SOME RESULTS ON $(LCS)_n$-MANIFOLDS

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Abstract. The object of the present paper is to study $(LCS)_n$-manifolds. Several interesting results on a $(LCS)_n$-manifold are obtained. Also the generalized Ricci recurrent $(LCS)_n$-manifolds are studied. The existence of such a manifold is ensured by several non-trivial new examples.

1. Introduction

Recently the present author [6] introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) with an example. The present paper deals with a study of various types of $(LCS)_n$-manifolds. After preliminaries, in Section 3 we study the fundamental results of $(LCS)_n$-manifolds and proved that in such a manifold the Ricci operator commutes with the structure tensor $\phi$. Section 4 is devoted to the study of conformally flat $(LCS)_n$-manifolds and it is proved that such a $(LCS)_n$-manifold is $\eta$-Einstein as well as a manifold of quasi constant curvature. The notion of generalized Ricci recurrent manifold was introduced by De, Guha, and Kamilya [2] in 1995. Section 5 is concerned with generalized Ricci recurrent $(LCS)_n$-manifolds and in the last section we investigate the existence of such a manifold and found various new examples of both in even and odd dimensions.

2. $(LCS)_n$-manifolds

An $n$-dimensional Lorentzian manifold $M$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where $T_p M$ denotes the tangent vector space of $M$ at $p$ and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p M$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp., $\leq 0$, $= 0$, $> 0$) [5]. The category to which a given vector falls is called its causal character.

Received July 13, 2006; Revised October 22, 2008.

2000 Mathematics Subject Classification. 53C15, 53C25.

Key words and phrases. $(LCS)_n$-manifold, conformally flat, generalized Ricci recurrent, $\eta$-Einstein, quasi constant curvature.

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Definition 2.1. In a Lorentzian manifold \((M, g)\) a vector field \(P\) defined by
\[
g(X, P) = A(X)
\]
for any \(X \in \chi(M)\) is said to be a concircular vector field if
\[
(\nabla_X A)(Y) = \alpha \{g(X, Y) + \omega(X)A(Y)\},
\]
where \(\alpha\) is a non-zero scalar and \(\omega\) is a closed 1-form.

Let \(M^n\) be a Lorentzian manifold admitting a unit timelike concircular vector field \(\xi\), called the characteristic vector field of the manifold. Then we have
\[
g(\xi, \xi) = -1.
\]
Since \(\xi\) is a unit concircular vector field, it follows that there exists a non-zero 1-form \(\eta\) such that for
\[
g(X, \xi) = \eta(X),
\]
the equation of the following form holds
\[
(\nabla X \eta)(Y) = \alpha \{g(X, Y) + \eta(X)\eta(Y)\} \quad (\alpha \neq 0)
\]
for all vector fields \(X, Y\), where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\) and \(\alpha\) is a non-zero scalar function satisfies
\[
\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho \eta(X),
\]
\(\rho\) being a certain scalar function given by \(\rho = -(\xi\alpha)\). If we put
\[
\phi X = \frac{1}{\alpha} \nabla X \xi,
\]
then from (2.3) and (2.5) we have
\[
\phi X = X + \eta(X)\xi,
\]
from which it follows that \(\phi\) is a symmetric (1, 1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold \(M^n\) together with the unit timelike concircular vector field \(\xi\), its associated 1-form \(\eta\) and (1, 1) tensor field \(\phi\) is said to be a Lorentzian concircular structure manifold (briefly \((LCS)_n\)-manifold) [6]. Especially, if we take \(\alpha = 1\), then we can obtain the LP-Sasakian structure of Matsumoto [4]. In a \((LCS)_n\)-manifold, the following relations hold [6]:
\[
\eta(\xi) = -1, \quad \phi \xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]
\[
\eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],
\]
\[
S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X),
\]
\[
R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y],
\]
\[
(\nabla X \phi)(Y) = \alpha \{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}
\]
for all vector fields $X, Y, Z$, where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold.

### 3. Fundamental results of $(LCS)_n$-manifolds

**Proposition 3.1.** A $(LCS)_n$-manifold of constant curvature is a manifold of constant curvature $(\alpha^2 - \rho)$.

**Proof.** If a $(LCS)_n$-manifold is of constant curvature $k$, say, then we have

$$R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y],$$

which yields by setting $Z = \xi$ that

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y].$$

This implies by virtue of (2.10) that $k = (\alpha^2 - \rho)$. Hence the proposition is proved. □

**Lemma 3.1.** In a $(LCS)_n$-manifold, the following relation holds:

$$X\rho = d\rho(X) = \beta\eta(X)$$

for any vector field $X$ and $\beta$ is a certain scalar function.

**Proof.** From (2.4), it follows that

$$\nabla (d\alpha)(Y, X) = \nabla_X (d\alpha)(Y) = X(Y\alpha) - ((\nabla_X Y)\alpha)$$

which implies that

$$\nabla (d\alpha)(X, Y) = (d\alpha)(Y, X).$$

Also

$$\nabla (d\alpha)(Y, X) = Y(d\alpha(X)) - d\alpha(\nabla_Y X),$$

which implies by virtue of (2.3) and (2.4) that

$$\nabla (d\alpha)(Y, X) = (Y\rho)\eta(X) + \rho\alpha [g(X, Y) + \eta(X)\eta(Y)].$$

This implies by virtue of (2.2) that

$$(X\rho)\eta(Y) = (Y\rho)\eta(X),$$

which yields

$$(X\rho) = \beta\eta(X),$$

where $\beta = -\xi\rho$ is a scalar function. Hence the result holds. □

**Lemma 3.2.** Let $M^n(\phi, \xi, \eta, g)$ be a $(LCS)_n$-manifold. Then for any $X, Y, Z$ on $M^n$, the following relation holds:

$$R(X, Y)\phi Z - \phi R(X, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi + \eta(Z)\{\eta(X)Y - \eta(Y)X\}.$$

**Proof.** From (2.3)-(2.7), (2.11) and the Ricci identity we can easily get (3.3). □
Lemma 3.3. Let \((M^n, g)\) be a \((LCS)_n\)-manifold. Then
\[
(3.4) \quad g(\phi R(\phi X, \phi Y)Z, \phi W) = g(R(X, Y)Z, W) + (\alpha^2 - \rho)[\{g(Y, W)\eta(Z)
- g(Y, Z)\eta(W)\}\eta(X) + \{g(X, W)\eta(Z)
- g(X, Z)\eta(W)\}\eta(Y)]
\]
for any vector field \(X, Y, Z, W\) on \(M^n\).

Proof. Using (2.6), (2.8) and \(\eta(\phi X) = 0\), we can calculate
\[
g(\phi R(\phi X, \phi Y)Z, \phi W) = g(R(\phi X, \phi Y)Z, W) = g(R(Z, W)\phi X, \phi Y)
= g(\phi R(Z, W)X, \phi Y) + (\alpha^2 - \rho)[g(W, \phi Y)\eta(X)\eta(Z)
- g(Z, \phi Y)\eta(X)\eta(W)].
\]
The relation (3.4) follows from this and
\[
g(R(Z, W)X, Y) = g(R(X, Y)Z, W). \quad \Box
\]

Lemma 3.4. Let \((M^n, g)\) be a \((LCS)_n\)-manifold. Then for any \(X, Y, Z\) on \(M^n\), the following relation holds:
\[
(3.5) \quad g(R(\phi X, \phi Y)\phi Z, \phi W) = g(R(X, Y)Z, W) + (\alpha^2 - \rho)[\{g(Y, W)\eta(Z)
- g(Y, Z)\eta(W)\}\eta(X) + \{g(X, W)\eta(Z)
- g(X, Z)\eta(W)\}\eta(Y)].
\]
Proof. Replacing \(X, Y\) by \(\phi X, \phi Y\) respectively in (3.3) and taking the inner product on both sides by \(\phi W\) we get
\[
g(R(\phi X, \phi Y)\phi Z, \phi W) = g(\phi R(\phi X, \phi Y)Z, \phi W).
\]
Using (3.4) in (3.6) we obtain (3.5). \quad \Box

Theorem 3.1. Let \((M^n, g)\) be a \((LCS)_n\)-manifold. Then the Ricci operator \(Q\) commutes with \(\phi\).

Proof. To prove the result, we shall show that
\[
(3.7) \quad Q\phi = \phi Q.
\]
From (3.2), it follows that
\[
(3.8) \quad \phi R(\phi X, \phi Y)\phi Z = R(X, Y)Z + (\alpha^2 - \rho)[\eta(X)\{\eta(Z)Y - g(Y, Z)\xi\}
+ \eta(Y)\{\eta(Z)X - g(X, Z)\xi\}].
\]
We now consider the following two cases:

(i) \(\dim M = n = \text{odd} = 2m + 1\),
(ii) \(\dim M = n = \text{even} = 2m + 2\).
Case (i): If \( n = 2m + 1 \), let \( \{ e_i, \phi e_i, \xi \} \), \( i = 1, 2, \ldots, m \) be an orthonormal frame at any point of the manifold. Then putting \( Y = Z = e_i \) in (3.8) and taking summation over \( i \) and using \( \eta(e_i) = 0 \), we get

\[
(3.9) \quad \sum_{i=1}^{m} \epsilon_i \phi R(\phi X, \phi e_i) \phi e_i = \sum_{i=1}^{m} \epsilon_i R(X, e_i) e_i - m(\alpha^2 - \rho)\eta(X)\xi,
\]

where \( \epsilon_i = g(e_i, e_i) \).

Again setting \( Y = Z = e_i \) in (3.8) and taking summation over \( i \) and then using \( \eta(e_i) = 0 \) and (2.1) we get

\[
(3.10) \quad \sum_{i=1}^{m} \epsilon_i \phi R(\phi X, e_i) e_i = \sum_{i=1}^{m} \epsilon_i R(X, e_i) e_i - (m+1)(\alpha^2 - \rho)\eta(X)\xi.
\]

Adding (3.9) and (3.10) and the definition of the Ricci operator, we obtain

\[
\phi(Q\phi X - R(\phi X, \xi)\xi) = QX - R(X, \xi)\xi - 2m(\alpha^2 - \rho)\eta(X)\xi.
\]

Using (2.10) and \( \phi \xi = 0 \) in the above relation we have

\[
\phi Q\phi X = QX - 2m(\alpha^2 - \rho)\eta(X)\xi.
\]

Operating both sides by \( \phi \) and using (2.1), symmetry of \( Q \), \( \phi \xi = 0 \) and (2.9) we get (3.7).

Case (ii): If \( n = 2m + 2 \), let \( \{ e_i, \phi e_i \} \), \( i = 1, 2, \ldots, m + 1 \) be an orthonormal frame such that each \( e_i \) is orthogonal to \( \xi \), i.e., \( \eta(e_i) = 0 \). Then putting \( Y = Z = e_i \) in (3.8) and taking summation over \( i \) and using \( \eta(e_i) = 0 \), we get

\[
(3.11) \quad \sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, \phi e_i) \phi e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, e_i) e_i - (m+1)(\alpha^2 - \rho)\eta(X)\xi,
\]

where \( \epsilon_i = g(e_i, e_i) \).

Again replacing \( Y \) and \( Z \) by \( \phi e_i \) in (3.8) and taking summation over \( i \) and then using \( \eta(e_i) = 0 \) and (2.1), it follows that

\[
(3.12) \quad \sum_{i=1}^{m+1} \epsilon_i \phi R(\phi X, e_i) e_i = \sum_{i=1}^{m+1} \epsilon_i R(X, e_i) e_i - (m+1)(\alpha^2 - \rho)\eta(X)\xi.
\]

Adding (3.11) and (3.12) and then proceeding similarly as in Case (i) we can easily obtain (3.7). This proves the theorem.

**Proposition 3.2.** In a \((LCS)_{n}\)-manifold the relation

\[
S(\phi X, \phi Y) = (n-1)(\alpha^2 - \rho)g(X, Y) + S(X, Y)
\]

holds.

**Proof.** The proposition follows from Theorem 3.1.
4. Conformally flat \((LCS)_n\)-manifolds

This section deals with conformally flat \((LCS)_n\) \((n \geq 4)\) manifolds.

**Definition 4.1.** A \((LCS)_n\)-manifold is said to be \(\eta\)-Einstein if its Ricci tensor \(S\) of type \((0, 2)\) is of the form
\[ S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \]
where \(a, b\) are the smooth functions over the manifold such that \(b\) is non-zero.

**Theorem 4.1.** A conformally flat \((LCS)_n\) \((n \geq 4)\) manifold is an \(\eta\)-Einstein manifold.

**Proof.** If a \((LCS)_n\) \((n \geq 4)\) manifold is conformally flat, then its curvature tensor is given by
\[
R(X, Y)Z = \frac{1}{n-2} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY]
- \frac{r}{(n-1)(n-2)} [g(Y, Z)X - g(X, Z)Y].
\]

Setting \(Z = \xi\) in (4.1) and then using (2.9) and (2.10) we obtain
\[
(\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y] = \frac{1}{n-2} [(n-1)(\alpha^2 - \rho)\eta(Y)X - \eta(X)Y]
+ \eta(Y)QX - \eta(X)QY
- \frac{r}{(n-1)(n-2)} [\eta(Y)X - \eta(X)Y].
\]

Again replacing \(Y\) by \(\xi\) in (4.2) we obtain by virtue of (2.9) that
\[
QX = \left[ \frac{r}{n-1} - (\alpha^2 - \rho) \right] X - \left[ \frac{r}{n-1} - n(\alpha^2 - \rho) \right] \eta(X)\xi
\]
which can also be written as
\[
S(X, Y) = \left[ \frac{r}{n-1} - (\alpha^2 - \rho) \right] g(X, Y) - \left[ \frac{r}{n-1} - n(\alpha^2 - \rho) \right] \eta(X)\eta(Y)
\]
which implies that the manifold is \(\eta\)-Einstein. \(\square\)

**Corollary 4.1.** A \((LCS)_3\) manifold is an \(\eta\)-Einstein manifold.

**Proof.** Since in a 3-dimensional Lorentzian manifold, the Weyl conformal curvature tensor vanishes, it follows that (4.1) holds for \(n = 3\) and hence it can be easily shown that a \((LCS)_3\) manifold is always an \(\eta\)-Einstein manifold. \(\square\)

**Definition 4.2.** A Riemannian manifold \((M^n, g)\) \((n \geq 4)\) is said to be of quasi-constant curvature if it is conformally flat and its curvature tensor \(\tilde{R}\) of type \((0, 4)\) has the following form:
\[
\tilde{R}(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]
+ b[g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)]
+ g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W),
\]
where $A$ is a 1-form and $a, b$ are scalars of which $b \neq 0$.

This notion of quasi-constant curvature was introduced by Chen and Yano [1].

**Theorem 4.2.** A conformally flat $(LCS)_n$ $(n \geq 4)$ manifold is of quasi-constant curvature.

**Proof.** By virtue of (4.3) and (4.4), the relation (4.1) takes the form

$$(4.6) \quad \tilde{R}(X, Y, Z, W) = \tilde{a}[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)]$$

$$+ \tilde{b}[g(Y, W)\eta(Y)\eta(Z) - g(Y, Z)\eta(W)]$$

$$+ g(Y, Z)\eta(W) - g(X, Z)\eta(Y),$$

where $\tilde{a} = \frac{1}{n-2}[\frac{r}{n-2} - 2(\alpha^2 - \rho)]$ and $\tilde{b} = \frac{1}{n-2}[\frac{r}{n-2} - n(\alpha^2 - \rho)]$ are smooth functions. Here $\tilde{b} \neq 0$ as for $\tilde{b} = 0$, (4.4) yields that the manifold is Einstein, but the manifold under consideration is $\eta$-Einstein. Hence comparing (4.5) and (4.6), the theorem is proved. \qed

5. Generalized Ricci recurrent $(LCS)_n$-manifold

**Definition 5.1.** A $(LCS)_n$-manifold is said to be generalized Ricci recurrent [2] if its Ricci tensor $S$ of type $(0, 2)$ satisfies the condition

$$(5.1) \quad (\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z),$$

where $A$ and $B$ are two non-zero 1-forms such that $A(X) = g(X, P)$ and $B(X) = g(X, L)$, $P$ and $L$ being associated vector fields of the 1-form $A$ and $B$, respectively.

**Theorem 5.1.** In a generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold, the 1-form $A$ and $B$ are related by

$$(5.2) \quad B(X) = (n - 1)(2\alpha \rho - \beta)\eta(X) - (\alpha^2 - \rho)A(X).$$

**Proof.** In a generalized Ricci recurrent $(LCS)_n$-manifold, we have the relation (5.1). Setting $Z = \xi$ in (5.1) we have

$$(5.3) \quad (\nabla_X S)(Y, \xi) = [(\alpha^2 - \rho)A(X) + B(X)]\eta(Y).$$

Again

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) = S(Y, \nabla_X \xi)$$

which yields by virtue of (2.3), (2.4), (2.9), and (3.1) that

$$(5.4) \quad (\nabla_X S)(Y, \xi) = (n - 1)[2\alpha \rho - \beta]\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X, Y)] - \alpha S(X, Y).$$

From (5.3) and (5.4), it follows that

$$(5.5) \quad \alpha S(X, Y) = (n - 1)[2\alpha \rho - \beta]\eta(X)\eta(Y) + \alpha(\alpha^2 - \rho)g(X, Y)$$

$$- (2\alpha^2 - \rho)A(X)\eta(Y)] - B(X)\eta(Y).$$

Replacing $Y$ by $\xi$ in (5.5) we obtain (5.2). This proves the theorem. \qed
Theorem 5.2. A generalized Ricci recurrent $(LCS)_n$-manifold is Einstein if and only if $\beta = 2\alpha\rho$.

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.5). Hence setting $Y = \phi Y$ in (5.5) and then using (2.7) we have

\[ S(X, Y) = (n - 1)(\alpha^2 - \rho)g(X, Y). \]

If the manifold under consideration is Einstein, then (5.6) implies $\alpha^2 - \rho = \text{constant}$ and hence $2\alpha\rho - \beta = 0$. Conversely, if $2\alpha\rho - \beta = 0$, then $\nabla_X(\alpha^2 - \rho) = 0$. Consequently $\alpha^2 - \rho = \text{constant}$. \(\square\)

Theorem 5.3. In an Einstein generalized Ricci recurrent $(LCS)_n$-manifold the associated 1-forms are linearly dependent and the vector fields of the associated 1-forms are of opposite direction for $\alpha^2 - \rho > 0$.

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.5). If such a manifold is Einstein, then $\alpha^2 - \rho$ is constant and hence $2\alpha\rho - \beta = 0$. Consequently (5.2) reduces to

\[ B(X) + kA(X) = 0, \]

where $k = (n - 1)(\alpha^2 - \rho) = \text{constant}$. This proves the theorem. \(\square\)

Theorem 5.4. A generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold is Ricci symmetric if and only if $\beta = 2\alpha\rho$.

Proof. In a generalized Ricci recurrent $(LCS)_n$-manifold we have the relation (5.6) from which it follows that

\[ (\nabla_X S)(Y, Z) = (n - 1)(2\alpha\rho - \beta)\eta(X)g(Y, Z). \]

If in a generalized Ricci recurrent $(LCS)_n$-manifold $\alpha^2 - \rho$ is constant, then the relation (5.7) holds. Hence using (5.7) in (5.1) we get

\[ (\nabla_X S)(Y, Z) = A(X)[S(Y, Z) - kg(Y, Z)]. \]

This implies by virtue of (5.6) that

\[ (\nabla_X S)(Y, Z) = 0. \]

Conversely, if (5.10) holds, then (5.8) implies that $2\alpha\rho - \beta = 0$ and hence $\alpha^2 - \rho = \text{constant}$. This proves the theorem. \(\square\)

Definition 5.2. The Ricci tensor of a generalized Ricci recurrent $(LCS)_n$-manifold is said to be $\eta$-parallel if it satisfies

\[ (\nabla_Z S)(\phi X, \phi Y) = 0 \]

for all vector fields $X, Y$ and $Z$ on $M$.

The notion of Ricci $\eta$-parallelity was first introduced by M. Kon [3] for the Sasakian manifolds.

Theorem 5.5. The Ricci tensor of a generalized Ricci recurrent $(LCS)_n$ $(n \geq 4)$ manifold is $\eta$-parallel if and only if the manifold is Einstein.
Proof. The Ricci tensor of a generalized Ricci recurrent \((LCS)_n\)-manifold is \(\eta\)-parallel if and only if the following relation holds \([6]\)
\[
(5.12) \quad (\nabla_z S)(X, Y) = \alpha[S(Y, Z)\eta(X) + S(X, Z)\eta(Y)] \\
- (n - 1)[(2\alpha - \beta)\eta(X)\eta(Y) + \eta(Z) \\
+ \alpha(\alpha^2 - \rho)(g(X, Z)\eta(Y) + g(Y, Z)\eta(X))].
\]
Again in a generalized Ricci recurrent \((LCS)_n\)-manifold, the relations \((5.5)\) and \((5.6)\) hold. Therefore, in view of \((5.6)\), \((5.8)\) and \((5.12)\) we obtain \(2\alpha - \beta = 0\) and hence \(\alpha^2 - \rho = \text{constant}\). Consequently \((5.6)\) implies that the manifold under consideration is Einstein. Conversely, if \(2\alpha - \beta = 0\), then \(\nabla_X (\alpha^2 - \rho) = 0\). Thus if a generalized Ricci recurrent \((LCS)_n\)-manifold is Einstein, then we have \(\alpha^2 - \rho = \text{constant}\) and hence the relation \((5.10)\) holds, which implies that
\[
(\nabla_z S)(\phi X, \phi Y) = 0
\]
for all \(X, Y\) and \(Z\) on \(M\). Therefore the Ricci tensor of the manifold under consideration is \(\eta\)-parallel. Thus the theorem is proved. \(\Box\)

6. Examples of \((LCS)_n\)-manifolds

Example 6.1. We consider the 3-dimensional manifold \(M = \{(x, y, z) \in \mathbb{R}^3\}\), where \((x, y, z)\) are the standard coordinates in \(\mathbb{R}^3\). Let \(\{e_1, e_2, e_3\}\) be linearly independent global frame on \(M\) given by
\[
e_1 = e^{-z} \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial y}\right), \quad e_2 = e^{-z} \frac{\partial}{\partial y}, \quad e_3 = e^{-2z} \frac{\partial}{\partial z}.
\]
Let \(g\) be the Lorentzian metric defined by \(g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, g(e_1, e_1) = g(e_2, e_2) = 1, g(e_3, e_3) = -1\). Let \(\eta\) be the 1-form defined by \(\eta(U) = g(U, e_3)\) for any \(U \in \chi(M)\). Let \(\phi\) be the \((1, 1)\) tensor field defined by \(\phi e_1 = e_1, \phi e_2 = e_2, \phi e_3 = 0\). Then using the linearity of \(\phi\) and \(g\) we have \(\eta(e_3) = -1, \phi^2 U = U + \eta(U)e_3\) and \(g(\phi U, \phi W) = g(U, W) + \eta(U)\eta(W)\) for any \(U, W \in \chi(M)\). Thus for \(e_3 = \xi, (\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \(M\).

Let \(\nabla\) be the Levi-Civita connection with respect to the Lorentzian metric \(g\) and \(R\) be the curvature tensor of \(g\). Then we have
\[
[e_1, e_2] = -e^{-z} e_2, \quad [e_1, e_3] = e^{-2z} e_1, \quad [e_2, e_3] = e^{-2z} e_2.
\]
Taking \(e_3 = \xi\) and using Koszul formula for the Lorentzian metric \(g\), we can easily calculate
\[
\nabla_{e_1}e_3 = e^{-2z} e_1, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_1 = e^{-2z} e_3, \\
\nabla_{e_2}e_3 = e^{-2z} e_2, \quad \nabla_{e_2}e_2 = e^{-2z} e_1 - e^{-z} e_1, \quad \nabla_{e_2}e_1 = e^{-2z} e_2, \\
\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_1 = 0.
\]
From the above it can be easily seen that \((\phi, \xi, \eta, g)\) is a \((LCS)_3\) structure on \(M\). Consequently \(M^3(\phi, \xi, \eta, g)\) is a \((LCS)_3\)-manifold with \(\alpha = e^{-2z} \neq 0\) such
that \((X\alpha) = \rho q(X)\), where \(\rho = 2e^{-4z}\). Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(e_2, e_3)e_3 = e^{-4z}e_2, \quad R(e_1, e_3)e_3 = e^{-4z}e_1, \quad R(e_1, e_2)e_2 = e^{-4z}e_1 - e^{-2z}e_1,
\]

\[
R(e_2, e_3)e_2 = e^{-4z}e_3, \quad R(e_1, e_3)e_1 = e^{-4z}e_3, \quad R(e_1, e_2)e_1 = -e^{-4z}e_2 + e^{-2z}e_2
\]

and the components which can be obtained from these by the symmetry properties from which, we can easily calculate the non-vanishing components of the Ricci tensor \(S\) as follows:

\[
S(e_1, e_1) = 2e^{-4z} - e^{-2z}, \quad S(e_2, e_2) = 2e^{-4z} - e^{-2z}, \quad S(e_3, e_3) = 2e^{-4z}.
\]

Since \(\{e_1, e_2, e_3\}\) is a frame field for \((LCS)_3\)-manifold, any vector field \(X, Y \in \chi(M)\) can be written as

\[
X = a_1e_1 + b_1e_2 + c_1e_3
\]

and

\[
Y = a_2e_1 + b_2e_2 + c_2e_3,
\]

where \(a_i, b_i, c_i \in \mathbb{R}^+ (= \text{the set of positive real numbers}), i = 1, 2, 3\), such that \(c_1c_2 \neq a_1a_2 + b_1b_2\). Hence

\[
S(X, Y) = 2(a_1a_2 + b_1b_2 + c_1c_2)e^{-4z} - (a_1a_2 + b_1b_2)e^{-2z}
\]

and

\[
g(X, Y) = a_1a_2 + b_1b_2 - c_1c_2.
\]

By virtue of the above we have the following:

\[
(\nabla_{e_1}S)(X, Y) = (a_1c_2 + a_2c_1)(e^{-4z} - 4e^{-6z}),
\]

\[
(\nabla_{e_2}S)(X, Y) = (b_1c_2 + b_2c_1)(e^{-4z} - 4e^{-6z})
\]

and

\[
(\nabla_{e_3}S)(X, Y) = 0.
\]

We shall show that this \((LCS)_3\)-manifold is a generalized Ricci recurrent, i.e., it satisfies the relation \((5.1)\). Let us now consider the 1-forms

\[
A(e_1) = \frac{(a_1c_2 + a_2c_1)}{2(a_1a_2 + b_1b_2 + c_1c_2)},
\]

\[
A(e_2) = \frac{(b_1c_2 + b_2c_1)}{2(a_1a_2 + b_1b_2 + c_1c_2)},
\]

\[
A(e_3) = 0,
\]

\[
B(e_1) = \frac{e^{-2z}(a_1c_2 + a_2c_1)[(a_1a_2 + b_1b_2)(1 - 8e^{-4z}) - 8c_1c_2e^{-4z}]}{2(a_1a_2 + b_1b_2 + c_1c_2)(a_1a_2 + b_1b_2 - c_1c_2)},
\]

\[
B(e_2) = \frac{e^{-2z}(b_1c_2 + b_2c_1)[(a_1a_2 + b_1b_2)(1 - 8e^{-4z}) - 8c_1c_2e^{-4z}]}{2(a_1a_2 + b_1b_2 + c_1c_2)(a_1a_2 + b_1b_2 - c_1c_2)},
\]

\[
B(e_3) = 0.
\]
at any point $x \in M$. In our $M^3$, (5.1) reduces with these 1-forms to the following equations:

(i) $\langle \nabla_{e_1} S \rangle (X, Y) = A(e_1) S(X, Y) + B(e_1) g(X, Y),$

(ii) $\langle \nabla_{e_2} S \rangle (X, Y) = A(e_2) S(X, Y) + B(e_2) g(X, Y),$

(iii) $\langle \nabla_{e_3} S \rangle (X, Y) = A(e_3) S(X, Y) + B(e_3) g(X, Y).$

This shows that the manifold under consideration is a generalized Ricci recurrent $(LCS)_3$-manifold which is neither Ricci-symmetric nor Ricci-recurrent. Hence we can state the following:

**Theorem 6.1.** There exists a generalized Ricci recurrent $(LCS)_3$-manifold which is neither Ricci-symmetric nor Ricci-recurrent.

**Example 6.2.** We consider the 4-dimensional manifold $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_4 \neq 0\}$, where $(x_1, x_2, x_3, x_4)$ are the standard coordinates in $\mathbb{R}^4$. Let \{e_1, e_2, e_3, e_4\} be linearly independent global frame on $M$ given by

\[
e_1 = x_4 \left( \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right), e_2 = x_4 \frac{\partial}{\partial x_2}, e_3 = x_4 \left( \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right), e_4 = (x_4)^3 \frac{\partial}{\partial x_4}.
\]

We define $\phi, \xi, \eta, g$ by $\phi e_1 = e_1, \phi e_2 = e_2, \phi e_3 = e_3, \phi e_4 = 0, \xi = (x_4)^3 \frac{\partial}{\partial x_2}, \eta(X) = g(X, e_4)$ for any $X \in \chi(M)$, $g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1$, $g(e_4, e_4) = -1, g(e_i, e_j) = 0$ for $i \neq j, i, j = 1, 2, 3, 4$.

Let $\nabla$ be the Levi-Civita connection with respect to the Lorentzian metric $g$. Then we have

\[\{e_1, e_2\} = -x_4 e_2, \quad \{e_1, e_4\} = -(x_4)^2 e_1, \quad \{e_2, e_4\} = -(x_4)^2 e_2, \quad \{e_3, e_4\} = -(x_4)^2 e_3.\]

Taking $e_4 = \xi$ and using Koszul formula for the Lorentzian metric $g$, we can easily calculate

\[
\nabla_{e_1} e_4 = -(x_4)^2 e_1, \quad \nabla_{e_2} e_1 = x_4 e_2, \quad \nabla_{e_1} e_1 = -(x_4)^2 e_4, \quad \nabla_{e_2} e_4 = -(x_4)^2 e_2,
\]

\[
\nabla_{e_3} e_4 = -(x_4)^2 e_3, \quad \nabla_{e_2} e_3 = -(x_4)^2 e_4, \quad \nabla_{e_3} e_2 = -(x_4)^2 e_4 - x_4 e_1.
\]

From the above it can be easily seen that $(\phi, \xi, \eta, g)$ is an $(LCS)_4$ structure on $M$. Consequently $M^4(\phi, \xi, \eta, g)$ is an $(LCS)_4$-manifold with $\alpha = -(x_4)^2 \neq 0$ such that $(X\alpha) = \rho g(X)$, where $\rho = 2(x_4)^4$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor as follows:

\[
R(e_1, e_4) e_1 = (x_4)^4 e_4, \quad R(e_2, e_4) e_2 = (x_4)^4 e_4, \quad R(e_3, e_4) e_3 = (x_4)^4 e_4,
\]

\[
R(e_1, e_3) e_3 = (x_4)^4 e_1, \quad R(e_1, e_3) e_1 = -(x_4)^4 e_3, \quad R(e_2, e_3) e_2 = -(x_4)^4 e_3,
\]

\[
R(e_1, e_4) e_4 = (x_4)^4 e_1, \quad R(e_2, e_4) e_4 = (x_4)^4 e_2, \quad R(e_1, e_2) e_2 = [(x_4)^4 - (x_4)^2] e_1,
\]

\[
R(e_2, e_3) e_3 = (x_4)^4 e_2, \quad R(e_3, e_4) e_4 = (x_4)^4 e_3, \quad R(e_1, e_2) e_1 = -[(x_4)^4 - (x_4)^2] e_2.
\]
We shall now show that this (\(LCS\)) manifold is a generalized Ricci recurrent, i.e., it satisfies the relation (5.1). Let us now consider the 1-forms

\[ A(e_1) = -\frac{(a_1d_2 + a_2d_1)}{3(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)}, \]

\[ A(e_2) = -\frac{(b_1d_2 + b_2d_1)}{3(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)}, \]

\[ A(e_3) = -\frac{(x_2d_2 + c_2d_1)}{(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)}, \quad A(e_4) = 0, \]

\[ B(e_1) = \frac{(x_4)(a_1d_2 + a_2d_1)[(a_1a_2 + b_1b_2)(18(x_4)^2 - 1) + 18(c_1c_2 + d_1d_2)(x_2)^2]}{3(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2)}, \]

\[ B(e_2) = \frac{(x_4)(b_1d_2 + b_2d_1)[(a_1a_2 + b_1b_2)(18(x_4)^2 - 1) + 18(c_1c_2 + d_1d_2)(x_4)^4]}{3(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2)}, \]

\[ B(e_3) = \frac{(x_4)(c_1d_2 + c_2d_1)(a_1a_2 + b_1b_2)}{(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2)(a_1a_2 + b_1b_2 + c_1c_2 - d_1d_2)}, \quad B(e_4) = 0. \]
at any point $x \in M$. In our $M^4$, (5.1) reduces with these 1-forms to the following equations:

(i) $\langle \nabla_{e_1} S(X, Y) \rangle = A(e_1) S(X, Y) + B(e_1) g(X, Y)$,
(ii) $\langle \nabla_{e_2} S(X, Y) \rangle = A(e_2) S(X, Y) + B(e_2) g(X, Y)$,
(iii) $\langle \nabla_{e_3} S(X, Y) \rangle = A(e_3) S(X, Y) + B(e_3) g(X, Y)$,
(iv) $\langle \nabla_{e_4} S(X, Y) \rangle = A(e_4) S(X, Y) + B(e_4) g(X, Y)$.

This shows that the manifold under consideration is a generalized Ricci recurrent $(LCS)_4$-manifold which is neither Ricci-symmetric nor Ricci-recurrent. This leads to the following:

**Theorem 6.2.** There exists a generalized Ricci recurrent $(LCS)_4$-manifold which is neither Ricci-symmetric nor Ricci-recurrent.

Acknowledgement. The author expresses his sincere thanks to the referees for their valuable suggestions in the improvement of the paper.

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