) \). Hence \( f^{-1}(A) \) is a \( P-I \)-open set and so \( f \) is \( P-I \)-continuous.

(ii) \( \Rightarrow \) (iii) Let \( x \in X \) and \( V \in \kappa \) containing \( f(x) \). Then there exist \( P-I \)-open \( W \) containing \( x \) such that \( f(W) \subset V \). It follows that \( W \subset f^{-1}(f(W)) \subset f^{-1}(V) \).

Since \( W \) is \( P-I \)-open,
\[
x \in W \subset p\text{Int}(W^*_p) \subset p\text{Int}(f^{-1}(V)^*_p) \subset f^{-1}(V)^*_p.
\]
Hence \( f^{-1}(V)^*_p \) is a \( P-I \)-neighborhood of \( x \).

(iii) \( \Rightarrow \) (iv) By Remark 2.9, it is straightforward.

(iv) \( \Rightarrow \) (i) Let \( A \) be an open set in \( Y \). If \( f^{-1}(A) = \emptyset \) then \( f^{-1}(A) \) is clearly \( P-I \)-open. Assume that \( f^{-1}(A) \neq \emptyset \) and let \( x \in f^{-1}(A) \). Then \( f(x) \in A \). Since \( f^{-1}(A)^*_p \) is a pre-neighborhood of \( x \), there exists a pre-open set \( H \) such that \( x \in H \subset f^{-1}(A)^*_p \).

Since \( H \) is pre-open, \( x \in H \subset p\text{Int}(H) \subset p\text{Int}(f^{-1}(A)^*_p) \) and so
$f^{-1}(A) \subset p\text{Int}(f^{-1}(A)^*).$ Hence $f^{-1}(A)$ is a $P$-$I$-open set. Therefore $f$ is $P$-$I$-continuous. $\square$

**Theorem 4.11.** For a mapping $f : (X, \tau, I) \rightarrow (Y, \kappa)$, the following statements are equivalent.

(i) $f$ is $P$-$I$-continuous.

(ii) The inverse image of each closed set in $Y$ is $P$-$I$-closed.

(iii) For each subset $A$ of $Y$, $f^{-1}(\text{Int}(A)) \subset p\text{Int}(f^{-1}(A)^*)$.

**Proof.** (i) $\Rightarrow$ (ii) Let $F$ be a closed subset of $X$. Then $F^c$ is open in $Y$. Since $f$ is $P$-$I$-continuous, $f^{-1}(F^c) = (f^{-1}(F))^c$ is $P$-$I$-open. Hence $f^{-1}(F)$ is $P$-$I$-closed.

(ii) $\Rightarrow$ (i) Let $G$ be an open set in $(Y, \kappa)$. Then $G^c$ is closed. By (ii), $f^{-1}(G^c) = (f^{-1}(G))^c$ is $P$-$I$-closed. Hence $f^{-1}(G)$ is $P$-$I$-open, and so $f$ is $P$-$I$-continuous.

(i) $\Rightarrow$ (iii) Suppose that $f$ is $P$-$I$-continuous. Let $A$ be a subset of $Y$. Then $f^{-1}(\text{Int}(A))$ is $P$-$I$-open. It follows that

$$f^{-1}(\text{Int}(A)) \subset p\text{Int}(f^{-1}(\text{Int}(A))^*) \subset p\text{Int}(f^{-1}(A)^*).$$

(iii) $\Rightarrow$ (i) Let $A$ be an open set in $(Y, \kappa)$. Then $f^{-1}(A) = f^{-1}(\text{Int}(A)) \subset p\text{Int}(f^{-1}(A)^*)$ by (iii). Hence $f^{-1}(A)$ is $P$-$I$-open. Therefore $f$ is $P$-$I$-continuous. $\square$

**Proposition 4.12.** Let $(X, \tau, I)$ be an ideal topological space. Then the following statements are equivalent.

(i) $X = X_p^*$.

(ii) $\tau^p \cap I = \{\emptyset\}$. ($\tau^p$ is a set of all pre-open sets in $(X, \tau)$).

(iii) If $A \in I$, then $p\text{Int}(A) = \emptyset$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $\tau^p \cap I \neq \{\emptyset\}$. Then there exists $G(\neq \emptyset) \in \tau^p \cap I$. Let $a \in G$, i.e., $a \notin X \setminus G$. Then $G \in \tau^p(a)$ and $X \cap G = G \in I$. Thus $a \notin X_p^*$ and so $X_p^* \subset X \setminus G$. Since $G \neq \emptyset$, $X_p^* \neq X$. This is a contradiction. Hence $\tau^p \cap I = \{\emptyset\}$.

(ii) $\Rightarrow$ (iii) Let $A \in I$. If $A = \emptyset$ then clearly $p\text{Int}(A) = \emptyset$. Assume that $A$ is not empty. Then for every $H \in \tau^p \setminus \{\emptyset\}$, we have $H \notin I$ by (ii) and so, $H \notin A$. Hence $p\text{Int}(A) = \emptyset$.

(iii) $\Rightarrow$ (i) Let $x \in X$. If there exist $G_x \in \tau^p(x)$ such that $G_x \cap X \in I$, then $G_x = p\text{Int}(G_x) = p\text{Int}(G_x \cap X) = \emptyset$ by (iii). It is a contradiction. Hence $G_x \cap X \notin I$ for every $G_x \in \tau^p(x)$ and so $x \in X_p^*$. This means that $X = X_p^*$. $\square$
Theorem 4.13. Let \((X, \tau, I)\) be an ideal topological space. If \(U \subseteq U_\tau^*\) for every pre-open \(U\), then \(X = X_\tau^*\).

Proof. Since \(X\) is always pre-open, \(X \subseteq X_\tau^*\) by the hypothesis. In general, \(X_\tau^* \subseteq X\). Hence \(X = X_\tau^*\). \(\square\)

The converse of Theorem 4.13 may not be true as seen in the following example.

Example 4.14. Let \(X = \{a, b, c, d\}\) with topology \(\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}\), ideal \(I = \{\emptyset, \{c\}\}\). Then \(\tau^I = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, X\}\). We know that \(X = X_\tau^*\) but there exist a pre-open set \(\{a, c\}\) such that \(\{a, c\} \notin \{a, c\}_\tau^* = \{a\}\).

Theorem 4.15. If \(f : (X, \tau, I) \to (Y, \kappa)\) is \(P-I\)-continuous, then \(X = X_\tau^*\).

Proof. Suppose that \(f : (X, \tau, I) \to (Y, \kappa)\) is \(P-I\)-continuous. Since \(Y\) is an open set in \((Y, \kappa)\) and \(f\) is \(P-I\)-continuous, \(f^{-1}(Y) = X\) is a \(P-I\)-open set and thus \(X \subseteq p\text{Int}(X_\tau^*) \subseteq X_\tau^*\). Hence \(X = X_\tau^*\) because \(X_\tau^* \subseteq X\) in general. \(\square\)

The converse of Theorem 4.15 may not be true as seen in the following example.

Example 4.16. Let \(X = \{a, b, c, d\}\) with topology \(\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}\), ideal \(I = \{\emptyset, \{c\}\}\). Let \(Y = \{1, 2, 3\}\) with a topology \(\kappa = \{\emptyset, Y, \{1\}, \{1, 2\}\}\), ideal \(J = \{\emptyset, \{2\}\}\). Consider a mapping \(f : X \to Y\) defined by \(f(a) = 2 = f(b), f(c) = 1, f(d) = 3\). Then \(X = X_\tau^*\) but \(f\) is not \(P-I\)-continuous.

Remark 4.17. By Proposition 4.12 and Theorem 4.15, we can deduce that if \(f : (X, \tau, I) \to (Y, \kappa)\) is \(P-I\)-continuous, then the following statements are valid.

(i) \(X = X_\tau^*\).
(ii) \(\tau^I \cap I = \{\emptyset\}\), \(\tau^I\) is a set of all pre-open sets in \((X, \tau)\).
(iii) If \(A \in I\), then \(p\text{Int}(A) = \emptyset\).

5. \(P-I\)-irresolute Mappings

Definition 5.1. A mapping \(f : (X, \tau, I) \to (Y, \kappa, J)\) is said to be \(P-I\)-irresolute if \(f^{-1}(V) \subseteq P\text{TO}(X, \tau, I)\) for all \(V \subseteq P\text{TO}(Y, \kappa, J)\).

Definition 5.2. A mapping \(f : (X, \tau, I) \to (Y, \kappa, J)\) is said to be \(I\)-irresolute if \(f^{-1}(V) \subseteq I\text{O}(X, \tau, I)\) for all \(V \subseteq I\text{O}(Y, \kappa, J)\).

Example 5.3. Let \((X, \tau, I)\) be an ideal topological space with \(X = \{a, b, c, d\}\), \(\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}\), \(I = \{\emptyset, \{a\}\}\), and let \((Y, \kappa, J)\) be an ideal topological space with \(Y = \{1, 2, 3, 4\}\), \(\kappa = \{\emptyset, Y, \{1\}, \{1, 2\}\}\) and \(J = \{\emptyset, \{2\}\}\). Then
IO(X, τ, I) = {∅, {b}, {c}, {a, b}, {b, c}, {a, b, c}, {b, c, d}, X},
PIO(X, τ, I) = {∅, {b}, {c}, {b, c}, {b, c, d}},
IO(Y, κ, J) = {∅, {1}, {1, 2}, {1, 3}, {1, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, Y},
PIO(Y, κ, J) = {∅, {1}, {1, 3}, {1, 4}, {1, 3, 4}}.

(a) A mapping f : (X, τ, I) → (Y, κ, J) given by f(a) = 2, f(b) = 1, f(c) = 4 = f(d) is both P-I-irresolute and I-irresolute.

(b) A mapping g : (X, τ, I) → (Y, κ, J) given by g(a) = 2 = g(d), g(b) = 1, g(c) = 3 P-I-irresolute which is not I-irresolute.

(c) A mapping h : (X, τ, I) → (Y, κ, J) given by h(a) = 3, h(b) = 1, h(c) = 2 = h(d) I-irresolute which is not P-I-irresolute.

(d) A mapping i : (X, τ, I) → (Y, κ, J) given by i(a) = 1, i(b) = 2 = i(c), i(d) = 3 is neither I-irresolute nor P-I-irresolute.

The above example shows that an I-irresolute mapping and a P-I-irresolute mapping are independent.

**Theorem 5.4.** If a mapping f : (X, τ, I) → (Y, κ) satisfy the following conditions,

- f is P-I-continuous.
- f⁻¹(V*) ⊂ f⁻¹(V) or f⁻¹(V*) ⊂ f⁻¹(V)ₚ for each V ⊂ Y.

then f is both an I-irresolute mapping and a P-I-irresolute mapping.

*Proof.* Assume that f satisfy two conditions. It is sufficient to show that the inverse image of I-open set is P-I-open set because every P-I-open set is an I-open set by Theorem 2.8. Let A be an I-open set. Then A ⊂ Int(A*). Since f is P-I-continuous, f⁻¹(Int(A*)) is P-I-open and hence f⁻¹(Int(A*)) ⊂ pInt(f⁻¹(Int(A*))ₚ). It follows from the second condition that

\[
f⁻¹(A) ⊂ pInt(f⁻¹(Int(A*))ₚ)
\]
\[
⊂ pInt(f⁻¹(A*)ₚ)
\]
\[
⊂ pInt(f⁻¹(A)*). 
\]

Hence f⁻¹(A) is P-I-open. Since every P-I-open set is an I-open set by Theorem 2.8, f is both an I-irresolute mapping and a P-I-irresolute mapping. □

The following example shows that a P-I-continuous mapping is neither an I-irresolute mapping nor a P-I-irresolute mapping.

**Example 5.5.** Consider two ideal topological spaces (X, τ, I) and (Y, κ, J) where X = {a, b, c, d}, τ = {∅, X, {a, b}, {a, b, c}}, I = {∅, {c}}, Y = {1, 2, 3, 4}, κ = {∅, Y, {3}, {1, 2}, {1, 2, 3}}, and J = {∅, {1}}. Define a mapping f : (X, τ, I) →
(Y, κ, J) by f(a) = 3, f(b) = 1, f(c) = 4 and f(d) = 2. Then f is a P-I-continuous mapping. Note that A = \{2\} is both an I-open set and a P-I-open set in (Y, κ, J). But f^{-1}(A) = \{d\} is neither an I-open set nor a P-I-open set. Hence f is neither an I- irresolute mapping nor a I- irresolute mapping.

**Theorem 5.6.** If a mapping f : (X, τ, I) → (Y, κ, J) satisfy the following conditions,

- f is I-continuous.
- f^{-1}(V^*) ⊆ f^{-1}(V) or f^{-1}(V^*) ⊆ f^{-1}(V)^* for each V ⊆ Y.

then f is an I- irresolute mapping.

**Proof.** Assume that f satisfy two given conditions. Let A be an I-open set. Then A ⊆ Int(A^*). Since f is I-continuous, f^{-1}(Int(A^*)) is I-open. It follows that

\[
f^{-1}(A) ⊆ f^{-1}(Int(A^*))
\]

\[
⊆ Int(f^{-1}(Int(A^*))^*)
\]

\[
⊆ Int(f^{-1}(A^*))^*
\]

\[
⊆ Int(f^{-1}(A)^*)
\]

so that f^{-1}(A) is I-open. Therefore f is an I- irresolute mapping. □

The following example shows that although a mapping f satisfy two conditions of Theorem 5.6, f may not be a P-I- irresolute mapping.

**Example 5.7.** Let X = \{a, b, c, d\}, τ = \{∅, X, \{a, b\}, \{a, b, c\}\} and I = \{∅, \{c\}\}. Let Y = \{1, 2, 3\}, κ = \{∅, Y, \{1\}, \{1, 2\}\}, and J = \{∅, \{2\}\}. A mapping f : (X, τ, I) → (Y, κ, J) given by f(a) = f(c) = 1, f(b) = 2, and f(d) = 3 is I- irresolute and satisfy the condition

\[
f^{-1}(V^*) ⊆ f^{-1}(V) or f^{-1}(V^*) ⊆ f^{-1}(V)^* for each V ⊆ Y.
\]

But f is not P-I- irresolute because f^{-1}(\{1\}) = \{a, c\} \notin PIO(Y, κ, J).

**Theorem 5.8.** Let f : (X, τ, I) → (Y, κ, J) be a mapping. If

\[
f^{-1}(A^*_p) ⊆ pInt(f^{-1}(A^*_p))
\]

for each A ⊆ Y, then f is a P-I- irresolute mapping.

**Proof.** Let A be a P-I-open set. Then A ⊆ pInt(A^*_p) which implies that

\[
f^{-1}(A) ⊆ f^{-1}(pInt(A^*_p)) ⊆ f^{-1}(A^*_p) ⊆ pInt(f^{-1}(A^*_p))
\]

Hence f is a P-I- irresolute mapping. □
The converse of above theorem may not be true as seen in the following example.

**Example 5.9.** Let \( X = \{a, b, c, d\} \), \( \tau = \varnothing, X, \{a, b\}, \{a, b, c\} \) and \( \mathcal{I} = \{\varnothing, \{c\}\} \). Let \( Y = \{1, 2, 3\} \), \( \kappa = \varnothing, Y, \{1\}, \{1, 2\} \), and \( \mathcal{J} = \{\{2\}\} \). A mapping \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(a) = 1 \), \( f(b) = 2 \), \( f(c) = 3 \) and \( f(d) = 3 \) is \( P-\mathcal{I}\)- irresolute. For a set \( A = \{1\} \), we have

\[
 f^{-1}(A^*_p) = X \not\in \mathrm{pInt}(f^{-1}(A^*_p)) = \{a\}.
\]

If \( f^{-1}(A^*_p) \subset \mathrm{pInt}(f^{-1}(A^*_p)) \) for each \( A \subset Y \), then is \( f \) a \( \mathcal{I}\)- irresolute mapping? The answer is negative as seen in the following example.

**Example 5.10.** Let \( X = \{a, b, c, d\} \), \( \tau = \varnothing, X, \{c\}, \{a, b\}, \{a, b, c\} \) and \( \mathcal{I} = \{\varnothing, \{a\}\} \). Let \( Y = \{1, 2, 3, 4\} \), \( \kappa = \varnothing, Y, \{1\}, \{2, 3\} \), and \( \mathcal{J} = \{\{2\}\} \). A mapping \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(a) = 2 \), \( f(b) = 1 \), \( f(c) = 3 \), is satisfied \( f^{-1}(A^*_p) \subset \mathrm{pInt}(f^{-1}(A^*_p)) \) for each \( A \subset Y \). But \( f \) is not a \( \mathcal{I}\)- irresolute mapping because \( f^{-1}(\{1, 2\}) = \{a, b, d\} \not\in \mathcal{I}O(X, \tau, \mathcal{I}) \) for \( \{1, 2\} \in \mathcal{I}O(Y, \kappa, \mathcal{J}) \).

**Theorem 5.11.** Let \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J}) \) be a mapping. If

\[
 f^{-1}(A^*) \subset \mathrm{Int}(f^{-1}(A^*))
\]

for each \( A \subset Y \), then \( f \) is an \( \mathcal{I}\)- irresolute mapping.

**Proof.** Let \( A \) be an \( \mathcal{I}\)-open set. Then \( A \subset \mathrm{Int}(A^*) \) which implies that

\[
 f^{-1}(A) \subset f^{-1}(\mathrm{Int}(A^*)) \subset f^{-1}(A^*) \subset \mathrm{Int}(f^{-1}(A^*)).
\]

Hence \( f \) is an \( \mathcal{I}\)- irresolute mapping. \( \square \)

The converse of above theorem may not be true as seen in the following example.

**Example 5.12.** Let \( X = \{a, b, c, d\} \), \( \tau = \varnothing, X, \{a, b\}, \{a, b, c\} \) and \( \mathcal{I} = \{\varnothing, \{c\}\} \). Let \( Y = \{1, 2, 3\} \), \( \kappa = \varnothing, Y, \{1\}, \{1, 2\} \), and \( \mathcal{J} = \{\{2\}\} \). A mapping \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(a) = 1 \), \( f(c) = 3 \) and \( f(d) = 3 \) is \( \mathcal{I}\)- irresolute. For a set \( A = \{3\} \), we obtain

\[
 f^{-1}(A^*_p) = \{d\} \not\in \mathrm{pInt}(f^{-1}(A^*)) = \emptyset.
\]

If \( f^{-1}(A^*) \subset \mathrm{Int}(f^{-1}(A^*)) \) for each \( A \subset Y \), then is \( f \) a \( P-\mathcal{I}\)- irresolute mapping? The answer is negative as seen in the following example.

**Example 5.13.** Let \( X = \{a, b, c, d\} \), \( \tau = \varnothing, X, \{a, b\}, \{a, b, c\} \) and \( \mathcal{I} = \{\varnothing, \{c\}\} \). Let \( Y = \{1, 2, 3, 4\} \), \( \kappa = \varnothing, Y, \{3\}, \{1\}, \{1, 2\}, \{1, 2, 3\} \), and \( \mathcal{J} = \{\{1\}, \{3\}, \{1, 3\}\} \). A mapping \( f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J}) \) given by \( f(a) = 2 \), \( f(c) = 3 \), \( f(b) = 1 \), \( f(d) = 3 \),
is satisfied $f^{-1}(A^*) \subseteq \text{Int}(f^{-1}(A^*))$ for each $A \subseteq Y$. But $f$ is not a $P$-$\mathcal{I}$-irresolute mapping because $f^{-1}(\{2\}) = \{a, c\} \notin \text{PTO}(X, \tau, \mathcal{I})$ for $\{2\} \in \text{PTO}(Y, \kappa, \mathcal{J})$.

**Lemma 5.14** ([18]). Let $A$ be a subset in an ideal topological space $(X, \tau, \mathcal{I})$. Then $p\text{Int}(A^*_p) \subseteq \text{Int}(A^*)$.

**Lemma 5.15** ([18]). For any subset $A$ of an ideal topological space $(X, \tau, \mathcal{I})$, we have

(i) $A^*_p \subseteq A^*$.

(ii) $A^*_p \subseteq p\text{Cl}(A)$.

**Corollary 5.16.** Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J})$ be a mapping. If $f^{-1}(A^*) \subseteq p\text{Int}(f^{-1}(A^*_p))$ for each $A \subseteq Y$, then $f$ is both an $\mathcal{I}$-irresolute mapping and a $P$-$\mathcal{I}$-irresolute mapping.

**Proof.** Since $A^*_p \subseteq A^*$ by Lemma 5.15(i), $f^{-1}(A^*_p) \subseteq f^{-1}(A^*)$. It follows that $f^{-1}(A^*_p) \subseteq f^{-1}(A^*) \subseteq p\text{Int}(f^{-1}(A^*_p)) \subseteq \text{Int}(f^{-1}(A^*))$ by the hypothesis and Lemma 5.14. Thus $f$ is both $P$-$\mathcal{I}$-irresolute and $\mathcal{I}$-irresolute by Theorem 5.8 and Theorem 5.11. □

**Theorem 5.17.** For two mappings $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J})$ and $g : (Y, \kappa, \mathcal{J}) \rightarrow (Z, \delta, \mathcal{H})$, the following statements are valid.

(i) If $f$ is $P$-$\mathcal{I}$-irresolute and $g$ is $P$-$\mathcal{I}$-irresolute, then $g \circ f$ is $P$-$\mathcal{I}$-irresolute.

(ii) If $f$ is $P$-$\mathcal{I}$-irresolute and $g$ is $P$-$\mathcal{I}$-continuous, then $g \circ f$ is $P$-$\mathcal{I}$-continuous.

(iii) If $f$ is $\mathcal{I}$-irresolute and $g$ is $\mathcal{I}$-irresolute, then $g \circ f$ is $\mathcal{I}$-irresolute.

(iv) If $f$ is $\mathcal{I}$-irresolute and $g$ is $\mathcal{I}$-continuous, then $g \circ f$ is $\mathcal{I}$-continuous.

(v) If $f$ is $\mathcal{I}$-irresolute and $g$ is $\mathcal{I}$-continuous, then $g \circ f$ is $\mathcal{I}$-continuous.

**Proof.** Straightforward. □

**Theorem 5.18.** Let mapping $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \kappa, \mathcal{J})$, $g : (Y, \kappa, \mathcal{J}) \rightarrow (Z, \delta, \mathcal{H})$. If $g$ is an injective mapping then the followings are valid.

(i) If $g$ is $P$-$\mathcal{I}$-open and $g \circ f$ is $P$-$\mathcal{I}$-irresolute, then $f$ is $P$-$\mathcal{I}$-continuous.

(ii) If $g$ is $\mathcal{I}$-open and $g \circ f$ is $\mathcal{I}$-irresolute, then $f$ is $\mathcal{I}$-continuous.

(iii) If $g$ is open and $g \circ f$ is $P$-$\mathcal{I}$-continuous, then $f$ is $P$-$\mathcal{I}$-continuous.

(iv) If $g$ is open and $g \circ f$ is $\mathcal{I}$-continuous, then $f$ is $\mathcal{I}$-continuous.

**Proof.** Straightforward. □
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