Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 13, No. 1, March 2009, 13–22

韓國數學敎育學會誌 시리즈 D:
<數學敎育硏究>
제 13 권 제 1 호 2009 년 3 월, 13–22

# The Learning of Mathematical Algorithms and Formulas without Understanding or Flair[1]

Suffolk, John

Science and Mathematics Education Department, University Brunei Darussalam,
Tungku Link, Gadong BE 1410, Brunei; Email: njsuffolk@hotmail.com

(Received August 13, 2008. Accepted November 30, 2008)

School children in Brunei Darussalam, as elsewhere, learn how to apply a lot of algorithms and formulas in mathematics. These include methods of finding the lowest common multiple and highest common multiple of numbers and methods of factorizing quadratics. Investigations and experience have shown that both able and less able students learn to do these mechanically and unimaginatively and in a way that is reliable when answering examination questions. Most of them do not, however, learn these algorithms and methods so as to develop a deeper insight of what they learn and thereby perform even more effectively in examinations. Yet it is possible to teach these and other methods for understanding in ways that are enjoyable and enable students to use them effectively and with flair.

*Keywords*: algorithms, secondary mathematics, primary mathematics, teaching and learning
*ZDM Classification*: D73, H33
*MSC2000 Classification*: 97D70

## INTRODUCTION

Students who are preparing at University Brunei Darussalam (UBD) to be teachers of secondary school mathematics take content courses that often do not require very much use of some of the mathematics that they will later be teaching in school. For example, the circle theorems, a standard part of the school programmer, are used very little at University. What is taught at university is that which is of interest to the teachers there, who are academic research mathematicians. The mathematics that was developed by

[1] This paper has been presented at the International Session of the 41st National Meeting on Mathematics Education at Donguei University, Busan, Korea; October 31–November 1, 2008.

Diophantus, Euler and Gauss, which is not now of much research interest, is not taught. Pedagogically, this is a pity because there is a lot of interesting work there, relevant to a deeper understanding of the mathematics that is taught in school.

For example, Diophantus' solution of the problem of finding two numbers given their sum and product is elegant. In order to find out what two numbers have a sum of 52 and a product of 667, let the numbers be

$$^{52}/_2 - x = 26 - x \text{ and } 26 + x.$$

Then

$$(26 - x)(26 + x) = 667,\ 676 - x^2 = 667,\ x^2 = 9 \text{ and } x = 3.$$

(Diophantus was not concerned with negative numbers.) The numbers are 23 and 29. This is much simpler than the manipulation required when the numbers are taken to be 52 and $52 - x$.

Moreover the mathematics that they have learnt in school is taken for granted and it is not re-considered. The methods that they've learnt in school remain in use, unchallenged. However, these methods are often learnt without any understanding of their rationale. There are two possible causes of this. One is that the teacher felt that the pressure to complete the examination syllabus did not warrant the time needed (Noridah & Clements, 2000; Suffolk, 2006). There is a preference for methods that can be applied across a wide range of diverse problems. Alternatively, the learners misunderstood or did not concern themselves with the explanation. Their confusion may not be the "fault" of the teacher but may be "fusions of unintended (by teacher or author) student stressing leading to a wide range of errors and bugs" (Mason, 1996).

The consequence is that those who are preparing to be teachers in Brunei Darussalam are capable of solving school examination mathematics problems but often have a poor understanding of the reasons for the methods they use and so they cannot bring much flair or flexibility to their teaching (Suffolk, 2004). This has also been observed in Thailand (Suffolk; Tananone & Clements, 2003).

This lack of understanding is demonstrated time after time and year after year in Methods of Teaching Mathematics classes. For example, students cannot explain why it is that, when solving an equation such as $x^2 + 5x = 6$, it has first to be written in the form $x^2 + 5x - 6 = 0$. It is something done because their teacher told them. For this reason the first lectures in the course looks at how teachers justify what they teach." In addition, the students are tested on their understanding of the school syllabus so as to enable the lecturers to identify and remove misconceptions before they are displayed in the classroom. One test was on school algebra (Suffolk, 2006). School algebra learnt in school is perhaps the area of school mathematics that is mostly used in University mathematics but it is only used as a manipulative tool which is assumed has been mastered and it is not explored. Methods learnt at school which perhaps have been learnt mechanically and without understanding are therefore not developed and improved, but remain unchanged. Mason's (1996) "bugs" are not removed.

## FACTORISING AND QUADRATIC EQUATIONS

We used one question which can examine pre-service teachers' ability to factorize quadratic expressions. The results are in Table 1.

**Table 1.** Performance of factorizing algebraic expressions

| Expressions | Correct | Common errors | Method of solution (%, where seen) | | |
|---|---|---|---|---|---|
| $x^2 + 31x + 30$ | 90% | - | Split 32 | Trial & error 39 | Formula 29 |
| $x^2 - 14x - 51$ | 86% | - | Split 24 | Trial & error 32 | Formula 45 |
| $12x^2 + 37x + 3$ | 78% | $(x + {}^1/_{12})(x + 3)$ 6% | Split 44 | Trial & error 29 | Formula 26 |
| $4x^2 - 9x - 9$ | 82% | - | Split 34 | Trial & error 26 | Formula 40 |
| $3x^2 - 12$ | 69% | $3(x^2 - 4)$ (10%)<br>$x = \pm 2$ (10%) | | | |

There are two conventional methods that students in Brunei Darussalam use to factorize quadratic expressions. The method that the author learnt at school is to find the right factors by trial and error. For example, in order to rewrite $12x^2 + 37x + 3$ in the form $(?x + ?)(?x + ?)$, the students must note that since the signs are both positive the numbers required must be positive. The two terms in x have a product of $12x^2$ and the two numbers in the bracket have a product of 3. Since 3 is a prime number, these numbers are 3 and 1. So

$$12x^2 + 37x + 3 = (?x + 1)(?x + 3).$$

For $12x^2$, possible pairs are $3x$ and $4x$, $2x$ and $6x$ or $12x$ and $x$. The last pair works so

$$12x^2 + 37x + 3 = (12x + 1)(x + 3).$$

The "Split" method is based on the fact that $(ax + b)(cx + d) = acx^2 + (ad + bc)x + bd$

The coefficient of the first term × the last term $(ac \times bd)$ is equal to the product of the two coefficients of $x\,(ad \times bc)$. So if two numbers exist whose product is the product of the numbers in the first and the last terms and whose sum is the same as the coefficient of the middle term, then the expression can be factorized.

In this case, the two numbers with a product of $12 \times 3 = 36$ and a sum of 37 are 1 and

36. Split $37x$ two terms $1x$ and $36x$, so

$$\begin{aligned} 12x^2 + 37x + 3 &= 12x^2 + 36x + x + 3 \\ &= 12x(x+3) + 1(x+3) \\ &= (x+3) + (12x+1). \end{aligned}$$

The author was introduced to this method by the secondary school students that he taught in Zambia almost forty years ago. This method was very effective when the coefficient of $x^2$ is more than one, but they used it when it was not needed, to factorize expressions whose $x^2$ coefficient is 1, such as $x^2 + 31x + 30$. The two numbers needed to have a sum of 31 and a product of 30. These numbers are 30 and 1, so as $x^2 + 31x + 30 = (x+30)(x+1)$.

That should be sufficient, but for the Zambian students, this was not sufficient. They wrote:

$$\begin{aligned} x^2 + 31x + 30 &= x^2 + x + 30x + 30 \\ &= x(x+1) + 30(x+1) \\ &= (x+1) + (x+30). \end{aligned}$$

This working is unnecessary, for once the numbers 1 and 30 have been found, the product of factors $(x+1)(x+30)$. The Zambian students did not think that this was sufficient, for they must always "show their working," even though in this case, the "working" is not really working and in practice often leads to errors.

The Bruneian trainee teachers could factorize quadratic expressions. Some used trial and error and some used the "split" method but many used the quadratic formula for solving quadratic equations. For example, in order to factorize $x^2 - 14x - 51$, 45% of those who showed their working solved the equation $x^2 - 14x - 51 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{14 \pm \sqrt{14^2 + 204}}{2} = \frac{14 \pm \sqrt{400}}{2} = 17 \text{ or } -3.$$

The factorized expression is, therefore, $(x-17)(x-(-3)) = (x-17)(x+3)$.

Not all the students showed their working, although there was space provided for it but Table 1 reveals that more than a quarter of those who did show their working used this method. The method has the advantage that it can be used to factorize all quadratics and be applied without thought in examinations, though at a great price in terms of lost understanding and time wasted and there are ample opportunities for calculation errors to be made. For example, solving the equation $12x^2 + 37x + 3 = 0$ leads to $(x = \frac{1}{12})$ or $(x = -3)$ and for some 6% of students, the wrong answer $(x + \frac{1}{12}) + (x+3)$.

The difficulty of a question is usually expressed in terms of the proportion that gets the question correct. By that definition, the most difficult factorization was the difference between two squares, $3x^2 - 12$. Yet this question is not difficult if the structure is recognized but those who learn by following methods blindly will not be adept at recognizing this. Whereas factorizing quadratics can be done methodologically, factorizing the difference of two squares requires students to observe and use the structure.

There was also question requiring solution of quadratics. The results of some parts of the question are in Table 2.

**Table 2.** Quadratic equation performance

| The equations | Correct | Common errors | Method of solution (where seen) |
|---|---|---|---|
| $x^2 + 3x - 180 = 0$ | 94% | - | Split 32% Trial & error 39% Formula 29% |
| $2x^2 - 15x + 27 = 0$ | 82% | - | Split 24% Trial & error 32% Formula 45% |
| $(x - 3)(3x - 1) = 11$ | 75% | 14 and 4 (10%) | Split 49% Trial & error 20% Formula 26% |
| $9p^2 = 6p - 1$ | 82% | - | Split 34% Trial & error 26% Formula 40% |
| $(x + 4)^2 = 36$ | 82% | 2 (10%) | Expand 62% Find $\sqrt{36}$ 24% |

The first two parts were solvable by factorization immediately. The third and fourth could be factorized, after expansion and rearrangement. Over a quarter of all pre-service teachers used the formula even when the coefficient of $x^2$ was 1. When the coefficient of $x^2$ was more than one the proportion was near to a half.

The last part was another question in which the ability to "see" the structure of the question. While most got the answer, only a quarter was able to do it in the most intelligent way, namely by taking the square root. Vaiyavutjamai (2004) found that Thai students were encouraged to expand, to rearrange and then to factorize when solving an equation such as $x^2 = 4$. Following one standard path, even if it is a long one, and using it for all problem types means that the learners have less time to understand and remember, but such questions then take much longer time to answer.

## MULTIPLES AND FACTORS

In the algebra test, pre-service teachers were also asked to find the highest common factor and lowest common multiple of $p^2q^3$ and $pq^4$. They could do this by inspection. The HCF must contain each letter p and q raised to the highest power that is in both terms and is $pq^3$. The L.C.M must contain each letter that is in both terms and is raised to the highest power in the terms and is $p^2q^4$. The results in Table 3 show that many pre-service teachers did not do this.

**Table 3.** Algebraic Factors and multiples performance

| The questions | Correct | Common errors | Method of solution (if seen) (One third showed working) | |
|---|---|---|---|---|
| Find the highest common factor of $p^2q^3$ and $pq^4$ | 76% | $p^2q^4$ (6%) | Scaffolding 95% | Listing 5% |
| Find the lowest common multiple of $p^2q^3$ and $pq^4$ | 63% | $pq$ (12%) | Scaffolding 84% | Listing 6% |

The "scaffolding" method that many pre-service teachers used is based on the method that is taught in primary schools here. These students used "Primary Mathematics for Brunei Darussalam Darjah 5A" (Tan, 1994) in their fifth year of primary school. The first method of finding highest common factors in this book requires children to list the factors of the numbers, note which are the common factors and then to pick the smallest of them. For example, the factors of 12 are 1, 2, 3, 4, 6 and 12 and the factors of 18 are 1, 2, 3, 6, 9 and 18. The common factors are 1, 2, 3 and 6. The highest common factor is 6.

Students also learn the scaffolding method to find the H.C.F. of two or more numbers. The method can be illustrated by finding the H.C.F. of 8 and 16, and of 24 and 30:

```
2 )  8,  16        2 )  24,  30
2 )  4,   8        3 )  12,  15
2 )  2,   4              4    5
     1,   2
```

The instructions of how to proceed are stated clearly. "Start with the smallest number that will divide the two given numbers exactly. Continue dividing the successive answers by a common number until they cannot be divided any further by a common number." (Tan, 1994, p. 39)

| | |
|---|---|
| "Since 2 can divide 8 and 16 in three successive divisions, the H.C.F. of 8 and 16 is (2 × 2 × 2) *i.e.*, 8." (p. 39) | "2, 3 and 6 (2 × 3) can divide 24 and 30 exactly. 2, 3 and 6 are common factors of 24 and 30. The H.C.F. of 24 and 30 is 6." (p. 39) |

The scaffolding method is also called the "stem and leaf method" and the "step method." The method is not given a name in the book but this is the method that children in Brunei learn to use. It is a method that the author saw that it is used in Korean textbooks that were on display at the ICME conference that took place in Copenhagen in 2004. It is not stated by Tan (1994) why the student should start with the smallest number that can be divided into both numbers. This is unfortunate, because recognizing a larger common factor that could be used to expedite the procedure. For example, finding the H.C.F. of 48 and 36 is much shorter if the students notice that both numbers are divisible by 12.

$$\begin{array}{r|rr} 2 & 48, & 36 \\ \hline 2 & 24, & 18 \\ \hline 2 & 12, & 9 \\ \hline & 4, & 3 \end{array} \qquad \begin{array}{r|rr} 2 & 48, & 36 \\ \hline & 4, & 3 \end{array}$$

Students in Brunei use the prime numbers as the divisors. Their using is specified in Question 4 of the exercise on page 44 (Tan, 1994, p. 44):

> "Use your exercise book and find the lowest common multiple of each pair of numbers in No. 1, 2 and 3 using prime numbers as divisors."

To find the L.C.M. divide both numbers by 2 until there are no factors of 2 in either number, then by then by 5 and so on until all that remains is 1 in each column. For illustration and comparison, the H.C.F. and L.C.M. of 30 and 36 are calculated below:

$$\begin{array}{r|rr} 2 & 30, & 36 \\ \hline 3 & 15, & 18 \\ \hline & 5, & 6 \end{array} \qquad \begin{array}{r|rr} 2 & 30, & 36 \\ \hline 2 & 15, & 18 \\ \hline 3 & 5, & 9 \\ \hline 3 & 5, & 3 \\ \hline 5 & 5, & 1 \\ \hline & 1, & 1 \end{array}$$

H.C.F. is $2 \times 3 = 6$.

L.C.M. is $2 \times 2 \times 3 \times 3 \times 5 = 180$.

It would be so much simpler to find the HCF to realise that the LCM is the product of the numbers down the left hand side and the numbers at the bottom of the HCF calculation, $2 \times 3 \times 5 \times 6$. It would be even simpler if the student pupil noticed that 30 and 36 are both divisible by 6 and thereby reduced the table to two lines:

$$\begin{array}{r|rr} 6 & 30, & 36 \\ \hline & 5, & 6 \end{array}$$

← 6 is a common divisor
There are no more common factors.

H.C.F. $= 6$

L.C.M. $= 6 \times 5 \times 6$ (Product of all divisors and quotients)

$= 180$

Students learn in primary school a method for finding the H.C.F. of the numbers, but they do learn to use the method intelligently. When they prepare to be primary and secondary mathematics teachers, they are often unaware of that if a is a multiple of b, then the H.C.F. of the numbers is b and their L.C.M. is a. They often do not realize that it is not necessary to divide by the smallest possible number each time when using the scaffolding method. They are unaware of the fact that the H.C.F. of two numbers is a factor of their difference. This often makes it possible to find the H.C.F. of two numbers easily by inspection. They do not realize that the product of H.C.F. and L.C.M. of two numbers is the same as the product of the numbers.

In the test only 54% could calculate correctly what L.C.M. of 24 and 48 is and 58% know that H.C.F. of 50 and 100 is 50.

## BODMAS

BODMAS[2] provides an interesting example of a topic where a rule is applied blindly. BODMAS stands for *Brackets, Of Division, Multiplication, Addition* and *Subtraction* to indicate the order in which arithmetical operations should be carried out. The order should be Brackets first, then "Of", then Division and Multiplication going from left to right when the operations are together, and, finally Addition and Subtraction going from left to right when the operations are together, This is often taken to mean that Addition should be done before Subtraction. It is unfortunate that the word Addition in English begins with a vowel. For that reason, and in order to make a pronounceable word of BODMAS (BODMSA is not easy to say) many children add before they subtract. Thus 39% of the primary teacher trainers tested (Suffolk, 2006) thought that $18-4+2=12$.

## WHAT CAN TO BE DONE

Teachers must try to teach their students in ways which lead to interesting and challenging work and lead to understand with flair and understand the ability to use what is learnt intelligently.

Students can use Algebra Tiles to learn how to factorize quadratic expressions. Each pair of students needs an envelope containing 4 big square papers ($x^2$), 20 paper strips $(1+x)$ and 10 one square unit pieces of paper. The teacher is careful to ensure that $x$ is not a whole number. She asks each pair to select 3 strips and 6 units and challenges them to arrange them to form a rectangle out of them. She asks what are the factors of $3x+6$? After the pairs have done several examples, she asks them to take put one large square, 5 strips and 6 units and challenges them to make a rectangle again. Discussion should lead

[2] http://www.bbc.co.uk/schools/ks3bitesize/maths/number/whole_numbers_2_3.shtml

to the students discovering how to factorize quadratics.

In a textbook series for Japanese primary school children that has recently been translated into English, Gakkohtosho (2008, p. 11) introduces multiples and factors through problem solving activities. Some children want to put some 2 cm by 3 cm pictures on a bulletin board with no gaps. What size should the board be? Possible answers for the lengths are the multiples of 3; possible values for the breadth are the multiples of 2. What happens when the bulletin board is square? The possible lengths and breadths have to be the same, in other words common multiples of 2 and 3. Common divisors (Gakkohtosho, 2008) are introduced by asking what size squares will fit along an 18 cm side without leaving any gaps. These are the factors of 18.

The number 2317 can be turned into the sentence $2 \times 3 + 1 = 7$. such numbers are called guilty numbers, as people are given sentences when they are guilty of a crime. A number such as 2239 which cannot be written as a sentence is called an innocent number. Children can investigate whether their number plates are innocent or guilty. A worksheet with carefully chosen numbers can draw children's attention to show how BODMAS works. For example, 3212 becomes $3 - 2 + 1 = 2$ and not 3210. This activity can be applied to numbers that are familiar to them, such as car registration numbers.

Further questions can be asked: why are all four digit palindromes guilty, why are all numbers with two zeros in them guilty? This year, 2008, is guilty; when is the next innocent year?

For each of the three areas of weakness that Bruneian children develop through an over reliance on rules leant by rote, an activity has been suggested that makes the learning and understanding of these topics interesting. This needs to be replicated across the whole syllabus.

## REFERENCES

Gakkohtosho (2008). *Study with your friends: Mathematics for Elementary 6th Grade.* Volume 1. Tokyo: Gakkohtosho.

Mason, J. (1996). Expressing generality and roots of algebra. In: N. Bednarz; C. Kieran & L. Lee (Eds.), *Approaches to algebra.* Dordrecht: Kluwer. MATHDI **1996e.**03211

Noridah bte Abdullah & Clements, M. A. (2000). An anthropological investigation into the influence of examinations on the teaching and learning of inverse functions. In: K. Y. Wong, H. Tairab & M. A. Clements (Eds.), *Science, Mathematics and Technical Education in the 20th and 21st centuries: Proceedings of the Fifth Annual Conference of the Department of Science and Mathematics Education.* Gadong, Brunei: Universiti Brunei Darussalam.

Quigley, M. & Suffolk, J. (2002). Bruneian primary school children's performance in pencil-and-paper addition and subtraction problems. In: H. S. Dhindsa, I. P.-A. Cheong, C. Tendencia & M. A. Clements (Eds.), *Realities in science mathematics and technical education* (pp. 236–245). Gadong, Brunei: Universiti Brunei Darussalam

Suffolk, J. A. (2004). Styles of teaching multiples and factors. In: I. Cheong, H. Dhindsa, I. Kyeleve & O. Chukwu (Eds.), *Globalisation Trends in Science, Mathematics and Technical Education*. Gadong, Brunei: Universiti Brunei Darussalam.

Suffolk, J. A. & Clements, M. A. (2003). Fractions concepts and skills of Form 1 and Form 2 students in Brunei Darussalam. In: H. S. Dhindsa, S. B. Lim, P. Achleitner & M. A. Clements (Eds.), *Studies in Science, Mathematics and Technical Education*. Gadong, Brunei: Universiti Brunei Darussalam.

Suffolk, J. A.; Tananone, A. & Clements, M. A. (2003). The mathematical knowledge of prospective mathematics teachers in Brunei Darussalam and Chiang Mai (Thailand). In: H. S. Dhindsa, S. B. Lim, P. Achleitner & M. A. Clements (Eds.), *Studies in Science, Mathematics and Technical Education*. Gadong, Brunei: Universiti Brunei Darussalam.

Tall, D. & Thomas, M. (1991). Concept image and definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics* **12**, 151–169.

_____ (1991). Encouraging versatile thinking in algebra using the computer. *Educ. Stud. Math.* **22(2)**, 125–147. MATHDI **1992a.**03686 ERIC EJ436603

Tan, W. (1994). *Primary mathematics for Brunei Darussalam. Darjah 5A*. Bandar Seri Begawan, Brunei: Curriculum Development Department, Ministry of Education.

Vaiyavutjamai, P. (2004). Effects of teaching on From 3 students' learning of equations and inequations. In: I. Cheong, H. Dhindsa, I. Kyeleve & O. Chukwu (Eds.), *Globalisation Trends in Science, Mathematics and Technical Education*. Gadong, Brunei: Universiti Brunei Darussalam.