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A NOTE ON GT-ALGEBRAS

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ABSTRACT. We introduce the notion of GT-algebras as a generalization of the concept of Tarski algebras. We introduce the notion of GT-filters in GT-algebras, and we prove some properties of GT-filters.

1. INTRODUCTION

The notion of Tarski algebras was introduced by J. C. Abbott in [2]. These algebras are an algebraic counterpart of the $\{\vee, \to\}$-fragment of the propositional classical calculus. S. A. Celani ([5]) introduced Tarski algebras equipped with a modal operator as a generalization of the concept of Boolean algebra with a modal operator which he researched into these fragments of the algebraic viewpoint. Properties of filters in Tarski algebras were treated by S. A. Celani ([5]) and the authors ([6]). Recently, the present authors ([6]) considered decompositions and expansions of filters in Tarski algebras, and also they have shown that there is no non-trivial quadratic Tarski algebras on a field X with $|X| \geq 3$. However, we feel that the concept of Tarski algebra is relatively too strong for filters. To deal with those, the algebraic structure should be treated in a more general setting, so-called a GT-algebra. In this paper, we shall introduce the notion of GT-algebras as a generalization of the concept of Tarski algebras. We introduce the notion of GT-filters in GT-algebras, and we shall prove some properties of GT-filters. Although the results of this paper are written in algebraic form, their own significance in theory of logics.

Let us review some definitions and results. By a *Tarski algebra* we mean an algebra $(X; \to, 1)$ of type $(2, 0)$ satisfying the following conditions:

(T1) $(\forall a \in X)(1 \to a = a)$.

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(T2) $(\forall a \in X)(a \to a = 1)$.
(T3) $(\forall a, b, c \in X)(a \to (b \to c) = (a \to b) \to (a \to c))$.
(T4) $(\forall a, b \in X)((a \to b) \to b = (b \to a) \to a)$.

In a Tarski algebra X we can define an order relation $\leq$ by setting $a \leq b$ if and only if $a \to b = 1$ for any $a, b \in X$. It is well known that $(X; \leq)$ is an ordered set and that X is a join-semilattice where the supremum of two elements $a, b \in X$ is defined by $a \vee b = (a \to b) \to b$ ([3]).

A non-empty subset F of a Tarski algebra X is said to be a *filter* if $1 \in F$, and $a \in F$ and $a \to b \in F$ imply $b \in F$ ([5]).

2. GENERALIZED TARSKI ALGEBRAS

Definition 2.1. By a *generalized Tarski algebra* (*GT-algebra*, for short) we mean an algebra $(X; \to, 1)$ of type $(2, 0)$ satisfying the following conditions: (T1), (T2), and (T3).

Example 2.2. (1) Every Tarski algebra is a GT-algebra.

(2) Let $X := \{a, b, c, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	1
a	1	1	c	1
b	1	1	c	1
c	1	1	1	1
1	a	b	c	1

It is routine to check that $(X; \to, 1)$ is a GT-algebra, which is not a Tarski algebra since $(a \to b) \to b = b \neq a = (b \to a) \to a$.

Proposition 2.3. *Let X be a GT-algebra. Then*

(p1) $(\forall a \in X)(a \to 1 = 1)$.
(p2) $(\forall a, b \in X)(a \to (b \to a) = 1)$.
(p3) $(\forall a, b \in X)(a \to (a \to b) = a \to b)$.
(p4) $(\forall a, b \in X)(a \to ((a \to b) \to b) = 1)$.
(p5) $(\forall a, b, c \in X)(a \to b = 1 \Rightarrow (c \to a) \to (c \to b) = 1)$.

Proof. (p1) Using (T2) and (T3), we have

$$a \to 1 = a \to (a \to a) = (a \to a) \to (a \to a) = 1.$$

(p2) Using (T2), (T3) and (p1), we get

$$a \to (b \to a) = (a \to b) \to (a \to a) = (a \to b) \to 1 = 1.$$

(p3) By (T1), (T2) and (T3), we have

$$a \to (a \to b) = (a \to a) \to (a \to b) = 1 \to (a \to b) = a \to b.$$

(p4) Using (T2), (T3) and (p3), we get $a \to ((a \to b) \to b) = (a \to (a \to b)) \to (a \to b) = (a \to b) \to (a \to b) = 1$.

(p5) Let $a, b \in X$ be such that $a \to b = 1$. Then we have

$$(c \to a) \to (a \to b) = c \to (a \to b) = c \to 1 = 1$$

for all $c \in X$. This completes the proof. □

A reflexive and transitive relation $\mathfrak{R}$ on a set X is called a *quasi-ordering* of X, and the couple $(X, \mathfrak{R})$ is called a *quasi-ordered set.*

We provide a method to make a GT-algebra from a quasi-ordered set.

Theorem 2.4. *Let $(X, \mathfrak{R})$ be a quasi-ordered set. Suppose that $1 \notin X$ and let $X1 = X \cup \{1\}$. Define a binary operation $\to$ on $X1$ as follows: $\forall a, b \in X1$,*

$$a \to b := \begin{cases} 1 & \text{if } (a, b) \in \mathfrak{R}, \\ b & \text{otherwise.} \end{cases}$$

Then $(X1; \to, 1)$ is a GT-algebra.

Proof. Since $(1, a) \notin \mathfrak{R}$ for every $a \in X1$, we have $1 \to a = a$ for all $a \in X1$. Thus $(X1; \to, 1)$ satisfies (T1). Since $\mathfrak{R}$ is reflexive, $a \to a = 1$ for all $a \in X$. This proves the condition (T2) holds. To verify the condition (T3), we consider the following four cases:

Case (1): $(a, b) \in \mathfrak{R}$ and $(b, c) \in \mathfrak{R}$ imply that $(a, c) \in \mathfrak{R}$, and so

$$a \to (b \to c) = a \to 1 = 1 = 1 \to 1 = (a \to b) \to (a \to c).$$

Case (2): $(a, b) \notin \mathfrak{R}$ and $(b, c) \in \mathfrak{R}$ imply that $a \to (b \to c) = a \to 1 = 1$. If $(a, c) \in \mathfrak{R}$, then $(a \to b) \to (a \to c) = b \to 1 = 1 = a \to (b \to c)$; if $(a, c) \notin \mathfrak{R}$, then $(a \to b) \to (a \to c) = b \to c = 1 = a \to (b \to c)$.

Case (3): $(a, b) \in \mathfrak{R}$ and $(b, c) \notin \mathfrak{R}$ imply that

$$a \to (b \to c) = a \to c = 1 \to (a \to c) = (a \to b) \to (a \to c).$$

Case (4): Let $(a, b) \notin \mathfrak{R}$ and $(b, c) \notin \mathfrak{R}$. If $(a, c) \in \mathfrak{R}$, then

$$a \to (b \to c) = a \to c = 1 = b \to 1 = (a, b) \to (a, c);$$

if $(a,c) \notin \mathfrak{R}$, then

$$a \to (b \to c) = a \to c = c = b \to c = (a,b) \to (a,c).$$

Hence the condition (T3) is true. This completes the proof. □

Employing the idea of Theorem 2.4, we construct a GT-algebra which is not a Tarski algebra.

Example 2.5. Let $X := \{a,b,c,d\}$ be a quasi-ordered set with the following relation

$$\begin{aligned}\mathfrak{R} \;:=\; & \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),\\ & (c,c),(c,d),(c,b),(d,d),(d,b),(d,c)\}.\end{aligned}$$

Then $(X1 := X \cup \{1\}; \to, 1)$ is a GT-algebra with the following Cayley table:

$\to$	a	b	c	d	1
a	1	1	1	1	1
b	a	1	1	1	1
c	a	1	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

Note that $X1$ is not a Tarski algebra since $(a \to b) \to b = b \neq 1 = (b \to a) \to a$.

Proposition 2.6. *Let $\mathfrak{R}X$ be a relation on a GT-algebra X defined by*

$$(x,y) \in \mathfrak{R}X \text{ if } x \to y = 1.$$

Then $\mathfrak{R}X$ is a quasi-ordering of X. Moreover,

(i) $(\forall a \in X)((a,1) \in \mathfrak{R}X)$.

(ii) *If $a \in X$ such that $(1,a) \in \mathfrak{R}X$, then $a = 1$.*

We call $\mathfrak{R}X$ the *induced quasi-ordering* of X.

Proof. Since $a \to a = 1$ for all $x \in X$, we get $(a,a) \in \mathfrak{R}X$, *i.e.*, $\mathfrak{R}X$ is reflexive. Let $a,b,c \in \mathfrak{R}X$ such that $(a,b) \in \mathfrak{R}X$ and $(b,c) \in \mathfrak{R}X$. Then $a \to b = 1$ and $b \to c = 1$. It follows from (p1), (T1) and (T3) that

$$a \to c = 1 \to (a \to c) = (a \to b) \to (a \to c) = a \to (b \to c) = a \to 1 = 1$$

so that $(a,c) \in \mathfrak{R}X$, *i.e.*, $\mathfrak{R}X$ is transitive.

(i) It is obvious by (p1).

(ii) Let $a \in X$ such that $(1,a) \in \mathfrak{R}X$. Then $a = 1 \to a = 1$. This completes the proof. □

In a GT-algebra X, we consider the following condition:

(P) $(\forall a, b, c \in X)(a \to b = 1 \Rightarrow (b \to c) \to (a \to c) = 1)$.

In Example 2.2 (2), $(X; \to, 1)$ satisfies the condition (P).

Proposition 2.7. *Let X be a GT-algebra with the condition (P) and $\mathfrak{R}X$ be the induced quasi-ordering of X. If $a, b \in X$ such that $(a, b) \in \mathfrak{R}X$, then $(c \to a, c \to b) \in \mathfrak{R}X$ and $(b \to c, a \to c) \in \mathfrak{R}X$ for all $c \in X$.*

Proof. Straightforward. □

For every quasi-ordering $\mathfrak{R}$ of a GT-algebra X, denote by $\mathfrak{C}\mathfrak{R}$ the relation on X given by

$$(a, b) \in \mathfrak{C}\mathfrak{R} \text{ iff } (a, b) \in \mathfrak{R} \text{ and } (b, a) \in \mathfrak{R}.$$

Obviously, $\mathfrak{C}\mathfrak{R}$ is an equivalence relation on X, which is called an *equivalence relation induced by* $\mathfrak{R}$. Denote by $[a]\mathfrak{C}\mathfrak{R}$ the equivalence class containing a and by $X/\mathfrak{C}\mathfrak{R}$ the collections of $[a]\mathfrak{C}\mathfrak{R}$, *i.e.*,

$$[a]\mathfrak{C}\mathfrak{R} := \{x \in X | (a, x) \in \mathfrak{C}\mathfrak{R}\}$$

and

$$X/\mathfrak{C}\mathfrak{R} := \{[a]\mathfrak{C}\mathfrak{R} | a \in X\}.$$

Define a relation $\leq \mathfrak{R}$ on $X/\mathfrak{C}\mathfrak{R}$ by

$$[a]\mathfrak{C}\mathfrak{R} \leq \mathfrak{R}[b]\mathfrak{C}\mathfrak{R} \text{ iff } (a, b) \in \mathfrak{R}.$$

Then $\leq \mathfrak{R}$ is a partial order on $X/\mathfrak{C}\mathfrak{R}$, and so $(X/\mathfrak{C}\mathfrak{R}, \leq \mathfrak{R})$ is a poset, which is called a *poset assigned to the quasi-ordered set* $(X, \mathfrak{R})$.

Let $\mathfrak{R}$ be a relation on a GT-algebra X. Then $\mathfrak{R}$ is said to be *compatible* if $(a \to e, b \to f) \in \mathfrak{R}$ whenever $(a, b) \in \mathfrak{R}$ and $(e, f) \in \mathfrak{R}$ for all $a, b, e, f \in X$. A compatible equivalence relation on X is said to be a *congruence* on X. The set

$$[1]\mathfrak{R} := \{x \in X | (1, x) \in \mathfrak{R}\}$$

is called the *kernel* of $\mathfrak{R}$.

Theorem 2.8. *Let X be a GT-algebra with the condition (P), $\mathfrak{R}X$ be the induced quasi-ordering of X, and let $\Theta = \mathfrak{C}\mathfrak{R}X$ be the equivalence relation induced by $\mathfrak{R}X$. Then*

(i) *Θ is a congruence relation on X with kernel $[1]\Theta = \{1\}$.*

(ii) *the quotient algebra $(X/\Theta; \Rrightarrow, [1]\Theta)$ is a GT-algebra, where the operation $\Rrightarrow$ on X/Θ is defined by*

$$[a]\Theta \Rrightarrow [b]\Theta := [a \to b]\Theta.$$

Proof. (i) Note that Θ is an equivalence relation on X. Let $a, b, e, f \in X$ such that $(a, b) \in \Theta$ and $(e, f) \in \Theta$. Then $(a, b) \in \mathfrak{R}X, (b, a) \in \mathfrak{R}X, (e, f) \in \mathfrak{R}X$ and $(f, e) \in \mathfrak{R}X$. So by Proposition 2.7, we get $(e \to a, f \to a) \in \mathfrak{R}X$ and $(f \to a, f \to b) \in \mathfrak{R}X$. By the transitivity of $\mathfrak{R}X$, we have $(e \to a, f \to b) \in \mathfrak{R}X$. Similarly, we have $(f \to b, e \to a) \in \mathfrak{R}X$. Hence

$$(a \to e, b \to f) \in \Theta.$$

i.e., Θ is a congruence relation on X. Now if $a \in [1]\Theta$, then $(1, a) \in \Theta$. It follows from Proposition 2.6 (ii) that $a = 1$. Therefore $[1]\Theta = \{1\}$.

(ii) Straightforward. □

Let X be a GT-algebra and $K(\neq \emptyset) \subseteq X$. Denote by ΘK the relation on X given by

$$(a, b) \in \Theta K \text{ iff } a \to b \in K \text{ and } b \to a \in K.$$

Lemma 2.9. *Let K be a nonempty subset K of a GT-algebra X. If ΘK is a reflexive relation on X, then $[1]\Theta K = K$.*

Proof. Let ΘK be a reflexive relation for a nonempty subset K of a GT-algebra X. Then $1 = a \to a \in K$. If $a \in K$, then $1 \to a = a \in K$ and $a \to 1 = 1 \in K$. Thus $(1, a) \in \Theta K$, *i.e.*, $a \in [1]\Theta K$. Conversely, if $a \in [1]\Theta K$ then $(1, a) \in \Theta K$ and so $a = 1 \to a \in K$. Therefore $[1]\Theta K = K$. □

3. GENERALIZED TARSKI-FILTERS

Definition 3.1. Let X be a GT-algebra. A nonempty subset F of X is called a *generalized Tarski-filter* (*GT-filter*, for short) of X if it satisfies the following conditions:

(F1) $(\forall a, b \in X)(b \in F \Rightarrow a \to b \in F)$.

(F2) $(\forall a, b \in X)(a \to b \in F, a \in F \Rightarrow b \in F)$.

Obviously, X and $\{1\}$ are GT-filters of X. Note that every GT-filter contains the element 1 by (T2) and (F1).

In Example 2.2 (2), the subset $\{a, b, 1\}$ is a GT-filter of X, but $\{a, 1\}$ is not a GT-filter of X.

Theorem 3.2. *Let K be a GT-filter of a GT-algebra X. Then the relation ΘK is* *an equivalence relation on X with the kernel* $[1]\Theta K = K$.

Proof. Since $x \to x = 1 \in K$ for all $x \in X$, we have $(x, x) \in \Theta K$. Obviously, ΘK is symmetric. Let $a, b, c \in X$ such that $(a, b) \in \Theta K$ and $(b, c) \in \Theta K$. Then $a \to b \in K, b \to a \in K, b \to c \in K$ and $c \to b \in K$. Since $b \to c \in K$, it follows from (T2) and (F1) that

$$(a \to b) \to (a \to c) = a \to (b \to c) \in K.$$

Using $a \to b \in K$ and (F2), we have $a \to c \in K$. Similarly, we have $c \to a \in K$. Therefore we obtain $(a, c) \in \Theta K$, *i.e.*, ΘK is an equivalence relation on X. By Lemma 2.9, we have $[1]\Theta K = K$. □

In Theorem 3.2, ΘK may not be compatible in general, as following example.

Example 3.3. Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	d	e	f	g	1
a	1	1	1	1	1	1	1	1
b	c	1	c	g	1	1	g	1
c	f	f	1	f	1	f	1	1
d	c	e	c	1	e	1	1	1
e	a	f	c	d	1	f	g	1
f	c	e	c	g	e	1	g	1
g	a	b	c	f	e	f	1	1
1	a	b	c	d	e	f	g	1

Then $(X; \to, 1)$ is a GT-algebra, and the subset $K := \{e, 1\}$ is a GT-filter of X. Moreover, we can find

$$\Theta K = \{(a, a), (b, b), (c, c), (d, d), (e, e), (e, 1), (f, f), (g, g), (1, e), (1, 1)\}.$$

It is routine to check that ΘK is an equivalence relation on X, which is not compatible since $(e, 1) \in \Theta K$ and $(b, b) \in \Theta K$, but $(e \to b, 1 \to b) = (f, b) \notin \Theta K$.

Theorem 3.4. *Let $\mathfrak{R}$ be a congruence relation on a GT-algebra X. Then the kernel $[1]\mathfrak{R}$ is a GT-filter of X.*

Proof. Let $a \in X$ and $b \in [1]\mathfrak{R}$. Then $(1, b) \in \mathfrak{R}$. Since $\mathfrak{R}$ is reflexive and compatible, it follows from (p1) that

$$(1, a \to b) = (a \to 1, a \to b) \in \mathfrak{R}$$

so that $a \to b \in [1]\mathfrak{R}$.

Let $a, b, c \in X$ such that $a \to b \in [1]\mathfrak{R}$ and $a \in [1]\mathfrak{R}$. Then $(1, a \to b) \in \mathfrak{R}$ and $(1, a) \in \mathfrak{R}$. Since $\mathfrak{R}$ is reflexive and compatible, we have

$$(b, a \to b) = (1 \to b, a \to b) \in \mathfrak{R}.$$

Since $\mathfrak{R}$ is symmetric and transitive, we have $(1, b) \in \mathfrak{R}$, and so $b \in [1]\mathfrak{R}$. □

The following example shows that the condition 'compatible' is necessary in the Theorem 3.4.

Example 3.5. Let $X := \{a, b, c, d, 1\}$ be a set with the following Cayley table:

$\to$	a	b	c	d	1
a	1	1	c	1	1
b	a	1	c	1	1
c	a	1	1	1	1
d	a	1	c	1	1
1	a	b	c	d	1

Then $(X; \to, 1)$ is a GT-algebra. Let

$$\mathfrak{R} := \{(a, a), (a, 1), (b, b), (b, d), (c, c), (d, b), (d, d), (1, a), (1, 1)\}.$$

It can be readily check that $\mathfrak{R}$ is an equivalence relation on X, which is not compatible since $(a, 1) \in \mathfrak{R}$ and $(d, d) \in \mathfrak{R}$ but $(a \to d, 1 \to d) = (1, d) \notin \mathfrak{R}$. Moreover, $[1]\mathfrak{R} = \{a, 1\}$ is not a GT-filter of X since $a \to d \in [1]\mathfrak{R}, d \notin [1]\mathfrak{R}$.

For any GT-algebra X and $x, y \in X$, we denote

$$A(x, y) := \{z \in X | x \to (y \to z) = 1\}.$$

Theorem 3.6. *Let X be a GT-algebra and $x, y \in X$. Then $A(x, y)$ is a GT-filter of X.*

Proof. Straightforward. □

Now, we give some characterization of GT-filters.

Theorem 3.7. *Let F be a nonempty subset of a GT-algebra X. Then F is a GT-filter of X if and only if for any $a, b \in F$, either $A(a, b) \subseteq F$ or $A(b, a) \subseteq F$.*

Proof. The necessity is straightforward. Suppose that for any $a, b \in F$, either $A(a, b) \subseteq F$ or $A(b, a) \subseteq F$. Let $c \in X$ and $d \in F$. Then it follows from (T3), (T2) and (p1) that $d \to (d \to (c \to d)) = d \to ((d \to c) \to (d \to d)) = d \to ((d \to c) \to 1) = d \to 1 = 1$, and so $c \to d \in F$ by assumption, which proves (F1). Let $c, d \in X$ such that $c \to d \in F$ and $c \in F$. Then we have $d \in A(c \to d, c)$. By assumption, we get $d \in F$, which proves (F2). Therefore, F is a GT-filter of X. □

Corollary 3.8. *Let F be a nonempty subset of a GT-algebra X. Then F is a GT-filter of X if and only if $F = \bigcup x, y \in FA(x,y)$.*

Theorem 3.9. *Let F be a nonempty subset of a GT-algebra X. Then F is a GT-filter of X if and only if it satisfies $1 \in F$ and (F2).*

Proof. The necessity is straightforward. Suppose that F satisfies $1 \in F$ and (F2). Let $x \in X$ and $a \in F$. Then $a \to (x \to a) = 1 \in F$ by (p2). It follows from the assumption and $a \in F$ that $x \to a \in F$. Therefore F is a GT-filter of X. □

Finally, we provide a method to make a GT-algebra from GT-filters.

Theorem 3.10. *Let $F(X)$ be the set of all GT-filters of a GT-algebra $(X; \to, 1)$. For any $F1, F2 \in F(X)$, we define*

$$F1 \Rrightarrow F2 := \{x \in X | [x) \cap F1 \subseteq F2\}.$$

Then $(F(X); \Rrightarrow, X)$ is a GT-algebra, where $[x) := \{z \in X | x \to z = 1\}$.

Proof. Let $F \in F(X)$. If $x \in X \Rrightarrow F$ then $x \in F$ and so $X \Rrightarrow F \subseteq F$. If $x \in F$ and $y \in [x) \cap X$ then we have $y \in F$, *i.e.*, $[x) \cap X \subseteq F$. Thus $x \in X \Rrightarrow F$. Hence $F \subseteq X \Rrightarrow F$. Therefore, $(F(X); \Rrightarrow, X)$ satisfies (T1).

For any $F \in F(X)$, obviously we get $F \Rrightarrow F \subseteq X$. If $x \in X$, then we have $[x) \cap F \subseteq F$, and so $x \in F \Rrightarrow F$. Hence $X \subseteq F \Rrightarrow F$. Therefore, $(F(X); \Rrightarrow, X)$ satisfies (T2).

To verify the condition (T3), we consider the following cases: Note that

$$\begin{aligned}
&F1 \Rrightarrow (F2 \Rrightarrow F3) \subseteq (F1 \Rrightarrow F2) \Rrightarrow (F1 \Rrightarrow F3)\\
\Leftrightarrow\ & x \in F1 \Rrightarrow (F2 \Rrightarrow F3) \Rightarrow x \in (F1 \Rrightarrow F2) \Rrightarrow (F1 \Rrightarrow F3)\\
\Leftrightarrow\ & [x) \cap F1 \subseteq F2 \Rrightarrow F3 \Rightarrow [x) \cap (F1 \Rrightarrow F2) \subseteq F1 \Rrightarrow F3\\
\Leftrightarrow\ & [x) \cap F1 \subseteq F2 \Rrightarrow F3 \;\&\; y \in [x) \cap (F1 \Rrightarrow F2) \Rightarrow y \in F1 \Rrightarrow F3\\
\Leftrightarrow\ & [x) \cap F1 \subseteq F2 \Rrightarrow F3 \;\&\; y \in [x) \cap (F1 \Rrightarrow F2) \;\&\; z \in [y) \cap F1 \Rightarrow z \in F3.
\end{aligned}$$

Let $[x) \cap F1 \subseteq F2 \Rrightarrow F3$, $y \in [x) \cap (F1 \Rrightarrow F2)$ and $z \in [y) \cap F1$ for all $F1, F2, F3 \in F(X)$. Then we have $y \in F1 \Rrightarrow F2$ implies that $z \in F2$. Since $z \in [x) \cap F1$, we have $z \in F2 \Rrightarrow F3$. Hence we get $z \in F3$. From the note above, we have $F1 \Rrightarrow (F2 \Rrightarrow F3) \subseteq (F1 \Rrightarrow F2) \Rrightarrow (F1 \Rrightarrow F3)$.

On the other hand, we can observe that

$$(F1 \Rrightarrow F2) \Rrightarrow (F1 \Rrightarrow F3) \subseteq F1 \Rrightarrow (F2 \Rrightarrow F3)$$

$$\Leftrightarrow [x) \cap (F1 \Rrightarrow F2) \subseteq F1 \Rrightarrow F3 \Rightarrow [x) \cap F1 \subseteq F2 \Rrightarrow F3$$

$$\Leftrightarrow [x) \cap (F1 \Rrightarrow F2) \subseteq F1 \Rrightarrow F3 \ \& \ y \in [x) \cap F1 \ \& \ z \in [y) \cap F2 \Rightarrow z \in F3.$$

Let $[x) \cap (F1 \Rrightarrow F2) \subseteq F1 \Rrightarrow F3$, $y \in [x) \cap F1$ and $z \in [y) \cap F2$ for all $F1, F2, F3 \in F(X)$. Since $y \in F1$ and $z \in [y)$ and $F1 \in F(X)$, we get $z \in F1$. Thus we have $z \in F1 \Rrightarrow F2$. Since $z \in [x) \cap (F1 \Rrightarrow F2)$, we get $z \in F1 \Rrightarrow F3$. Hence we obtain $z \in F3$ by $z \in [z) \cap F1$. Therefore the reverse inclusion $(F1 \Rrightarrow F2) \Rrightarrow (F1 \Rrightarrow F3) \subseteq F1 \Rrightarrow (F2 \Rrightarrow F3)$ holds. Hence (T3) follows for $(F(X); \Rrightarrow, X)$. □

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