ON GENERALIZED UPPER SETS IN $BE$-ALGEBRAS

SUN SHIN AHN AND KEUM SOOK SO

Abstract. In this paper, we develop the idea of a generalized upper set in a $BE$-algebra. Furthermore, these sets are considered in the context of transitive and self distributive $BE$-algebras and their ideals, providing characterizations of one type, the generalized upper sets, in terms of the other type, ideals.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: $BCK$-algebras and $BCI$-algebras ([5, 6]). It is known that the class of $BCK$-algebras is a proper subclass of the class of $BCI$-algebras. In [3, 4], Q. P. Hu and X. Li introduced a wide class of abstract algebras: $BCH$-algebras. They have shown that the class of $BCI$-algebras is a proper subclass of the class of $BCH$-algebras. J. Neggers and H. S. Kim ([10]) introduced the notion of $d$-algebras which is another generalization of $BCK$-algebras. S. S. Ahn and Y. H. Kim ([1]) gave some constructions of implicative commutative $d$-algebras which are not $BCK$-algebras. Y. B. Jun, E. H. Roh, and H. S. Kim ([7]) introduced the notion of $BH$-algebra, which is a generalization of $BCH/BCI/BCK$-algebras. In [8], H. S. Kim and Y. H. Kim introduced the notion of a $BE$-algebra as a dualization of generalization of a $BCK$-algebra. Using the notion of upper sets they provided an equivalent condition describing filters in $BE$-algebras. Using the notion of upper sets they gave an equivalent condition for a subset to be a filter in $BE$-algebras. In [2], we introduced the notion of ideals in $BE$-algebras, and then stated and proved several characterizations of such ideals.

In this paper, we generalize the notion of upper sets in $BE$-algebras, and discuss properties of the characterizations of generalized upper sets $A_n(u, v)$ while relating them to the structure of ideals in transitive and self distributive $BE$-algebras.

2. Preliminaries

We recall some definitions and results (See [2, 8]).
Definition 2.1. An algebra \((X; *, 1)\) of type \((2, 0)\) is called a BE-algebra \([8]\) if

- (BE1) \(x * x = 1\) for all \(x \in X\);
- (BE2) \(x * 1 = 1\) for all \(x \in X\);
- (BE3) \(1 * x = x\) for all \(x \in X\);
- (BE4) \(x * (y * z) = y * (x * z)\) for all \(x, y, z \in X\). (exchange)

We introduce a relation “\(\leq\)” on \(X\) by \(x \leq y\) if and only if \(x * y = 1\). Note that if \((X; *, 1)\) is a BE-algebra, then \(x * (y * x) = 1\) for any \(x, y \in X\).

Example 2.2 \([8]\). Let \(X := \{1, a, b, c, d, 0\}\) be a set with the following table:

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Then \((X; *, 1)\) is a BE-algebra.

Definition 2.3. A BE-algebra \((X, *, 1)\) is said to be self distributive \([8]\) if \(x * (y * z) = (x * y) * (x * z)\) for all \(x, y, z \in X\).

Example 2.4 \([8]\). Let \(X := \{1, a, b, c, d\}\) be a set with the following table:

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It is easy to see that \(X\) is a BE-algebra satisfying self distributivity.

Note that the BE-algebra in Example 2.2 is not self distributive, since \(d * (a * 0) = d * d = 1\), while \((d * a) * (d * 0) = 1 * a = a\).

Definition 2.5 \([2]\). A non-empty subset \(I\) of \(X\) is called an ideal of \(X\) if

- (I1) \(x \in X\) and \(a \in I\) imply \(x * a \in I\), i.e., \(X * I \subseteq I\);
- (I2) \(x \in X\), \(a, b \in I\) imply \((a * (b * x)) * x \in I\).

In Example 2.2, \(\{1, a, b\}\) is an ideal of \(X\), but \(\{1, a\}\) is not an ideal of \(X\), since \((a * (a * b)) * b = (a * a) * b = 1 * b = b \notin \{1, a\}\).

It was proved that every ideal \(I\) of a BE-algebra \(X\) contains 1, and if \(a \in I\) and \(x \in X\), then \((a * x) * x \in I\). Moreover, if \(I\) is an ideal of \(X\) and if \(a \in I\) and \(a \leq x\), then \(x \in I\) (see \([2]\)).

Lemma 2.6 \([2]\). Let \(I\) be a subset of \(X\) such that...
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(I3) $1 \in I$;

(I4) $x * (y * z) \in I$ and $y \in I$ imply $x * z \in I$ for all $x, y, z \in X$.

If $a \in I$ and $a \leq x$, then $x \in I$.

\textbf{Definition 2.7.} A \textit{BE}-algebra $(X; *, 1)$ is said to be \textit{transitive} ([2]) if for any $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

\textbf{Example 2.8 ([2])}. Let $X := \{1, a, b, c\}$ be a set with the following table:

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Then $X$ is a transitive \textit{BE}-algebra.

\textbf{Proposition 2.9 ([2])}. If $X$ is a self distributive \textit{BE}-algebra, then it is transitive.

The converse of Proposition 2.9 need not be true in general. In Example 2.8, $X$ is a transitive \textit{BE}-algebra, but $a \ast (a \ast b) = a \ast a = 1$, while $(a \ast a) \ast (a \ast b) = 1 \ast a = a$, showing that $X$ is not self distributive.

\textbf{Theorem 2.10 ([2])}. Let $X$ be a transitive \textit{BE}-algebra. A subset $I (\neq \emptyset)$ of $X$ is an ideal of $X$ if and only if it satisfies conditions (I3) and (I4).

\section{Main results}

In what follows let $X$ denote a \textit{BE}-algebra unless otherwise specified. For any elements $u$ and $v$ of $X$ and $n \in \mathbb{N}$, we use the notation $u^n \ast v$ instead of $u * (\cdots (u * v) \cdots)$ in which $u$ occurs $n$ times. Let $X$ be a \textit{BE}-algebra and let $u, v \in X$. Define

$$A(u, v) := \{ z \in X | u * (v * z) = 1 \}$$

We call $A(u, v)$ an \textit{upper set} ([8]) of $u$ and $v$. It is easy to see that $1, u, v \in A(u, v)$ for any $u, v \in X$. We generalize the notion of the upper set $A(u, v)$ using the concept of $u^n \ast v$ as follows.

For any $u, v \in X$, consider a set

$$A_n(u, v) := \{ z \in X | u^n \ast (v * z) = 1 \}.$$ 

We call $A_n(u, v)$ an \textit{generalized upper set} of $u$ and $v$ in a \textit{BE}-algebra $X$. In Example 2.2, the set $A_n(1, a) = \{1, a\}$ is not an ideal of $X$. Hence we know that $A_n(u, v)$ may not be an ideal of $X$ in general.

\textbf{Theorem 3.1}. If $X$ is a self distributive \textit{BE}-algebra, then $A_n(u, v)$ is an ideal of $X$, $\forall u, v \in X$, where $n \in \mathbb{N}$. 

Proof. Let \( a \in A_n(u, v) \) and \( x \in X \). Then \( u^n \ast (v \ast a) = 1 \). It follows from the self distributivity law that
\[
\begin{align*}
\quad u^n \ast (v \ast (x \ast a)) \\
= u^{n-1} \ast [u \ast (v \ast (x \ast a))] \\
= u^{n-1} \ast [u \ast ((u \ast (x \ast a))) \ast (v \ast x)] \\
= u^{n-1} \ast ([u \ast (v \ast x)] \ast [u \ast (v \ast a)]) \\
= (u^{n-1} \ast [u \ast (v \ast x)]) \ast (u^{n-1} \ast [u \ast (v \ast a)]) \\
= (u^{n-1} \ast (u \ast (v \ast x))) \ast (u^{n-1} \ast (u \ast (v \ast a))) \\
= 1 \\
\end{align*}
\]
whence \( x \ast a \in A_n(u, v) \). Thus, (I) holds.

Let \( a, b \in A_n(u, v) \) and \( x \in X \). Then \( u^n \ast (v \ast a) = 1 \) and \( u^n \ast (v \ast b) = 1 \). It follows from the self distributivity law that
\[
\begin{align*}
\quad u^n \ast (v \ast ((a \ast (b \ast x)) \ast x)) \\
= u^{n-1} \ast (u \ast [v \ast ((a \ast (b \ast x)) \ast x)]) \\
= u^{n-1} \ast (u \ast [(v \ast (a \ast (b \ast x))) \ast (v \ast x)]) \\
= u^{n-1} \ast ([u \ast (v \ast (a \ast (b \ast x)))) \ast (u \ast (v \ast x))] \\
= (u^{n-1} \ast ((u \ast (v \ast a)) \ast (u \ast (v \ast (b \ast x)))))) \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [(u^{n-1} \ast (u \ast (v \ast a))) \ast (u^{n-1} \ast [u \ast (v \ast x)])] \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [1 \ast (u^{n-1} \ast (u \ast (v \ast (b \ast x))))]) \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [u^{n-1} \ast (u \ast ((v \ast b) \ast (v \ast x))))] \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= ([u^{n-1} \ast (u \ast (v \ast b)) \ast (u^{n-1} \ast (v \ast x))]) \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [(u \ast (v \ast b)) \ast (u^{n-1} \ast (v \ast x))] \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [1 \ast (u^{n-1} \ast (v \ast x))] \ast (u^{n-1} \ast [u \ast (v \ast x)]) \\
= [u^{n-1} \ast (v \ast x)] \ast [u^{n-1} \ast (u \ast (v \ast x))] \\
= u^{n-1} \ast [[(v \ast x) \ast (u \ast (v \ast x))] \\
= u^{n-1} \ast [u \ast ((v \ast x) \ast (v \ast x))] \\
= u^{n-1} \ast (u \ast 1) \\
= u^{n-1} \ast 1 = 1
\end{align*}
\]
The proof is straightforward.

Let $A_n(u,v) = A_n(y,x)$ for all $x \in X$, where $n \in \mathbb{N}$.

Proof. The proof is straightforward.

Example 3.3. Let $X := \{1, a, b, c, d\}$ be a set with the following table:

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Then $X$ is a self distributive $BE$-algebra. By Lemma 3.2, we have $A_n(x,d) = A_n(d,x) = X$ for all $x \in X$. Furthermore, we have that $A_n(1,1) = 1$, $A_n(1,a) = A_n(a,1) = A_n(a,a) = A_n(a,b) = \{1, a\}$, $A_n(1,b) = A_n(b,1) = A_n(b,b) = \{1, b\}$, $A_n(1,c) = A_n(a,c) = A_n(c,1) = A_n(c,a) = A_n(c,c) = \{1, a, c\}$, $A_n(b,a) = \{1, a, b\}$, and $A_n(c,b) = X$ are ideals of $X$, where $n \in \mathbb{N}$.

Using the notion of upper set $A(u,v)$, we given an equivalent condition for a non-empty subset to be an ideal in $BE$-algebras.

Theorem 3.4. Let $X$ be a transitive $BE$-algebra. A subset $I (\neq \emptyset)$ of $X$ is an ideal of $X$ if and only $A_n(u,v) \subseteq I, \forall u, v \in I$, where $n \in \mathbb{N}$.

Proof. Assume that $I$ is an ideal of $X$. If $z \in A_n(u,v)$, then $u^n \ast (v \ast z) = 1$ and so $z = 1 \ast z = (u^n \ast (v \ast z)) \ast z \in I$ by (I2). Hence $A_n(u,v) \subseteq I$.

Conversely, suppose that $A_n(u,v) \subseteq I$ for all $u, v \in I$. Note that $1 \in A_n(u,v) \subseteq I$. Hence (I3) holds. Let $x, y, z \in X$ with $x \ast (y \ast z), y \in I$. Since

$$
(x \ast (y \ast z))^n \ast (y \ast (x \ast z)) = (x \ast (y \ast z))^{n-1} \ast [(x \ast (y \ast z)) \ast (y \ast (x \ast z))] \\
= (x \ast (y \ast z))^{n-1} \ast [(x \ast (y \ast z)) \ast (x \ast (y \ast z))] \\
= (x \ast (y \ast z))^{n-1} \ast 1 = 1,
$$

we have $x \ast z \in A_n(x \ast (y \ast z), y) \subseteq I$. Hence (I4) holds. By Theorem 2.10, $I$ is an ideal of $X$.

Corollary 3.5. Let $X$ be a self distributive $BE$-algebra. A subset $I (\neq \emptyset)$ of $X$ is an ideal of $X$ if and only $A_n(u,v) \subseteq I, \forall u, v \in I$, where $n \in \mathbb{N}$.

Proof. The proof follows from Proposition 2.9 and Theorem 2.10.
Theorem 3.6. Let $X$ be a transitive BE-algebra. If $I$ is an ideal of $X$, then

$$I = \bigcup_{u,v \in I} A_n(u,v),$$

where $n \in \mathbb{N}$.

Proof. Let $I$ be an ideal of $X$ and let $x \in I$. Obviously, $x \in A_n(u,1)$ and so

$$I \subseteq \bigcup_{x \in I} A_n(x,1) \subseteq \bigcup_{u,v \in I} A_n(u,v).$$

Now, let $y \in \bigcup_{u,v \in I} A(u,v)$. Then there exist $a, b \in I$ such that $y \in A_n(a,b) \subseteq I$ by Theorem 3.4. Hence $y \in I$. Therefore $\bigcup_{u,v \in I} A_n(u,v) \subseteq I$. This completes the proof. \[\square\]

Corollary 3.7. Let $X$ be a self distributive BE-algebra. If $I$ is an ideal of $X$, then

$$I = \bigcup_{u,v \in I} A_n(u,v),$$

where $n \in \mathbb{N}$.

Proof. The proof follows from Proposition 2.9 and Theorem 3.6. \[\square\]

Corollary 3.8. Let $X$ be a transitive BE-algebra. If $I$ is an ideal of $X$, then

$$I = \bigcup_{w \in I} A_n(w,1),$$

where $n \in \mathbb{N}$.

Corollary 3.9. Let $X$ be a self distributive BE-algebra. If $I$ is an ideal of $X$, then

$$I = \bigcup_{w \in I} A_n(w,1),$$

where $n \in \mathbb{N}$.

References


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