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LEFSCHETZ FIXED POINT THEORY FOR COMPACT ABSORBING CONTRACTIVE ADMISSIBLE MAPS

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ABSTRACT. New Lefschetz fixed point theorems for compact absorbing contractive admissible maps between Fréchet spaces are presented. Also we present new results for condensing maps with a compact attractor. The proof relies on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces.

1. INTRODUCTION

This paper presents new Lefschetz fixed point theorems for compact absorbing contractive maps between Fréchet spaces. In addition we will discuss condensing maps with a compact attractor. The proofs rely on fixed point theory in Banach spaces and viewing a Fréchet space as the projective limit of a sequence of Banach spaces. In the literature [1, 2], one usually assumes the map F is defined on a subset X of a Fréchet space E and its restriction (again called F) is well defined on $\overline{X_n}$ (see Section 2). In general, of course, for Volterra operators, the restriction is always defined on X_n and in most applications it is in fact defined on $\overline{X_n}$ and usually even on E_n (see Section 2). In this paper, we make use of the fact that the restriction is well defined on X_n and we only assume it admits an extension (satisfying certain properties) on $\overline{X_n}$. We also show in Section 2 and Section 3 how easily one can extend fixed point theory in Banach spaces to fixed point theory in Fréchet spaces.

The existence in Section 2 and Section 3 will be based on some Lefschetz type fixed point theory. Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

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(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;

(ii) p is a proper map, i.e., for every compact $A \subseteq X$, we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$, where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that

$$q' \circ f = q, \ \ p' \circ f = p, \ \ q \circ g = q', \ \ p \circ g = p'.$$

The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. Let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$, a set $\phi(x) = q\,p^{-1}(x)$, where $\phi = [(p,q)]$, is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now, let $N(f) = \{x \in E : \ f^{(n)}(x) = 0 \text{ for some } n\}$, where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$, we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\,\tilde{E} < \infty$; for such f, we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$, where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if

(i) all f_q are admissible;

(ii) almost all $\tilde{E}_q$ are trivial.

For such f, we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q \, Tr\,(f_q).$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f: X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q}: H_q(X) \to H_q(X)$.

The Čech homology functor can be extended to a category of morphisms (see [5, p. 364]) and also note the homology functor H extends over this category, i.e., for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y,$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$.

Let $\phi : X \to Y$ be a multivalued map (note that, for each $x \in X$, we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:

(i) p is a Vietoris map;

(ii) $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.1. A upper semicontinuous map $\phi : X \to Y$ is said to be *admissible* (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ.

Definition 1.2. A map $\phi \in Ad(X, X)$ is said to be a *Lefschetz map* if, for each selected pair $(p, q) \subset \phi$, the linear map $q_\star p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}(\phi)$ (or $\mathbf{\Lambda}_X(\phi)$) by

$$\mathbf{\Lambda}(\phi) = \left\{\Lambda(q_\star p_\star^{-1}) : (p, q) \subset \phi\right\}.$$

Definition 1.3. A multivalued map $\phi : X \to 2^X$ is called a *compact absorbing contraction* if there exists an open set $U \subseteq X$ such that $\overline{\phi(U)}$ is a compact subset of U and $X \subseteq \cup_{i=0}^{\infty} \phi^{-i}(U)$, where 2^X denotes the family of nonempty subsets of X.

Definition 1.4. We say $\phi \in CAC(X, X)$ if $\phi \in Ad(X, X)$ and is a compact absorbing contraction.

Definition 1.5. A Hausdorff topological space X is said to be a *Lefschetz space* provided every $\phi \in CAC(X, X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Example 1.1. If $X \in ANR$, then X is a Lefschetz space (see [4, p. 208]).

Let (X, d) be a metric space and S be a nonempty subset of X. For any $x \in X$ let $d(x, S) = \inf_{y \in S} d(x, y)$. Also, $diam\, S = \sup\{d(x, y) : x, y \in S\}$. We let $B(x, r)$ denote the open ball in X centered at x of radius r and by $B(S, r)$ we denote $\cup_{x \in S} B(x, r)$. For two nonempty subsets S_1 and S_2 of X, we define the generalized Hausdorff distance H to be

$$H(S_1, S_2) = \inf\{\epsilon > 0 : \ S_1 \subseteq B(S_2, \epsilon), \ S_2 \subseteq B(S_1, \epsilon)\}.$$

Now, suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then G is said to be hemicompact if each sequence $\{x_n\}_{n \in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

In Section 3, we discuss condensing single valued maps. Now, with this in mind, let H be the singular homology functor (with coefficients in the field K) from the category of topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional singular homology group of X. For a continuous map $f : X \to Y$, $H(f)$ is the induced linear map $f_\star = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

Definition 1.6. A continuous map $f : X \to X$ is called a *Lefschetz map* provided $f_\star : H(X) \to H(X)$ is a Leray endomorphism. For such f, we define the Lefschetz number $\Lambda(f)$ (or $\Lambda_X(f)$) of f by putting $\Lambda(f) = \Lambda(f_\star)$.

Let X a a topological space and $f : X \to X$ be a continuous map with $x \in X$. Then the set

$$O(x) = \{x, \ f(x), \ \cdots, \ f^m(x), \cdots\}$$

is called the orbit of x under f.

Definition 1.7. We say that a compact set A is an *attractor* for $f : X \to X$ if, for every $x \in X$, we have

$$\overline{O(x)} \cap A \neq \emptyset,$$

where $\overline{O(x)}$ denotes the closure of $O(x)$ in X.

Let (X, d) be a metric space and Ω_X be the bounded subsets of X. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_X \to [0, \infty]$ defined by (where $A \in \Omega_X$)

$$\alpha(A) = \inf\{r > 0 : \ A \subseteq \cup_{i=1}^n A_i \ \text{ and } \ diam\,(A_i) \leq r\}.$$

Let S be a nonempty subset of X. Then the single valued map $G : S \to X$ is

(i) k–set contractive (here $k \geq 0$) if $G(S)$ is bounded and $\alpha(G(W)) \leq k\,\alpha(W)$ for all bounded sets W of S;

(ii) condensing if $G(S)$ is bounded, G is 1–set contractive and $\alpha(G(W)) < \alpha(W)$ for all bounded sets W of S with $\alpha(W) \neq 0$;

(iii) hemicompact if each sequence $\{x_n\}_{n\in N}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature [1].

Theorem 1.1. *Let (Y, d) be a metric space, D be a nonempty complete subset of Y and $G : D \to Y$ a condensing map. Then G is hemicompact.*

Definition 1.8. A space X is said to be a *CA Lefschetz space* provided any continuous condensing map $f : X \to X$ with a compact attractor is a Lefschetz map and $\Lambda_X(f) \neq 0$ implies f has a fixed point.

The following result is due to Nussbaum [3].

Example 1.2. If X is an open subset of a Banach space, then X is a CA Lefschetz space.

We say a closed bounded subset X of a Banach space E is a special ANR if there exists an open $U \subseteq E$ and a continuous map $r : U \to X$ with $X \subseteq U$, $r(x) = x$ for every $x \in X$ and, for every $A \subseteq U$, we have $\alpha(r(A)) \leq \alpha(A)$.

Definition 1.9. A space X is said to be a *special Lefschetz space* provided any continuous condensing map $f : X \to X$ is a Lefschetz map and $\Lambda_X(f) \neq 0$ implies f has a fixed point.

The following result is due to Gorniewicz [3].

Example 1.3. If X is a special ANR, then X is a special Lefschetz space.

Now, let I be a directed set with order $\leq$ and $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I$, $\beta \in I$ for which $\alpha \leq \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{ x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : \; x_\alpha = \pi_{\alpha,\beta}(x_\beta), \; \forall\, \alpha,\, \beta \in I,\, \alpha \leq \beta \right\}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow} \{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection ([7, pp. 439]) $\bigcap_{\alpha \in I} E_\alpha$.)

2. FIXED POINT THEORY IN FRÉCHET SPACES

Let $E = (E, \{|\cdot|_n\}_{n\in N})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in N\}$, where $N = \{1, 2, \cdots\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \cdots \quad \text{for every } x \in E. \tag{2.1}$$

A subset X of E is bounded if, for every $n \in N$, there exists $r_n > 0$ such that $|x|_n \le r_n$ for all $x \in X$. For any $r > 0$ and $x \in E$, we denote $B(x,r) = \{y \in E : |x-y|_n \le r, \ \forall n \in N\}$. To E, we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in N$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{if and only if} \quad |x-y|_n = 0. \tag{2.2}$$

We denote by $\mathbf{E}^n = (E \,/\sim_n, |\cdot|_n)$ the quotient space and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now, since (2.1) is satisfied, the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also, (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m \,/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. Now, $\mu_{n,m}\,\mu_{m,k} = \mu_{n,k}$ if $n \le m \le k$ and $\mu_n = \mu_{n,m}\,\mu_m$ if $n \le m$. We now assume the following condition holds:

$$\begin{cases} \text{for each } n \in N, \text{ there exists a Banach space } (E_n, |\cdot|_n) \\ \text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \to E_n. \end{cases} \tag{2.3}$$

Remark 2.1. (1) For convenience, the norm on E_n is denoted by $|\cdot|_n$.

(2) In our applications, $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in N$.

(3) Note that, if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However, if $x \in E_n$, then x is not necessaily in E and, in fact, E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example, if $E = C[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n = C[0,n]$.

Finally, we assume that

$$(2.4)\qquad \begin{cases} E_1 \supseteq E_2 \supseteq \cdots \text{ and, for each } n \in N, \\ |j_n\, \mu_{n,n+1}\, j_{n+1}^{-1}\, x|_n \leq |x|_{n+1},\ \forall\, x \in E_{n+1} \end{cases}$$

(where we use the notation from [7], i.e., decreasing in the generalized sense). Let $\lim_{\leftarrow} E_n$ (or $\cap_1^\infty E_n$, where $\cap_1^\infty$ is the generalized intersection [7]) denote the projective limit of $\{E_n\}_{n\in N}$ (note $\pi_{n,m} = j_n\, \mu_{n,m}\, j_m^{-1} : E_m \to E_n$ for $m \geq n$) and note $\lim_{\leftarrow} E_n \cong E$, so, for convenience, we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in N$, we set $X_n = j_n\, \mu_n(X)$, and we let $\overline{X_n}$, $int\, X_n$ and ∂X_n denote, respectively, the closure, the interior and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also, the pseudo-interior of X is defined by

$$pseudo - int\,(X) = \{x \in X :\ j_n\, \mu_n(x) \in \overline{X_n} \setminus \partial X_n \text{ for every } n \in N\}.$$

The set X is pseudo-open if $X = pseudo - int\,(X)$. For $r > 0$ and $x \in E_n$, we denote $B_n(x,r) = \{y \in E_n :\ |x - y|_n \leq r\}$.

Let $M \subseteq E$ and consider the map $F : M \to 2^E$. Assume that, for each $n \in N$ and $x \in M$, $j_n\, \mu_n\, F\,(x)$ is closed. Let $n \in N$ and $M_n = j_n\, \mu_n(M)$. Since we only consider Volterra type operators, we assume

$$(2.5)\qquad \text{if } x,\, y \in E \text{ with } |x - y|_n = 0 \text{ then } H_n(F\,x, F\,y) = 0,$$

where H_n denotes the appropriate generalized Hausdorff distance (alternatively, we could assume that, for all $n \in N$ and $x,\, y \in M$, if $j_n\, \mu_n\, x = j_n\, \mu_n\, y$, then $j_n\, \mu_n\, F\, x = j_n\, \mu_n\, F\, y$ and, of course, here we do not need to assume that $j_n\, \mu_n\, F\,(x)$ is closed for each $n \in N$ and $x \in M$). Now, (2.5) guarantees that we can define (a well defined) F_n on M_n as follows:

For any $y \in M_n$ there exists $x \in M$ with $y = j_n\, \mu_n(x)$ and we let

$$F_n\, y = j_n\, \mu_n\, F\, x$$

(we could of course call it $F\, y$ since it is clear in the situation we use it); note that $F_n : M_n \to C(E_n)$ and, if there exists $z \in M$ with $y = j_n\, \mu_n(z)$, then $j_n\, \mu_n\, F\, x = j_n\, \mu_n\, F\, z$ from (2.5) (where $C(E_n)$ denotes the family of nonempty closed subsets of E_n). In this paper, we assume that F_n will be defined on $\overline{M_n}$, i.e., we assume that the F_n described above admits an extension (again, we call it F_n) $F_n : \overline{M_n} \to 2^{E_n}$ (we will assume certain properties on the extension).

Now, we present some Lefschetz type theorems in Fréchet spaces. Let E and E_n be as described above.

Definition 2.1. A set $A \subseteq E$ is said to be PRLS if, for each $n \in N$, $A_n \equiv j_n \mu_n (A)$ is a Lefschetz space.

Definition 2.2. A set $A \subseteq E$ is said to be CPRLS if, for each $n \in N$, $\overline{A_n}$ is a Lefschetz space.

Example 2.1. Let A be pseudo-open. Then A is a PRLS.

To see this, fix $n \in N$. We now show

$$A_n \text{ is a open subset of } E_n.$$

First, notice that $A_n \subseteq \overline{A_n} \setminus \partial A_n$. In fact, if $y \in A_n$, then there exists $x \in A$ with $y = j_n \mu_n(x)$ and this together with $A = pseudo-int\, A$ yields $j_n \mu_n(x) \in \overline{A_n} \setminus \partial A_n$, i.e., $y \in \overline{A_n} \setminus \partial A_n$. In addition, notice that

$$\overline{A_n} \setminus \partial A_n = (int\, A_n \cup \partial A_n) \setminus \partial A_n = int\, A_n \setminus \partial A_n = int\, A_n$$

since $int\, A_n \cap \partial A_n = \emptyset$. Consequently, we have

$$A_n \subseteq \overline{A_n} \setminus \partial A_n = int\, A_n, \quad \text{so} \quad A_n = int\, A_n.$$

As a result, A_n is open in E_n. Thus A_n is a Lefschetz space (see Example 1.1), so A is a PRLS.

Theorem 2.1. *Let E and E_n be as described above, $C \subseteq E$ is an PRLS and $F : C \to 2^E$. Also, assume that, for each $n \in N$ and $x \in C$, $j_n \mu_n F(x)$ is closed and $F_n : \overline{C_n} \to 2^{E_n}$ as described above is a closed map with $x \notin F_n(x)$ in E_n for any $x \in \partial C_n$. Suppose that the following conditions are satisfied:*

$$(2.6) \qquad \begin{cases} \text{for each } n \in N, \ F_n \in CAC(C_n, C_n) \text{ and} \\ F_n : \overline{C_n} \to 2^{E_n} \text{ is hemicompact;} \end{cases}$$

$$(2.7) \qquad \text{for each } n \in N, \ \mathbf{\Lambda}_{C_n}(F_n) \neq \{0\};$$

$$(2.8) \qquad \begin{cases} \text{for each } n \in \{2, 3, \cdots\} \text{ if } y \in C_n \text{ solves } y \in F_n y \text{ in } E_n \\ \text{then } j_k \mu_{k,n} j_n^{-1}(y) \in C_k \text{ for } k \in \{1, \cdots, n-1\}. \end{cases}$$

Then F has a fixed point in E.

Proof. For each $n \in N$, there exists $y_n \in C_n$ with $y_n \in F_n y_n$ in E_n. Let's look at $\{y_n\}_{n \in N}$. Notice that $y_1 \in C_1$ and $j_1 \mu_{1,k} j_k^{-1}(y_k) \in C_1$ for $k \in N \setminus \{1\}$ from (2.8). Note that $j_1 \mu_{1,n} j_n^{-1}(y_n) \in F_1 (j_1 \mu_{1,n} j_n^{-1}(y_n))$ in E_1; to see note, for each $n \in N$ fixed, there exists $x \in E$ with $y_n = j_n \mu_n(x)$, so $j_n \mu_n(x) \in F_n(y_n) = j_n \mu_n F(x)$

on E_n, so on E_1, we have

$$\begin{aligned} j_1\,\mu_{1,n}\,j_n^{-1}(y_n) &= j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x) \in j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,F(x) \\ &= j_1\,\mu_{1,n}\,\mu_n\,F(x) = j_1\,\mu_1\,F(x) = F_1(j_1\,\mu_1\,(x)) \\ &= F_1(j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x)) = F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n)). \end{aligned}$$

Now, (2.6) guarantees that there exists is a subsequence $N_1^\star$ of N and $z_1 \in \overline{C_1}$ with $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 \in F_1\,z_1$ since F_1 is a closed map. Note that $z_1 \in C_1$ since $x \notin F_1(x)$ in E_1 for any $x \in \partial\,C_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Now, $j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n) \in C_2$ for each $n \in N_1$ together with (2.6) guarantees that there exists subsequence $N_2^\star$ of N_1 and $z_2 \in \overline{C_2}$ with $j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 \in F_2\,z_2$. Also, $z_2 \in C_2$. Note that, from (2.4) and the uniqueness of limits, $j_1\,\mu_{1,2}\,j_2^{-1}\,z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note that $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) = j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n)$ for each $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots, \quad N_k^\star \subseteq \{k, k+1, \cdots\}$$

and $z_k \in \overline{C_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k \in F_k\,z_k$. Also, $z_k \in C_k$. Note that $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \cdots\}$. Also, let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now, $z_k \in F_k\,z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = \cdots = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for every $m \geq k$. We can do this for each $k \in N$. As a result, $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in C_k$ for each $k \in N$. Thus, for each $k \in N$, we have

$$j_k\,\mu_k\,(y) = z_k \in F_k\,z_k = j_k\,\mu_k\,F\,y \quad \text{in} \quad E_k,$$

so $y \in F\,y$ in E. □

Remark 2.2. Of course, one could remove $x \notin F_n(x)$ in E_n for any $x \in \partial\,C_n$ for each $n \in N$ if C is a closed subset of E. The proof follows as in Theorem 2.1 except in this case $z_k \in \overline{C_k}$ (but not necessarily in C_k). Also, from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$, we can conclude that $y \in \overline{C} = C$ (note $q \in \overline{C}$ if and only if, for every $k \in N$, there exists $(x_{k,m}) \in C$, $x_{k,m} = \pi_{k,n}\,(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k\,\mu_k\,(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k\,\mu_k\,(y) \in C_k$ and so $j_k\,\mu_k\,(y) \in j_k\,\mu_k\,F\,(y)$ in E_k as before. Note in fact we can remove the assumption that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Remark 2.3. If we replace $F_n : \overline{C_n} \to 2^{E_n}$ is hemicompact in (2.6) with $F_n : C_n \to 2^{E_n}$ is hemicompact, then one can remove $x \notin F_n(x)$ in E_n for any $x \in \partial C_n$ and $F_n : \overline{C_n} \to 2^{E_n}$ is a closed map for each $n \in N$ in the statement of Theorem 2.1 since if, for each $n \in N$, $F_n : C_n \to 2^{E_n}$ is hemicompact, then we automatically have that $z_k \in C_k$.

Essentially, the same reasoning as in Theorem 2.1 (with Remark 2.2) yields the following result.

Theorem 2.2. *Let E and E_n be as described above, $C \subseteq E$ be an CPRLS and $F : C \to 2^E$. Also, assume that C is a closed subset of E and, for each $n \in N$ and $x \in C$, $j_n \mu_n F(x)$ is closed and also for each $n \in N$ that $F_n : \overline{C_n} \to 2^{E_n}$ is as described above. Suppose that the following conditions are satisfied:*

$$\text{for each } n \in N,\ F_n \in CAC(\overline{C_n}, \overline{C_n}) \text{ is hemicompact;} \tag{2.9}$$

$$\text{for each } n \in N,\ \Lambda_{\overline{C_n}}(F_n) \neq \{0\}; \tag{2.10}$$

$$\begin{cases} \text{for each } n \in \{2, 3, \cdots\} \text{ if } y \in \overline{C_n} \text{ solves } y \in F_n y \text{ in } E_n \\ \text{then } j_k \mu_{k,n} j_n^{-1}(y) \in \overline{C_k} \text{ for } k \in \{1, \cdots, n-1\}. \end{cases} \tag{2.11}$$

Then F has a fixed point in E.

Remark 2.4. Note that we can remove the assumption in Theorem 2.2 that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Remark 2.5. Of course, there are analogue results for compact absorbing contractive morphisms (see the ideas here and in [4, p. 243, 6]).

Remark 2.6. The results in Theorem 2.1 and Theorem 2.2 hold if admissible in Definition 1.4 is replaced by permissible (see [4, p. 276]).

3. Fixed Point Theory for Condensing Maps in Fréchet Spaces

Let E and E_n be as in Section 2. We consider single valued maps. Let $M \subseteq E$ and consider the map $F : M \to E$. Let $n \in N$ and $M_n = j_n \mu_n(M)$. Since in this section we only consider Volterra type operators, we assume

$$\text{(3.1)} \qquad \text{if } x,\, y \in E \text{ with } |x-y|_n = 0 \text{ then } |F\,x - F\,y|_n = 0.$$

Now, (3.1) guarantees that we can define (a well defined) F_n on M_n as follows: For any $y \in M_n$ there exists a $x \in M$ with $y = j_n\,\mu_n(x)$ and we let

$$F_n\,y = j_n\,\mu_n\,F\,x$$

(we could of course call it $F\,y$ since it is clear in the situation we use it in; note that, if there exists a $z \in M$ with $y = j_n\,\mu_n(z)$, then $j_n\,\mu_n\,F\,x = j_n\,\mu_n\,F\,z$ from (3.1)). In this paper, we assume that F_n will be defined on $\overline{M_n}$ i.e. we assume that the F_n described above admits an extension (again we call it F_n) $F : \overline{M_n} \to E_n$ (we will assume certain properties on the extension).

Now, we present some Lefschetz type theorems in Fréchet spaces.

Definition 3.1. A set $A \subseteq E$ is said to be PRCALS if, for each $n \in N$, $A_n \equiv j_n\,\mu_n\,(A)$ is a CA Lefschetz space.

Definition 3.2. A set $A \subseteq E$ is said to be CPRCALS if, for each $n \in N$, $\overline{A_n}$ is a CA Lefschetz space.

Example 3.1. Let A be pseudo-open. Then A is a PRCALS.

To see this, fix $n \in N$. We know (see Example 2.1) that

$$A_n \text{ is a open subset of } E_n.$$

Thus A_n is a CA Lefschetz space (see Example 1.2), so A is a PRCALS.

Theorem 3.1. *Let E and E_n be as described in Section 2, $C \subseteq E$ be an PRCALS and $F : C \to E$. Also, assume that, for each $n \in N$, $F_n : \overline{C_n} \to E_n$ as described above is a continuous map with $x \neq F_n(x)$ in E_n for any $x \in \partial\, C_n$. Suppose that the following conditions are satisfied:*

$$\text{(3.2)} \qquad \begin{cases} \textit{for each } n \in N,\ F_n : C_n \to C_n \textit{ is a continuous map} \\ \textit{with a compact attractor and } F_n : \overline{C_n} \to E_n \textit{ is condensing;} \end{cases}$$

$$\text{(3.3)} \qquad \textit{for each } n \in N,\ \Lambda_{C_n}(F_n) \neq 0;$$

$$\text{(3.4)} \qquad \begin{cases} \textit{for each } n \in \{2,3,\cdots\} \textit{ if } y \in C_n \textit{ solves } y = F_n\,y \textit{ in } E_n \\ \textit{then } j_k\,\mu_{k,n}\,j_n^{-1}(y) \in C_k \textit{ for } k \in \{1,\cdots,n-1\}. \end{cases}$$

Then F has a fixed point in E.

Proof. For each $n \in N$, there exists $y_n \in C_n$ with $y_n = F_n\,y_n$ in E_n. Let's look at $\{y_n\}_{n \in N}$. Notice that $y_1 \in C_1$ and $j_1\,\mu_{1,k}\,j_k^{-1}\,(y_k) \in C_1$ for each $k \in N \backslash \{1\}$ from

(3.4). Note that

$$j_1\,\mu_{1,n}\,j_n^{-1}(y_n) = F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n))$$

in E_1; to see note, for each $n \in N$ fixed, there exists $x \in E$ with $y_n = j_n\,\mu_n\,(x)$, so $j_n\,\mu_n\,(x) = F_n\,(y_n) = j_n\,\mu_n\,F(x)$, on E_n, so, on E_1, we have

$$\begin{aligned} j_1\,\mu_{1,n}\,j_n^{-1}(y_n) &= j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x) = j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,F(x) \\ &= j_1\,\mu_{1,n}\,\mu_n\,F(x) = j_1\,\mu_1\,F(x) = F_1(j_1\,\mu_1\,(x)) \\ &= F_1(j_1\,\mu_{1,n}\,j_n^{-1}\,j_n\,\mu_n\,(x)) = F_1\,(j_1\,\mu_{1,n}\,j_n^{-1}(y_n)). \end{aligned}$$

Now, (3.2) guarantees that there exists is a subsequence $N_1^\star$ of N and $z_1 \in \overline{C_1}$ with $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) \to z_1$ in E_1 as $n \to \infty$ in $N_1^\star$ and $z_1 = F_1\,z_1$ since F_1 is a continuous map. Note that $z_1 \in C_1$ since $x \neq F_1(x)$ in E_1 for any $x \in \partial C_1$. Let $N_1 = N_1^\star \setminus \{1\}$. Now, $j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n) \in C_2$ for each $n \in N_1$ together with (3.2) guarantees that there exists a subsequence $N_2^\star$ of N_1 and $z_2 \in \overline{C_2}$ with $j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n) \to z_2$ in E_2 as $n \to \infty$ in $N_2^\star$ and $z_2 = F_2\,z_2$. Also, $z_2 \in C_2$. Note from (2.4) and the uniqueness of limits that $j_1\,\mu_{1,2}\,j_2^{-1}\,z_2 = z_1$ in E_1 since $N_2^\star \subseteq N_1$ (note $j_1\,\mu_{1,n}\,j_n^{-1}\,(y_n) = j_1\,\mu_{1,2}\,j_2^{-1}\,j_2\,\mu_{2,n}\,j_n^{-1}\,(y_n)$ for each $n \in N_2^\star$). Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots, \quad N_k^\star \subseteq \{k, k+1, \cdots\}$$

and $z_k \in \overline{C_k}$ with $j_k\,\mu_{k,n}\,j_n^{-1}\,(y_n) \to z_k$ in E_k as $n \to \infty$ in $N_k^\star$ and $z_k = F_k\,z_k$. Also, $z_k \in C_k$. Note that $j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = z_k$ in E_k for $k \in \{1, 2, \cdots\}$. Also, let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in N$. Now, $z_k = F_k\,z_k$ in E_k. Note as well that

$$\begin{aligned} z_k &= j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,z_{k+1} = j_k\,\mu_{k,k+1}\,j_{k+1}^{-1}\,j_{k+1}\,\mu_{k+1,k+2}\,j_{k+2}^{-1}\,z_{k+2} \\ &= j_k\,\mu_{k,k+2}\,j_{k+2}^{-1}\,z_{k+2} = \cdots = j_k\,\mu_{k,m}\,j_m^{-1}\,z_m = \pi_{k,m}\,z_m \end{aligned}$$

for each $m \geq k$. We can do this for each $k \in N$. As a result, $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and also note $z_k \in C_k$ for each $k \in N$. Thus, for each $k \in N$, we have

$$j_k\,\mu_k\,(y) = z_k = F_k\,z_k = j_k\,\mu_k\,F\,y \text{ in } E_k$$

so $y = F\,y$ in E. □

Remark 3.1. Of course, one could remove $x \neq F_n(x)$ in E_n for any $x \in \partial C_n$ for each $n \in N$ if C is a closed subset of E. The proof follows as in Theorem 3.1 except in this case $z_k \in \overline{C_k}$ (but not necessarily in C_k). Also, from $y = (z_k) \in \lim_{\leftarrow} E_n = E$ and $\pi_{k,m}\,(y_m) \to z_k$ in E_k as $m \to \infty$, we can conclude that

$y \in \overline{C} = C$ (note that $q \in \overline{C}$ if and only if, for each $k \in N$, there exists $(x_{k,m}) \in C$, $x_{k,m} = \pi_{k,n}(x_{n,m})$ for $n \geq k$ with $x_{k,m} \to j_k \mu_k(q)$ in E_k as $m \to \infty$). Thus $z_k = j_k \mu_k(y) \in C_k$ and so $j_k \mu_k(y) = j_k \mu_k F(y)$ in E_k as before. Note in fact that we can remove the assumption that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Remark 3.2. If we replace $F_n : \overline{C_n} \to E_n$ is condensing in (3.2) with $F_n : C_n \to C_n$ is condensing and hemicompact, then one can remove $x \neq F_n(x)$ in E_n for any $x \in \partial C_n$ and $F_n : \overline{C_n} \to E_n$ is a continuous map for each $n \in N$ in the statement of Theorem 3.1 since if, for each $n \in N$, $F_n : C_n \to C_n$ is hemicompact, then we automatically have that $z_k \in C_k$.

Essentially, the same reasoning as in Theorem 3.1 (with Remark 3.1) yields the following result.

Theorem 3.2. *Let E and E_n be as described in Section 2, $C \subseteq E$ is an CPRCALS and $F : C \to E$. Also, assume that C is a closed subset of E and, for each $n \in N$ $F_n : \overline{C_n} \to E_n$ is as described above. Suppose that the following conditions are satisfied:*

$$\text{(3.5)} \qquad \begin{cases} \text{for each } n \in N,\ F_n : \overline{C_n} \to \overline{C_n} \text{ is a continuous} \\ \text{condensing map with a compact attractor;} \end{cases}$$

$$\text{(3.6)} \qquad \text{for each } n \in N,\ \Lambda_{\overline{C_n}}(F_n) \neq 0;$$

$$\text{(3.7)} \qquad \begin{cases} \text{for each } n \in \{2, 3, \cdots\}, \text{ if } y \in \overline{C_n} \text{ solves } y = F_n\, y \text{ in } E_n \\ \text{then } j_k\, \mu_{k,n}\, j_n^{-1}(y) \in \overline{C_k} \text{ for each } k \in \{1, \cdots, n-1\}. \end{cases}$$

Then F has a fixed point in E.

Remark 3.3. Note that we can remove the assumption in Theorem 3.2 that C is a closed subset of E if we assume $F : Y \to 2^E$ with $C \subseteq Y$ and $\overline{C_n} \subseteq Y_n$ for each $n \in N$.

Definition 3.3. A set $A \subseteq E$ is said to be SPRLS if, for each $n \in N$, $A_n \equiv j_n\, \mu_n(A)$ is a special Lefschetz space.

Definition 3.4. A set $A \subseteq E$ is said to be SCPRLS if, for each $n \in N$, $\overline{A_n}$ is a special Lefschetz space.

Essentially, the same reasoning as in Theorem 3.1 yields the following results (we also have an analogue of Remarks 3.1, 3.2 and 3.3).

Theorem 3.3. *Let E and E_n be as described in Section 2, $C \subseteq E$ is an SPRLS and $F : C \to E$. Also, assume that, for each $n \in N$, $F_n : \overline{C_n} \to E_n$ as described above is a continuous map with $x \neq F_n(x)$ in E_n for any $x \in \partial C_n$. Suppose that* (3.3), (3.4) *and the following condition is satisfied:*

$$(3.8) \qquad \begin{cases} \text{for each } n \in N, \ F_n : C_n \to C_n \text{ is a continuous map} \\ \text{and } F_n : \overline{C_n} \to E_n \text{ is condensing.} \end{cases}$$

Then F has a fixed point in E.

Theorem 3.4. *Let E and E_n be as described in Section 2, $C \subseteq E$ is an SCPRLS and $F : C \to E$. Also, assume that C is a closed subset of E and, for each $n \in N$, $F_n : \overline{C_n} \to E_n$ is as described above. Suppose that* (3.6), (3.7) *and the following condition is satisfied:*

$$(3.9) \qquad \text{for each } n \in N, \ F_n : \overline{C_n} \to \overline{C_n} \text{ is a continuous condensing map.}$$

Then F has a fixed point in E.

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