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ON THE STABILITY OF MODULE LEFT DERIVATIONS IN BANACH ALGEBRAS

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ABSTRACT. In this paper, we improve the generalized Hyers-Ulam stability and the superstability of module left derivations due to the results of [7].

1. INTRODUCTION

Let A be an algebra over the real or complex field F and M a left A-module (respectively, A-bimodule). An additive map $\delta : A \to M$ is said to be a module left derivation (respectively, module derivation) if $\delta(xy) = x\delta(y) + y\delta(x)$ (respectively, $\delta(xy) = x\delta(y) + \delta(x)y$) holds for all $x, y \in A$. Since A is a left A-module (respectively, A-bimodule) with the product of A giving the module multiplication (respectively, two module multiplications), the module left derivation (respectively, module derivation) $\delta : A \to A$ is said to be a ring left derivation (respectively, ring derivation) on A.

Recently, T. Miura et al. [8] considered the stability of ring derivations on Banach algebras: Under suitable conditions, every approximate ring derivation f on a Banach algebra A is an exact ring derivation. In particular, if A is a commutative semisimple Banach algebra with the maximal ideal space without isolated points, then f is identically zero. The first stability result concerning derivations between operator algebras was obtained by P. Semrl [11].

The study of stability problems originated from a famous talk given by S.M. Ulam [12] in 1940: Under what condition does there exists a homomorphism near an approximate homomorphism? In the next year 1941, D.H. Hyers [5] was answered affirmatively the question of Ulam for Banach spaces, which states that if $\delta > 0$ and

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$f : X \to Y$ is a map with X a normed space, Y a Banach space such that

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for all $x, y \in X$, then there exists a unique additive map $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \delta$$

for all $x \in X$. This stability phenomenon is called the Hyers-Ulam stability of the additive functional equation

$$h(x+y) = h(x) + h(y).$$

A generalized version of the theorem of Hyers for approximate additive maps was given by T. Aoki [1] in 1950. In 1978, Th.M. Rassias [10] independently introduced the unbounded Cauchy difference and was the first to prove the stability of the linear mapping between Banach spaces. If there exist $\theta \ge 0$ and $0 \le p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X$, then there exist a unique additive map $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$. Moreover, if $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X, where $\mathbb{R}$ denotes the set of the real numbers, then T is linear. Due to this fact, many mathematicians say that the additive functional equation

$$f(x+y) = f(x) + f(y)$$

has the Hyers-Ulam-Rassias stability property. Since then, a great deal of work has been done by a number of authors (for instances, [3, 6, 7]). In 1991, Z. Gajda [2] answered the question for the case $p > 1$, which was raised by Rassias. Gajda [2] also gave an example that the Rassias' stability result is not valid for $p = 1$.

On the other hand, J.M. Rassias [9] generalized the Hyers' stability result by presenting a weaker condition controlled by a product of different powers of norms. That is, assume that there exist constants $\theta \ge 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \ne 1$, and $f : X \to Y$ is a map with X a normed space, Y a Banach space such that the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta\|x\|^{p_1}\|y\|^{p_2}$$

for all $x, y \in X$, then there exist a unique additive map $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{\theta}{2 - 2^p}\|x\|^p$$

for all $x \in X$. If, in addition, $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x in X, then T is linear. A counter-example for a singular case of this result was given by P. Găvruţă [4].

Our purpose in this paper is to deal with the stability problems of module left derivations and to improve the results in [7].

2. Stability of Module Left Derivations in the Sense of J.M. Rassias

In this section, $\mathbb{N}$ will denote the set of the natural numbers.

Theorem 2.1. *Let A be a normed algebra and let M be a Banach left A-module. Suppose that $f : A \to M$ is a map such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta \|x\|^{p_1} \|y\|^{p_2}, \tag{2.1}$$

$$\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon \|x\|^{q_1} \|y\|^{q_2} \tag{2.2}$$

for some $\theta, \varepsilon \geq 0$ and some $p_1, p_2, q_1, q_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1, q_1 \neq 1, q_2 \neq 1$, and all $x, y \in A \backslash \{0\}$. If $p < 1, q_2 < 1$ or $p > 1, q_2 > 1$, then there exists a unique module left derivation $\delta : A \to M$ such that

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{|2 - 2^p|} \|x\|^p \tag{2.3}$$

for all $x \in A \backslash \{0\}$ and $f(0) = \delta(0)$.

Proof. Assume that $p < 1$, $q_2 < 1$ or $p > 1$, $q_2 > 1$. Set $\tau = 1$ if $p < 1$, $q_2 < 1$ and $\tau = -1$ if $p > 1$, $q_2 > 1$. By the J.M. Rassias' result [9], the inequality (2.1) guarantees that there exists a unique additive map $\delta : A \to M$ defined by

$$\delta(x) := \begin{cases} \lim_{n \to \infty} 2^{-\tau n} f(2^{\tau n} x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases} \tag{2.4}$$

such that (2.3) holds for all $x \in A \backslash \{0\}$. We claim that $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in A$. Since δ is additive, we see that $\delta(x) = 2^{-\tau n} \delta(2^{\tau n} x)$ for all $x \in A$ and all $n \in \mathbb{N}$. From (2.3), we have

$$\begin{aligned} \|f(0) - \delta(0)\| &= \|f((kx) + (-kx)) - f(kx) - f(-kx)\| \\ &\quad + \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\| \\ &\leq \theta \, |k|^p \, \|x\|^p \left(1 + \frac{2}{|2 - 2^p|}\right) \end{aligned}$$

for all $x \in A\backslash\{0\}$ and all $k \in \mathbb{R}\backslash\{0\}$ from which we deduce $f(0) = \delta(0)$. Using (2.2), (2.3) and considering the fact that M is a Banach left A-module, there exists a constant $K > 0$ such that

$$\begin{aligned}
&\|\delta(xy) - x\delta(y) - yf(x)\| \\
&\leq \|\delta(xy) - 2^{-\tau n} f(2^{\tau n} xy)\| + \|2^{-\tau n} f(2^{\tau n} xy) - 2^{-\tau n} x f(2^{\tau n} y) - yf(x)\| \\
&\quad + \|2^{-\tau n} x f(2^{\tau n} y) - x\delta(y)\| \\
&\leq 2^{\tau(p-1)n} \frac{\theta}{|2-2^p|} \|x\|^p \|y\|^p + 2^{\tau(q_2-1)n} \varepsilon \|x\|^{q_1} \|y\|^{q_2} \\
&\quad + 2^{\tau(p-1)n} K \frac{\theta}{|2-2^p|} \|x\|^p \|y\|^p \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

which implies that

$$\delta(xy) = x\delta(y) + yf(x) \tag{2.5}$$

for all $x, y \in A\backslash\{0\}$. From (2.5),

$$\begin{aligned}
\delta(xy) &= 2^{-\tau n} \delta(2^{\tau n} xy) \\
&= 2^{-\tau n} 2^{\tau n} x\delta(y) + 2^{-\tau n} y f(2^{\tau n} x) \\
&= x\delta(y) + 2^{-\tau n} y f(2^{\tau n} x)
\end{aligned}$$

and

$$\delta(xy) = \lim_{n\to\infty} (x\delta(y) + 2^{-\tau n} y f(2^{\tau n} x)) = x\delta(y) + y\delta(x)$$

for all $x, y \in A\backslash\{0\}$. That is, δ is a module left derivation, as claimed and the proof is complete. □

Let A be an algebra. A left A-module M is said to be *unitary* if A has a unit element e and $eu = u$ for all $u \in M$.

Corollary 2.2. *Let A be a unital normed algebra and let M be a unitary Banach left A-module. Suppose that $f : A \to M$ is a map satisfying (2.1) and (2.2) for some $\theta, \varepsilon \geq 0$ and some $p_1, p_2, q_1, q_2 \in \mathbb{R}$ such that $p = p_1 + p_2 \neq 1, q_2 \neq 1$. If $p, q_2 < 1$ or $p, q_2 > 1$, then f is a module left derivation.*

Proof. By Theorem 2.1, the inequalities (2.1) and (2,2) guarantee that there exists a unique module left derivation $\delta : A \to M$ satisfying (2.5) for all $x \in A\backslash\{0\}$ and $f(0) = \delta(0)$. Since $\delta(e) = 0$, it follows from (2.5) that

$$\delta(x) = \delta(xe) = x\delta(e) + ef(x) = f(x)$$

for all $x \in A\backslash\{0\}$. This completes the proof. □

Theorem 2.3. *Let A be a normed algebra and let M be a Banach left A-module. Suppose that $f : A \to M$ is a map such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\|x\|^p\|y\|^p,$$
$$\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon\|x\|^{q_1}\|y\|^{q_2}$$

for some $\theta, \varepsilon \geq 0$ and some $p, q_1, q_2 \in \mathbb{R}$ such that $p < 0, q_1 \neq 1, q_2 \neq 1$, and all $x, y \in A\backslash\{0\}$. Then f is a module left derivation.

Proof. By Theorem 2.1, the inequalities (2.1) and (2.2) guarantee that there exists a unique module left derivation $\delta : A \to M$ such that

$$\|f(x) - \delta(x)\| \leq \frac{\theta}{2 - 2^p}\|x\|^{2p}$$

for all $x \in A\backslash\{0\}$ and $f(0) = \delta(0)$. Since $\delta : A \to M$ is an additive map,

$$\begin{aligned}\|f(x) - \delta(x)\| &= \|f(x) - f((k+1)x) - f(-kx)\| \\ &\quad + \|f((k+1)x) - \delta((k+1)x)\| + \|f(-kx) - \delta(-kx)\| \\ &\leq |k+1|^p|k|^p\|x\|^{2p} + \frac{\theta}{2 - 2^p}(|k+1|^{2p} + |k|^{2p})\|x\|^{2p}, \\ \|f(0) - \delta(0)\| &= \|f((kx) + (-kx)) - f(kx) - f(-kx)\| \\ &\quad + \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\| \\ &\leq \theta|k|^{2p}\|x\|^{2p}\left(1 + \frac{2}{2 - 2^p}\right)\end{aligned}$$

for all $x \in A\backslash\{0\}$ and all $k \in \mathbb{R} \setminus \{0\}$. Taking $k \to \infty$ in the above relations, we get

$$f(x) = \delta(x)$$

for all $x \in A$ which completes the proof. □

3. Stability of Module Left Derivations in the Sense of Th.M. Rassias

Theorem 3.1. *Let A be a normed algebra and let M be a Banach left A-module. Suppose that $f : A \to M$ is a map such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3.1}$$

$$\|f(xy) - xf(y) - yf(x)\| \leq \varepsilon(\|x\|^q + \|y\|^q) \tag{3.2}$$

for some $\theta, \varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p \neq 1, q \neq 2$, and all $x, y \in A\backslash\{0\}$.

If $p < 1, q < 2$ or $p > 1, q > 2$, then there exists a unique module left derivation $\delta : A \to M$ such that

$$\|f(x) - \delta(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p \tag{3.3}$$

for all $x \in A\backslash\{0\}$ and $f(0) = \delta(0)$.

Proof. Set $\tau = 1$ if $p < 1, q < 2$ and $\tau = -1$ if $p > 1, q > 2$. By the Th.M. Rassias' theorem [10], the inequality (3.1) guarantees that there exists a unique additive map $\delta : A \to M$ satisfying (3.3) holds for all $x \in A\backslash\{0\}$ and $\delta(x)$ is defined as (2.4). We claim that $\delta(xy) = x\delta(y) + y\delta(x)$ for all $x, y \in A$. Since δ is additive, we see that $\delta(x) = 2^{-\tau n}\delta(2^{\tau n}x)$ for all $x \in A$ and all $n \in \mathbb{N}$. From (3.1) and (3.3), we have

$$\begin{aligned}\|f(0) - \delta(0)\| &= \|f((kx) + (-kx)) - f(kx) - f(-kx)\| \\ &\quad + \|f(kx) - \delta(kx)\| + \|f(-kx) - \delta(-kx)\| \\ &\leq \theta\,|k|^p\,\|x\|^p\left(2 + \frac{4}{|2 - 2^p|}\right)\end{aligned}$$

for all $x \in A\backslash\{0\}$ and all $k \in \mathbb{R}\backslash\{0\}$, hence $f(0) = \delta(0)$. Since f satisfies (3.2), we get

$$\begin{aligned}&\|2^{-2\tau n}f(2^{2\tau n}xy) - 2^{-\tau n}xf(2^{\tau n}y) - 2^{-\tau n}yf(2^{\tau n}x)\| \\ &= 2^{-2\tau n}\|f((2^{\tau n}x)(2^{\tau n}y)) - 2^{\tau n}xf(2^{\tau n}y) - 2^{\tau n}yf(2^{\tau n}x)\| \\ &\leq 2^{\tau n(q-2)}\varepsilon(\|x\|^q + \|y\|^q)\end{aligned}$$

for all $x, y \in A\backslash\{0\}$ and all $n \in \mathbb{N}$. By reminding of $\tau(q - 2) < 0$, we see that

$$\|2^{-2\tau n}f(2^{2\tau n}xy) - 2^{-\tau n}xf(2^{\tau n}y) - 2^{-\tau n}yf(2^{\tau n}x)\| \to 0 \quad \text{as} \quad n \to \infty.$$

which implies that

$$\delta(xy) = \lim_{n\to\infty}(x\delta(y) + 2^{-\tau n}yf(2^{\tau n}x)) = x\delta(y) + y\delta(x)$$

for all $x, y \in A\backslash\{0\}$. Since $\delta(0) = 0$, δ is a module left derivation, as claimed and the proof is complete. □

Theorem 3.2. *Let A be a unital normed algebra and let M be a unitary Banach left A-module. Suppose that $f : A \to M$ is a map satisfying* (3.1) *and* (3.2) *for some $\theta, \varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p < 1, q < 1$, and all $x, y \in A\backslash\{0\}$. Then f is a module left derivation.*

Proof. Let e be a unit element of A. By Theorem 3.1, there exists a unique module left derivation $\delta : A \to M$ such that (3.3) is true. Recall that δ is additive, and hence it is easy to see that $\delta(2x) = 2\delta(x)$ for all $x \in A$.

The inequality (3.2) yields that

$$\left\| \frac{f(2^{n+j}e) - 2^j f(2^n e) - 2^n f(2^j e)}{2^n} \right\| \leq \frac{\varepsilon(\|2^n e\|^q + \|2^j e\|^q)}{2^n} \tag{3.4}$$

for all $n \in \mathbb{N}$. Passing to $n \to \infty$ in (3.4), we get

$$f(2^j e) = 2^j \delta(e) - f(2^j e) = -f(2^j e), \quad j \in \mathbb{N}.$$

and so

$$f(2^j e) = 0, \quad j \in \mathbb{N}. \tag{3.5}$$

Now it follows from (3.2) and (3.5) that

$$\begin{aligned} \|f(2x) - 2f(x)\| &\leq \frac{\|f(2^{n+1}x) - 2xf(2^n e) - 2^n f(2x)\|}{2^n} \\ &\quad + \frac{\| - f(2^{n+1}x) + xf(2^{n+1}e) + 2^{n+1}f(x)\|}{2^n} \\ &\leq \frac{\varepsilon(\|2x\|^q + \|2^n e\|^q + \|x\|^q + \|2^{n+1}e\|^q)}{2^n} \end{aligned} \tag{3.6}$$

for all $x \in A$ and all $n \in \mathbb{N}$. Taking $n \to \infty$ in (3.6), we see that $f(2x) = 2f(x)$ for all $x \in A$ which gives

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n} = f(x)$$

for all $x \in A$. This completes the proof. □

Corollary 3.3. *Let A be a unital normed algebra and let M be a unitary Banach left A-module. Suppose that $f : A \to M$ is a map satisfying* (3.1) *and* (3.2) *for some $\varepsilon \geq 0$ and some $p, q \in \mathbb{R}$ such that $p < 0, q < 2$, and all $x, y \in A\backslash\{0\}$. Then f is a module left derivation.*

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