J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 16, Number 1 (February 2009), Pages 147–154

SOME GENERALIZED HIGHER SCHWARZIAN OPERATORS

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ABSTRACT. Tamanoi proposed higher Schwarzian operators which include the classical Schwarzian derivative as the first nontrivial operator. In view of the relations between the classical Schwarzian derivative and the analogous differential operator defined in terms of Peschl's differential operators, we define the generating function of our generalized higher operators in terms of Peschl's differential operators and obtain recursion formulas for them. Our generalized higher operators include the analogous differential operator to the classical Schwarzian derivative. A special case of our generalized higher Schwarzian operators turns out to be the Tamanoi's operators as expected.

1. PRELIMINARIES

In this paper, we consider three Riemann surfaces, the Riemann sphere $\widehat{\mathbb{C}}$, the complex plane $\mathbb{C}$, and the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. We use the notation $\mathbb{C}_\varepsilon$ for these three surfaces; $\widehat{\mathbb{C}}$ for $\varepsilon = 1$, $\mathbb{C}$ for $\varepsilon = 0$, and $\mathbb{D}$ for $\varepsilon = -1$. $\mathbb{C}_\varepsilon$ is equipped with the canonical metric $\lambda_\varepsilon(z)|dz| = |dz|/(1+\varepsilon|z|^2)$.

For a holomorphic map $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta, \varepsilon = 1, 0, -1)$, we consider invariant operators of $f(z)$ associated with $\mathbb{C}_\delta$ and $\mathbb{C}_\varepsilon$, $D^n f(z)$ due to Peschl [9], which is defined by the power series expansion

$$\frac{f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right) - f(z)}{1+\varepsilon\overline{f(z)}f\left(\frac{\zeta+z}{1-\delta\bar{z}\zeta}\right)} = \sum_{n=1}^{\infty} \frac{D^n f(z)}{n!} \cdot \zeta^n \tag{1.1}$$

around $\zeta = 0$.

For $n = 1, 2, 3$,

$$D^1 f(z) = \frac{(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2},$$

Received by the editors November 26, 2008. Revised January 20, 2009. Accepted February 4, 2009.
2000 *Mathematics Subject Classification.* Primary 30F45, Secondary 58C10.
Key words and phrases. Schwarzian derivative, Peschl's invariant differential operators, higher Schwarzian operators.

$$D^2 f(z) = \frac{(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2} + \frac{2\delta\bar{z}(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} - \frac{2\varepsilon(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2},$$

$$D^3 f(z) = \frac{(1+\delta|z|^2)^3 f'''(z)}{1+\varepsilon|f(z)|^2} - \frac{6\varepsilon(1+\delta|z|^2)^3\overline{f(z)}f'(z)f''(z)}{(1+\varepsilon|f(z)|^2)^2} + \frac{6\delta\bar{z}(1+\delta|z|^2)^2 f''(z)}{1+\varepsilon|f(z)|^2}$$

$$+\frac{6\delta^2\bar{z}^2(1+\delta|z|^2)f'(z)}{1+\varepsilon|f(z)|^2} - \frac{12\delta\varepsilon\bar{z}(1+\delta|z|^2)^2\overline{f(z)}f'(z)^2}{(1+\varepsilon|f(z)|^2)^2} + \frac{6\varepsilon^2(1+\delta|z|^2)^3\overline{f(z)}^2 f'(z)^3}{(1+\varepsilon|f(z)|^2)^3}.$$

When $\delta = \varepsilon = 0$, $D^n f(z)$ is just n-th derivative $f^{(n)}(z)$ of $f(z)$. These operators appeared in [3] for $\delta = -1, \varepsilon = 0$ and in [4], [7] for $\delta = -1, \varepsilon = 1$ and in [8] for $\delta = -1, \varepsilon = -1$. These are also generalized by Minda and Schippers [10] and Kim and Sugawa [5] for arbitrary conformal metrics. These operators are invariant in the sense that

$$|D^n(L \circ f \circ M)| = |D^n f| \circ M$$

for $L \in \mathrm{Isom}^+(\mathbb{C}_\varepsilon)$ and $M \in \mathrm{Isom}^+(\mathbb{C}_\delta)$. Note that the group $\mathrm{Isom}^+(\mathbb{C}_\varepsilon)$ of sense-preserving isometries of $\mathbb{C}_\varepsilon$ consists of the maps $L(\zeta) = \eta(\zeta - a)/(1 + \varepsilon\bar{a}\zeta)$ for some $a \in \mathbb{C}_\varepsilon$ and $\eta \in \mathbb{C}$ with $|\eta| = 1$, where $L(\zeta) = -\eta/\zeta$ for $\varepsilon = 1$ and $a = \infty$.

The classical Schwarzian derivative S_f for a non-constant meromorphic function $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta = -1, \varepsilon = 1, 0, -1)$, is expressed in terms of Peschl's differential operators ([3], [4], [8]).

$$\frac{D^3 f}{D^1 f} - \frac{3}{2}\left(\frac{D^2 f}{D^1 f}\right)^2 = (1-|z|^2)^2 S_f.$$

Here, the classical Schwarzian derivative S_f is defined by

$$S_f = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2.$$

It is well known that the formula

$$S_{g\circ f} = (S_g \circ f)\cdot(f')^2 + S_f$$

holds for the composition $g \circ f$ of non-constant meromorphic functions f and g. $S_f = 0$ on a domain if and only if f is (a restriction of) a Möbius transformation and hence $S_{M\circ f\circ L} = (S_f \circ L)\cdot(L')^2$ holds for Möbius transformations M and L. The classical Schwarzian derivative plays an important role in the theory of univalent functions. For example, Nehari ([2], Theorem 8.12) showed that if $(1-|z|^2)^2|S_f(z)| \le 2$, then f is univalent.

Various higher Schwarzian operators have been introduced and studied by Aharonov [1], Tamanoi [12] and Schippers ([10], [11]). Recently, Kim and Sugawa [6] also introduced the invariant Schwarzian derivatives of higher order in the polynomia forms. In this paper, we observe Tamanoi's higher Schwarzian operators and show that they can be generalized in terms of Peschl's differential operators as in the case of the classical Schwarzian derivative. In fact, we generalize Tamanoi's higher Schwarzian operators by giving a definition of the generating function of our new operators in terms of Peschl's differential operators. Our generalized higher Schwarzian operators turn out to be the Tamanoi's operators in the special case.

2. Tamanoi's Higher Schwarzian Operators

Tamanoi constructs a family of Möbius invariant nonlinear differential operators which are called higher Schwarzian operators [12]. The classical Schwarzian derivative is included as a nontrivial lowest order operator. He proves that any Möbius invariant differential operator in one complex variable can be derived from the classical Schwarzian derivative. In this section, we give a brief account of the generating function of Tamanoi's higher Schwarzian operators for an analytic function defined on plane region and the recursion formula of them. Tamanoi [12] introduces higher Schwarzian operators $S_n f$ in the following way. For an analytic function f defined on a domain Ω in $\mathbb{C}$, fix a point $z \in \Omega$ where $f'(z)$ does not vanish. Take a Möbius transformation $T_{f,z}$ satisfying that

$$T_{f,z}(0) = f(z),\ T'_{f,z}(0) = f'(z),\ T''_{f,z}(0) = f''(z).$$

Then expand $V(z,w) = (T_{f,z}^{-1} \circ f)(z+w)$ as a power series around $w = 0$:

$$V(z,w) = \frac{f'(z)(f(z+w) - f(z))}{\frac{1}{2}f''(z)(f(z+w) - f(z)) + f'(z)^2} = \sum_{n=0}^{\infty} S_n f(z) \frac{w^n}{n!} \tag{2.1}$$

Here, our notation is slightly different from that of Tamanoi; our $S_n f$ is written as $S_{n-1}[f]$ in [12]. By the above choice of $T_{f,z}$,

$$S_0 f = 0,\ S_1 f = 1,\ S_2 f = 0$$

and $S_3 f$ is the classical Schwarzian derivative S_f.

By direct partial differentiations of $V(z,w)$, we can show

$$\frac{\partial V}{\partial w} - \frac{\partial V}{\partial z} = 1 + \tfrac{1}{2} S_3 f(z) V^2. \tag{2.2}$$

We obtain the recursion formula for $S_n f$ by inserting the power series expansion of $V(z,w)$ into the equation (2.2) and comparing the coefficients of w^n as follows.

$$S_{n+1}f = (S_nf)' + \tfrac{1}{2}S_3f\sum_{k=0}^{n}\binom{n}{k}S_kfS_{n-k}f, \quad n \geq 3. \tag{2.3}$$

More detailed proof is found in [6].

Tamanoi's higher Schwarzian operators are Möbius invariant, that is, for any holomorphic function f defined on Ω with nonvanishing first derivative, we have $S_n[T \circ f](z) = S_nf(z)$ for any Möbius transformation T.

3. Generalized Higher Schwarzian Operators

Let $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta, \varepsilon = 1, 0, -1)$ be a holomorphic mapping. Now, we generalize Tamanoi's higher Schwarzian operators in terms of the Peschl's differential operators in the following way. We establish a generating function $W(z, w)$ for our new higher Schwarzian operators and define our generalized higher Schwarzian operators $\Phi^n f$ through expanding $W(z, w)$ as a power series for small w:

$$W(z,w) = \frac{D^1f(z)g(z,w)}{\frac{1}{2}D^2f(z)g(z,w) + [D^1f(z)]^2} = \sum_{n=0}^{\infty}\Phi^n f(z)\frac{w^n}{n!}. \tag{3.1}$$

Here,

$$g(z,w) = (M_{f(z),\varepsilon} \circ f \circ M_{-z,\delta})(w),$$

and

$$M_{-z,\delta}(w) = \frac{w+z}{1-\delta\bar{z}w}.$$

Explicitly,

$$g(z,w) = \frac{f\left(\frac{w+z}{1-\delta\bar{z}w}\right) - f(z)}{1+\varepsilon\overline{f(z)}f\left(\frac{w+z}{1-\delta\bar{z}w}\right)} = \sum_{n=1}^{\infty}\frac{D^nf(z)}{n!}w^n. \tag{3.2}$$

Tamanoi's generating function in (2.1) corresponds to the case where $\delta = \varepsilon = 0$ in the above definition of the generalized higher Schwarzian operator.

The next theorem gives an analogous expression of our higher Schwarzian operator to the classical Schwarzian derivative.

Theorem 3.1. *Let $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta, \varepsilon = 1, 0, -1)$ be a nonconstant holomorphic mapping with the canonical metric λ_δ and λ_ε, respectively. Then,*

$$\Phi^0 f = 0, \quad \Phi^1 f = 1, \quad \Phi^2 f = 0,$$

$$\Phi^3 f = \frac{D^3f}{D^1f} - \frac{3}{2}\left(\frac{D^2f}{D^1f}\right)^2.$$

Proof. By (3.1) and (3.2), we obtain

$$g(z,0)=0, \quad D^n f=\left(\frac{\partial^n g}{\partial w^n}\right)_{w=0},$$
$$\Phi^0 f=W(z,0)=0, \quad \Phi^n f=\left(\frac{\partial^n W}{\partial w^n}\right)_{w=0}.$$

Now, we explicitly calculate $(\frac{\partial^n W}{\partial w^n})_{w=0}$ and write them in terms of $D^n f$ for $n=1,2,3$ to obtain the above assertion in this theorem. To make calculations easier, we let

$$B(z,w)=\frac{D^2 f(z)}{2D^1 f(z)}g(z,w)+D^1 f(z).$$

Then, $B(z,0)=D^1 f(z)$ and,

$$\frac{\partial W}{\partial w}=\frac{D^1 f}{B(z,w)^2}\frac{\partial g}{\partial w},$$
$$\frac{\partial^2 W}{\partial w^2}=\frac{D^1 f}{B(z,w)^2}\frac{\partial^2 g}{\partial w^2}-\frac{D^2 f}{B(z,w)^3}\left(\frac{\partial g}{\partial w}\right)^2,$$
$$\frac{\partial^3 W}{\partial w^3}=\frac{D^1 f}{B(z,w)^2}\frac{\partial^3 g}{\partial w^3}-\frac{3D^2 f}{B(z,w)^3}\frac{\partial g}{\partial w}\frac{\partial^2 g}{\partial w^2}+\frac{3}{2B(z,w)^4}\frac{(D^2 f)^2}{D^1 f}\left(\frac{\partial g}{\partial w}\right)^3.$$

Substituting $w=0$ into the above three relations, we complete the proof. □

We note that the classical Schwarzian derivative has an analogous form to that of $\Phi^3 f$ in terms of $D^n f$ for $n=1,2,3$. Using the next lemma, we establish the recursion formula for the generalized higher Schwarzian operators $\Phi^n f$.

Lemma 3.2. *Let $f:\mathbb{C}_\delta\to\mathbb{C}_\varepsilon$ $(\delta,\varepsilon=1,0,-1)$ be a nonconstant holomorphic mapping with the canonical metric λ_δ and λ_ε, respectively. Then,*

$$(1-\delta\bar{z}w)\frac{\partial W}{\partial w}-(1+\delta|z|^2)\frac{\partial W}{\partial z}=1+\tfrac{1}{2}\Phi^3 f(z)W^2-\delta\bar{z}W. \tag{3.3}$$

Proof. For the proof of (3.3), the following property of $g(z,w)$ is crucial.

$$(1-\delta\bar{z}w)\frac{\partial g}{\partial w}-(1+\delta|z|^2)\frac{\partial g}{\partial z}=(1-\varepsilon\overline{f(z)}g(z,w))D^1 f(z). \tag{3.4}$$

This follows from the direct calculations of next derivatives,

$$\frac{\partial g}{\partial w}=\frac{(1+\delta|z|^2)(1+\varepsilon|f(z)|^2)}{(1-\delta\bar{z}w)^2}\frac{f'(u)}{(1+\varepsilon\overline{f(z)}f(u))^2},$$

and

$$\frac{\partial g}{\partial z}=\frac{-f'(z)}{1+\varepsilon\overline{f(z)}f(u)}+\frac{(1+\varepsilon|f(z)|^2)f'(u)}{[1+\varepsilon\overline{f(z)}f(u)]^2(1-\delta\bar{z}w)}.$$

Here,

$$u=\frac{w+z}{1-\delta\bar{z}w}.$$

Since

$$\frac{\partial W}{\partial w} = \frac{D^1 f}{B(z,w)^2}\frac{\partial g}{\partial w}$$

and

$$\frac{\partial W}{\partial z} = \frac{1}{B(z,w)^2}\left[D^1 f\frac{\partial g}{\partial z} - g\frac{\partial(D^1 f)}{\partial z} - g^2\frac{\partial}{\partial z}\left(\frac{1}{2}\frac{D^2 f}{D^1 f}\right)\right],$$

we have

$$\begin{aligned}&(1-\delta\bar{z}w)\frac{\partial W}{\partial w} - (1+\delta|z|^2)\frac{\partial W}{\partial z}\\ &= \frac{1}{B(z,w)^2}(1-\varepsilon\overline{f(z)}g)(D^1 f)^2\\ &+ \frac{1}{B(z,w)^2}(1+\delta|z|^2)\left[g\frac{\partial(D^1 f)}{\partial z} + g^2\frac{\partial}{\partial z}\left(\frac{1}{2}\frac{D^2 f}{D^1 f}\right)\right]\end{aligned}$$

by using (3.4). To get the partial derivatives of $D^1 f$ and $D^2 f$, we now use the following recursion relations for $D^n f(z)$([5], Corollary 7.3): For $n \geq 1$,

$$(1+\delta|z|^2)\frac{\partial(D^n f)}{\partial z} = D^{n+1}f(z) - \delta n\bar{z}D^n f(z) + \varepsilon\overline{f(z)}D^1 f(z)D^n f(z).$$

Therefore, we have

$$(1+\delta|z|^2)\frac{\partial(D^1 f)}{\partial z} = D^2 f - \delta\bar{z}D^1 f(z) + \varepsilon\overline{f(z)}(D^1 f)^2,$$

and

$$(1+\delta|z|^2)\frac{\partial}{\partial z}\left(\frac{D^2 f}{D^1 f}\right) = \frac{D^3 f}{D^1 f} - \left(\frac{D^2 f}{D^1 f}\right)^2 - \delta\bar{z}\frac{D^2 f}{D^1 f}.$$

Now, it only takes simple rearrangements of terms to prove the lemma. □

Next, we obtain the recursion formula for $\Phi^n f$ from Lemma 3.2.

Theorem 3.3. *Let $f : \mathbb{C}_\delta \to \mathbb{C}_\varepsilon$ $(\delta, \varepsilon = 1, 0, -1)$ be a nonconstant holomorphic mapping with the canonical metric λ_δ and λ_ε, respectively. Then*

(3.5)

$$\Phi^{n+1}f = \left[(1+\delta|z|^2)\frac{\partial}{\partial z} - (n-1)\delta\bar{z}\right]\Phi^n f + \frac{1}{2}\Phi^3 f\sum_{k=0}^{n}\binom{n}{k}\Phi^k f\Phi^{n-k}f, \quad n \geq 3.$$

Proof. Since

$$W(z,w) = \sum_{n=0}^{\infty}\Phi^n f(z)\frac{w^n}{n!},$$

we have

$$\frac{\partial W}{\partial w} = \sum_{n=1}^{\infty}\Phi^n f(z)\frac{w^{n-1}}{(n-1)!}, \quad \frac{\partial W}{\partial z} = \sum_{n=0}^{\infty}\frac{\partial}{\partial z}(\Phi^n f(z))\frac{w^n}{n!}$$

and

$$W^2 = \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \Phi^k f(z) \Phi^{n-k} f(z) \right] \frac{w^n}{n!}.$$

By inserting these three relations into the equation (3.3) of Lemma 3.2, we have

$$(1 - \delta \bar{z} w) \sum_{n=1}^{\infty} \Phi^n f(z) \frac{w^{n-1}}{(n-1)!} - (1 + \delta |z|^2) \sum_{n=0}^{\infty} \frac{\partial}{\partial z} (\Phi^n f(z)) \frac{w^n}{n!}$$
$$= 1 + \tfrac{1}{2} \Phi^3 f(z) \sum_{n=0}^{\infty} \left[\sum_{k=0}^{n} \binom{n}{k} \Phi^k f(z) \, \Phi^{n-k} f(z) \right] \frac{w^n}{n!} - \delta \bar{z} \sum_{n=0}^{\infty} \Phi^n f(z) \frac{w^n}{n!}.$$

Comparing the coefficients of w^n on both sides of the above relation, we obtain the recursion formula for the generalized higher Schwarzian operators. □

The recursion formula (3.5) is similar to the recursion formula (2.3) for Tamanoi's higher Schwarzian operators. They look slightly different because of the first term of the right-hand side of (3.5).

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