ON THE STABILITY OF MODULE LEFT DERIVATIONS IN BANACH ALGEBRAS

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\textbf{Abstract.} In this paper, we improve the generalized Hyers-Ulam stability and the superstability of module left derivations due to the results of [7].

1. \textbf{Introduction}

Let $A$ be an algebra over the real or complex field $F$ and $M$ a left $A$-module (respectively, $A$-bimodule). An additive map $\delta : A \to M$ is said to be a module left derivation (respectively, module derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (respectively, $\delta(xy) = x\delta(y) + \delta(x)y$) holds for all $x, y \in A$. Since $A$ is a left $A$-module (respectively, $A$-bimodule) with the product of $A$ giving the module multiplication (respectively, two module multiplications), the module left derivation (respectively, module derivation) $\delta : A \to A$ is said to be a ring left derivation (respectively, ring derivation) on $A$.

Recently, T. Miura et al. [8] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation $f$ on a Banach algebra $A$ is an exact ring derivation. In particular, if $A$ is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then $f$ is identically zero. The first stability result concerning derivations between operator algebras was obtained by P. Semrl [11].

The study of stability problems originated from a famous talk given by S.M. Ulam [12] in 1940: Under what condition does there exists a homomorphism near an approximate homomorphism? In the next year 1941, D.H. Hyers [5] was answered affirmatively the question of Ulam for Banach spaces, which states that if $\delta > 0$ and

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$f : X \to Y$ is a map with $X$ a normed space, $Y$ a Banach space such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \delta
\]
for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \delta
\]
for all $x \in X$. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation
\[
h(x + y) = h(x) + h(y).
\]
A generalized version of the theorem of Hyers for approximate additive maps was given by T. Aoki [1] in 1950. In 1978, Th.M. Rassias [10] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces. If there exist $\theta \geq 0$ and $0 \leq p < 1$ such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)
\]
for all $x, y \in X$, then there exist a unique additive map $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2p}\|x\|^p
\]
for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x$ in $X$, where $\mathbb{R}$ denotes the set of the real numbers, then $T$ is linear. Due to this fact, many mathematicians say that the additive functional equation
\[
f(x + y) = f(x) + f(y)
\]
has the Hyers-Ulam-Rassias stability property. Since then, a great deal of work has been done by a number of authors (for instances, [3, 6, 7]). In 1991, Z. Gajda [2] answered the question for the case $p > 1$, which was raised by Rassias. Gajda [2] also gave an example that the Rassias’ stability result is not valid for $p = 1$.

On the other hand, J.M. Rassias [9] generalized the Hyers’ stability result by presenting a weaker condition controlled by a product of different powers of norms. That is, assume that there exist constants $\theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1$, and $f : X \to Y$ is a map with $X$ a normed space, $Y$ a Banach space such that the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta\|x\|^{p_1}\|y\|^{p_2}
\]
for all $x, y \in X$, then there exist a unique additive map $T : X \to Y$ such that
\[
\|f(x) - T(x)\| \leq \frac{\theta}{2 - 2p}\|x\|^p
\]
for all \( x \in X \). If, in addition, \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \) in \( X \), then \( T \) is linear. A counter-example for a singular case of this result was given by P. Găvrută [4].

Our purpose in this paper is to deal with the stability problems of module left derivations and to improve the results in [7].

2. Stability of Module Left Derivations
in the Sense of J.M. Rassias

In this section, \( \mathbb{N} \) will denote the set of the natural numbers.

Theorem 2.1. Let \( A \) be a normed algebra and let \( M \) be a Banach left \( A \)-module. Suppose that \( f : A \to M \) is a map such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^{p_1} \|y\|^{p_2},
\]

\[
\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon \|x\|^{q_1} \|y\|^{q_2}
\]

for some \( \theta, \varepsilon \geq 0 \) and some \( p_1, p_2, q_1, q_2 \in \mathbb{R} \) such that \( p = p_1 + p_2 \neq 1, q_1 \neq 1, q_2 \neq 1 \), and all \( x, y \in A \setminus \{0\} \). If \( p < 1, q_2 < 1 \) or \( p > 1, q_2 > 1 \), then there exists a unique module left derivation \( \delta : A \to M \) such that

\[
\|f(x) - \delta(x)\| \leq \frac{\theta}{|2 - 2p|} \|x\|^p
\]

for all \( x \in A \setminus \{0\} \) and \( f(0) = \delta(0) \).

Proof. Assume that \( p < 1, q_2 < 1 \) or \( p > 1, q_2 > 1 \). Set \( \tau = 1 \) if \( p < 1, q_2 < 1 \) and \( \tau = -1 \) if \( p > 1, q_2 > 1 \). By the J.M. Rassias' result [9], the inequality (2.1) guarantees that there exists a unique additive map \( \delta : A \to M \) defined by

\[
\delta(x) := \begin{cases} 
\lim_{n \to \infty} 2^{-\tau n} f(2^{\tau n} x) & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}
\]

such that (2.3) holds for all \( x \in A \setminus \{0\} \). We claim that \( \delta(xy) = x\delta(y) + y\delta(x) \) for all \( x, y \in A \). Since \( \delta \) is additive, we see that \( \delta(x) = 2^{-\tau n} \delta(2^{\tau n} x) \) for all \( x \in A \) and all \( n \in \mathbb{N} \). From (2.3), we have

\[
\|f(0) - \delta(0)\| = \|f((kx) + (-kx)) - f(kx) - f(-kx)\|
\]

\[
+ \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\|
\]

\[
\leq \theta |k|^p \|x\|^p \left(1 + \frac{2}{|2 - 2p|}\right)
\]
for all \( x \in A \setminus \{0\} \) and all \( k \in \mathbb{R} \setminus \{0\} \) from which we deduce \( f(0) = \delta(0) \). Using (2.2), (2.3) and considering the fact that \( M \) is a Banach left \( A \)-module, there exists a constant \( K > 0 \) such that

\[
\|\delta(xy) - x\delta(y) - yf(x)\|
\leq \|\delta(xy) - 2^{-\tau n} f(2^{\tau n} xy)\|
+ \|2^{-\tau n} f(2^{\tau n} xy) - 2^{-\tau n} x f(2^{\tau n} y) - y f(x)\|
+ \|2^{-\tau n} x f(2^{\tau n} y) - x \delta(y)\|
\leq 2^{\tau(p-1)n} \frac{\theta}{|2 - 2p|} \|x\|^p \|y\|^p
+ 2^{\tau(q_2-1)n} \varepsilon \|x\|^{q_1} \|y\|^{q_2}
+ 2^{\tau(p-1)n} K \frac{\theta}{|2 - 2p|} \|x\|^p \|y\|^p \to 0 \quad \text{as} \quad n \to \infty
\]

which implies that

(2.5) \[\delta(xy) = x\delta(y) + yf(x)\]

for all \( x, y \in A \setminus \{0\} \). From (2.5),

\[
\delta(xy) = 2^{-\tau n} \delta(2^{\tau n} xy)
= 2^{-\tau n} 2^{\tau n} x \delta(y) + 2^{-\tau n} y f(2^{\tau n} x)
= x \delta(y) + 2^{-\tau n} y f(2^{\tau n} x)
\]

and

\[
\delta(xy) = \lim_{n \to \infty} (x \delta(y) + 2^{-\tau n} y f(2^{\tau n} x)) = x \delta(y) + y \delta(x)
\]

for all \( x, y \in A \setminus \{0\} \). That is, \( \delta \) is a module left derivation, as claimed and the proof is complete. \( \square \)

Let \( A \) be an algebra. A left \( A \)-module \( M \) is said to be unitary if \( A \) has a unit element \( e \) and \( eu = u \) for all \( u \in M \).

**Corollary 2.2.** Let \( A \) be a unital normed algebra and let \( M \) be a unitary Banach left \( A \)-module. Suppose that \( f : A \to M \) is a map satisfying (2.1) and (2.2) for some \( \theta, \varepsilon \geq 0 \) and some \( p_1, p_2, q_1, q_2 \in \mathbb{R} \) such that \( p = p_1 + p_2 \neq 1, q_2 \neq 1 \). If \( p, q_2 < 1 \) or \( p, q_2 > 1 \), then \( f \) is a module left derivation.

**Proof.** By Theorem 2.1, the inequalities (2.1) and (2.2) guarantee that there exists a unique module left derivation \( \delta : A \to M \) satisfying (2.5) for all \( x \in A \setminus \{0\} \) and \( f(0) = \delta(0) \). Since \( \delta(e) = 0 \), it follows from (2.5) that

\[
\delta(x) = \delta(xe) = x \delta(e) + ef(x) = f(x)
\]

for all \( x \in A \setminus \{0\} \). This completes the proof. \( \square \)
Theorem 2.3. Let $A$ be a normed algebra and let $M$ be a Banach left $A$-module. Suppose that $f : A \to M$ is a map such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta \|x\|^p \|y\|^p,
\]
\[
\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon \|x\|^{q_1} \|y\|^{q_2}
\]

for some $\theta, \varepsilon \geq 0$ and some $p, q_1, q_2 \in \mathbb{R}$ such that $p < 0, q_1 \neq 1, q_2 \neq 1$, and all $x, y \in A \setminus \{0\}$. Then $f$ is a module left derivation.

Proof. By Theorem 2.1, the inequalities (2.1) and (2.2) guarantee that there exists a unique module left derivation $\delta : A \to M$ such that

\[
\|f(x) - \delta(x)\| \leq \frac{\theta}{2 - 2^p} \|x\|^{2p}
\]

for all $x \in A \setminus \{0\}$ and $f(0) = \delta(0)$. Since $\delta : A \to M$ is an additive map,

\[
\|f(x) - \delta(x)\| = \|f(x) - f((k + 1)x) - f(-kx)\| \\
+ \|f((k + 1)x) - \delta((k + 1)x)\| + \|f(-kx) - \delta(-kx)\|
\]

\[
\leq \|k + 1\|^{p} \|k\|^{p} \|x\|^{2p} + \frac{\theta}{2 - 2^p} (\|k + 1\|^{2p} + \|k\|^{2p}) \|x\|^{2p},
\]

\[
\|f(0) - \delta(0)\| = \|f((kx) + (-kx)) - f(kx) - f(-kx)\| \\
+ \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\|
\]

\[
\leq \theta \|k\|^{2p} \|x\|^{2p} \left(1 + \frac{2}{2 - 2^p}\right)
\]

for all $x \in A \setminus \{0\}$ and all $k \in \mathbb{R} \setminus \{0\}$. Taking $k \to \infty$ in the above relations, we get

\[
f(x) = \delta(x)
\]

for all $x \in A$ which completes the proof. \qed

3. Stability of Module Left Derivations 
   in the Sense of Th.M. Rassias

Theorem 3.1. Let $A$ be a normed algebra and let $M$ be a Banach left $A$-module. Suppose that $f : A \to M$ is a map such that

\[
\|f(x + y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p) \tag{3.1}
\]
\[
\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon (\|x\|^{q} + \|y\|^{q}) \tag{3.2}
\]

for some $\theta, \varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p \neq 1, q \neq 2$, and all $x, y \in A \setminus \{0\}$. 
If $p < 1, q < 2$ or $p > 1, q > 2$, then there exists a unique module left derivation $\delta : A \to M$ such that

\begin{equation}
\|f(x) - \delta(x)\| \leq \frac{2\theta}{2 - 2p}\|x\|^p
\end{equation}

for all $x \in A \setminus \{0\}$ and $f(0) = \delta(0)$.

**Proof.** Set $\tau = 1$ if $p < 1, q < 2$ and $\tau = -1$ if $p > 1, q > 2$. By the Th.M. Rassias’ theorem [10], the inequality (3.1) guarantees that there exists a unique additive map $\delta : A \to M$ satisfying (3.3) holds for all $x \in A \setminus \{0\}$ and $\delta(x)$ is defined as (2.4). We claim that $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in A$. Since $\delta$ is additive, we see that $\delta(x) = 2^{-\tau n}\delta(2^{\tau n}x)$ for all $x \in A$ and all $n \in \mathbb{N}$. From (3.1) and (3.3), we have

\[
\|f(0) - \delta(0)\| = \|f((kx) + (-kx)) - f(kx) - f(-kx)\| \\
+ \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\| \\
\leq \theta |k|^p \|x\|^p \left( 2 + \frac{4}{|2 - 2p|} \right)
\]

for all $x \in A \setminus \{0\}$ and all $k \in \mathbb{R} \setminus \{0\}$, hence $f(0) = \delta(0)$. Since $f$ satisfies (3.2), we get

\[
\|2^{-2\tau n}f(2^{2\tau n}xy) - 2^{-\tau n}xf(2^{\tau n}y) - 2^{-\tau n}yf(2^{\tau n}x)\| \\
= 2^{-2\tau n}\|f((2^{\tau n}x)(2^{\tau n}y)) - 2^{\tau n}xf(2^{\tau n}y) - 2^{\tau n}yf(2^{\tau n}x)\| \\
\leq 2^{\tau n(q-2)}\varepsilon(\|x\|^q + \|y\|^q)
\]

for all $x, y \in A \setminus \{0\}$ and all $n \in \mathbb{N}$. By reminding of $\tau(q - 2) < 0$, we see that

\[
\|2^{-2\tau n}f(2^{2\tau n}xy) - 2^{-\tau n}xf(2^{\tau n}y) - 2^{-\tau n}yf(2^{\tau n}x)\| \to 0 \quad \text{as} \quad n \to \infty.
\]

which implies that

\[
\delta(xy) = \lim_{n \to \infty} (x\delta(y) + 2^{-\tau n}yf(2^{\tau n}x)) = x\delta(y) + y\delta(x)
\]

for all $x, y \in A \setminus \{0\}$. Since $\delta(0) = 0$, $\delta$ is a module left derivation, as claimed and the proof is complete.

**Theorem 3.2.** Let $A$ be a unital normed algebra and let $M$ be a unitary Banach left $A$-module. Suppose that $f : A \to M$ is a map satisfying (3.1) and (3.2) for some $\theta, \varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p < 1, q < 1$, and all $x, y \in A \setminus \{0\}$. Then $f$ is a module left derivation.

**Proof.** Let $e$ be a unit element of $A$. By Theorem 3.1, there exists a unique module left derivation $\delta : A \to M$ such that (3.3) is true. Recall that $\delta$ is additive, and hence it is easy to see that $\delta(2x) = 2\delta(x)$ for all $x \in A$. 

\[\square\]
The inequality (3.2) yields that
\[
\left\| \frac{f(2^{n+j}e) - 2^j f(2^n e) - 2^n f(2^j e)}{2^n} \right\| \leq \frac{\varepsilon(\|2^n e\|^q + \|2^j e\|^q)}{2^n}
\]
for all $n \in \mathbb{N}$. Passing to $n \to \infty$ in (3.4), we get
\[
f(2^j e) = 2^j \delta(e) - f(2^j e) = -f(2^j e), \quad j \in \mathbb{N}.
\]
and so
\[
f(2^j e) = 0, \quad j \in \mathbb{N}.
\]
Now it follows from (3.2) and (3.5) that
\[
\|f(2x) - 2f(x)\| \leq \frac{\|f(2^{n+1}x) - 2x f(2^n e) - 2^n f(2x)\|}{2^n} + \frac{\| - f(2^{n+1}x) + x f(2^{n+1} e) + 2^{n+1} f(x)\|}{2^n} \\
\leq \frac{\varepsilon(\|2x\|^q + \|2^n e\|^q + \|x\|^q + \|2^{n+1} e\|^q)}{2^n}
\]
for all $x \in A$ and all $n \in \mathbb{N}$. Taking $n \to \infty$ in (3.6), we see that $f(2x) = 2f(x)$ for all $x \in A$ which gives
\[
\delta(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} = f(x)
\]
for all $x \in A$. This completes the proof. \hfill \qed

**Corollary 3.3.** Let $A$ be a unital normed algebra and let $M$ be a unitary Banach left $A$-module. Suppose that $f : A \to M$ is a map satisfying (3.1) and (3.2) for some $\varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p < 0, q < 2$, and all $x, y \in A \setminus \{0\}$. Then $f$ is a module left derivation.

**References**


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